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Abstract

Let $G = (V, E)$ be a simple and undirected graph. For some real number α with $0 < \alpha \leq 1$, a set $D \subseteq V$ is called an *α-dominating set* in G if every vertex v outside D has at least $\alpha \cdot d_v$ neighbor(s) in S where d_v is the degree of v. The cardinality of a minimum α-dominating set in a graph G is called the *α-domination number* of G and denoted by $\gamma_\alpha(G)$. In this paper, we introduce a generalization of α-dominating set, that we call it *f_{deg}-dominating set*. Given a function f_{deg} where f_{deg} is as $f_{deg} : \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and f_{deg} may not be an integer-value function. A set $D \subseteq V$ is called an f_{deg}-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \geq f_{deg}(d_v)$. In this paper, for this new concept, we will present some results on the its NP-completeness, APX-completeness and inapproximability.

1 Introduction

Let $G = (V, E)$ be an undirected and simple graph. A set $D \subseteq V$ is called a *dominating* set if every vertex outside D has at least one neighbor in D. The cardinality of a minimum dominating set is called the *domination number* of G denoted by $\gamma(G)$. In 2000, Dunbar et al. [5], introduced the concept of α-domination. Let α be a real number with $0 < \alpha \leq 1$. A set $D \subseteq V$ is called an α-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \geq \alpha \times d_v$ where $N(v)$ is the set of all neighbors of v in G, and $d_v := |N(v)|$ is the degree of v. Also, let k be a real number with $k \geq 1$. A set $D \subseteq V$ is called a *k-dominating* set in G if for every vertex v outside D, $|N(v) \cap D| \geq k$.

Now consider the definition of α-dominating. One generalization of this concept is that instead of having at least $\alpha \times d_v$ neighbors in D for each vertex $v \notin D$, we have at least $f(d_v)$ neighbors in D, for some special function f. By selecting $f(x) = \alpha x$, the definition match the α-dominating. It seems that this generalization is much near to the reality. Hence, in this paper, we define the *f_{deg}-dominating* set. Given a function f_{deg} where f_{deg} is as $f_{deg} : \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and f_{deg} may not be an integer-value function. A set $D \subseteq V$ is called an f_{deg}-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \geq f_{deg}(d_v)$. In this paper, we consider the graphs with no isolated vertices. We can easily extend the results for the graphs with isolated vertices. In this paper, we prove the NP-completeness of the following problem: given a graph G and a positive integer k, decide whether G has an f_{deg}-dominating set S with $|S| \leq k$. Moreover, we prove that the problem of finding a minimum f_{deg}-dominating set when $f_{deg}(x) = k$ (in the other words, the k-dominating set) for any integer $k \geq 1$ is APX-complete (there is no PTAS). Also, we present some inapproximability result for the problem of finding a minimum f_{deg}-dominating set for constant function $f_{deg}(x) = k$.

2 NP-completeness result

In this section, we will prove that the problem of finding the f_{deg}-domination number of a graph is NP-complete, for every given function f_{deg} with some special properties. It is well known that the following decision problem, denoted by 3-REGULAR DOMINATION (3RDM), is NP-complete [6]: given a 3-regular graph $G = (V, E)$ and a positive integer k, does G has a dominating set S with $|S| \leq k$? Now, consider the following decision problem, denoted by f-DOMINATION (fDM): given a graph $G = (V, E)$ without isolated vertices and a positive integer k, does G has an f_{deg}-dominating set S with $|S| \leq k$?

We will show that fDM is NP-complete for some special functions. We will extend the proof of the result in which that α-domination is NP-complete (see [5]).

Theorem 2.1. *. If an increasing function f_{deg} with domain $\mathbb{N}$ satisfies*
a. $\forall x \in \mathbb{N}, 0 < f_{deg}(x) \leq x$,
b. $\exists x_0 > 0$ *such that* $\forall x \geq x_0,\ x + 1 \geq f_{deg}(x + 3)$.
c. *For every two integers x and y,* $f_{deg}(y + x) \leq f_{deg}(y) + f_{deg}(x)$,
d. *For a given* $x \in \mathbb{N}$, *there is* $y \in \mathbb{N}$, *such that* $y > x$ *and* $f_{deg}(y) \leq x$,
then, the problem fDM is an NP-complete problem.

Sketch of Proof. Let f_{deg} be an arbitrary function that has the conditions of the theorem. We fix the function f. We can easily see that $fDM \in NP$. Now, we proof the completeness. We make a transformation from 3RDM to fDM. Suppose that x is the smallest integer such that $(x+1) \geq f_{deg}(x+3)$, and y is the largest integer with $y > x$ and $x \geq f_{deg}(y)$. Consider the complete graph K_{y+1} and assume that $U = \{v_1, v_2, \ldots, v_x\}$ is a subset of vertices of K_{y+1} with x elements. We call the vertex set of K_{y+1} by W.

We transform a 3-regular graph G to a graph denoted by $\hat{G}$ by joining each vertex of set U to all vertices of G. Assume that S is a dominating set in G such that $|S| \leq k$. Consider the set $D = S \cup U$. Using the conditions **b** and **d**, it is easy to see that D is an f_{deg}-dominating set in $\hat{G}$ with $|D| \leq x + k$.

Now, we assume that D is an f_{deg}-dominating set in $\hat{G}$ with $|D| \leq x+k$. Among all f_{deg}-dominating set in $\hat{G}$ with $|D| \leq x + k$, we suppose that D is the one with maximum $|D \cap U|$. Also, without loss of generality we can suppose that there is a vertex in $W - U$ that is outside D. Using conditions **a, b, c,** and **d**, it is not hard to prove that the set $D \cap V(G)$ is a dominating set in G with $|D \cap V(G)| \leq k$. Because 3RDM is NP-complete [6], fDM is also NP-complete for the function f that satisfies the conditions of Theorem 2.1. □

There are many functions that satisfy the conditions of Theorem 2.1, such as $\sqrt{x}$, $\ln x$ and $\frac{x}{2}$.

3 APX-completeness result

In this section, we prove that the problem of finding a minimum f_{deg}-dominating set of a graph with maximum degree $k + 2$ and $f_{deg}(x) = k$ for any $k \geq 1$ is APX-complete (there is no PTAS). We denote the problem of finding a minimum f_{deg}-dominating set of a graph where $f_{deg}(x) = k$ by MIN k-DOM SET, and when the problem is restricted to the graphs with maximum degree $k + 2$, we call it MIN k-DOM SET-$(k + 2)$.

At first, we recall the L-reduction.

Definition 3.1. (L-reduction)[2]. Given two NP optimization problems F and G and a polynomial transformation f from instances of F to instances of G, we say that f is an L-reduction if there are two positive constants α and β such that for every instance x of F

1. $opt_G(f(x)) \leq \alpha opt_F(x)$
2. for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can, in polynomial time, find a solution y' of x with $m_F(f(x), y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta |opt_G(f(x)) - c_2|$.

To prove that a problem F is APX-complete, it is sufficient to prove that $F \in$APX and there is an L-reduction from some APX-complete problem to problem F.

Theorem 3.2 (**[?]**). *For a graph G,* MIN k-DOM SET *can be approximated in polynomial time by a factor of* $\ln(2\Delta(G)) + 1$ *where* $\Delta(G)$ *is the maximum degree of G.*

Theorem 3.3. MIN k-DOM SET-$(k + 2)$ *is an APX-complete problem for any* $k \geq 1$.

Sketch of Proof. The case $k = 1$ proved in [1]. Consider $k > 1$. Clearly, by Theorem 3.2, if the degree of vertices of the graph is bounded by a constant then the approximation ratio is constant. Thus the problem MIN k-DOM SET-$(k + 2)$ is in APX. Suppose that $G = (V, E)$ is a graph of bounded degree 3. Construct a graph $G_k = (V_k, E_k)$ of bounded degree $k + 2$ as follows. Create a set S_v of $k - 1$ new vertices for each vertex v. Join each vertex $v \in V$ to $k-1$ vertices of S_v. Given a k-dominating set D_k of $G_k = f_k(G)$ (f_k is a transformation from G to G_k. Recall Definition 3.1), we can find a dominating set D in G as $D = D_k - \left(\bigcup_{v \in V(G)} S_v\right)$. So $\gamma(G) \leq |D| = |D_k| - (k - 1)n$, where $n = |V|$. Also, given a dominating set D of G, clearly the set $D_k = \left(\bigcup_{v \in V(G)} S_v\right) \cup D$ is a k-dominating set in G_k. So $\gamma_k(G_k) \leq |D_k| = |D| + (k - 1)n$. Hence, we can easily conclude that $\gamma_k(G_k) = \gamma(G) + (k - 1)n$.

Finally, using the above argument, we can find an L-reduction with parameters $\alpha = 4k - 3$ and $\beta = 1$. So, the problem MIN k-DOM SET-$(k + 2)$ is APX-complete. □

4 Inapproximability result on MIN k-DOM SET

In this section, we presents some inapproxmabillity result for MIN k-DOM SET.

Theorem 4.1 ([3])**.** *For any constant $\epsilon > 0$ there is no polynomial time algorithm approximating* MIN 1-DOM SET *within a factor of* $(1 - \epsilon) \ln n$ *unless NP* $\subseteq$ *DTIME($n^{O(\log \log n)}$). The same result holds for bipartite graphs.*

Theorem 4.2. *For every $k \geq 1$ and every $\epsilon > 0$, there is no polynomial time algorithm approximating s̈c Min k-*DOM SET *for bipartite graphs within a factor of* $(1 - \epsilon) \ln n$*, unless NP* $\subseteq$ *DTIME($n^{O(\log \log n)}$).*

Sketch of Proof. It is sufficient that, we make some modifications in the proof of Theorem 4.1. We make a reduction from domination on a bipartite graph G with n vertices such that $n + 2k - 2 \leq n^{1+\epsilon}$ and $\gamma(G). \geq \frac{2(k-1)(1+2\epsilon)}{\epsilon^2}$. Then we transform the bipartite graph $G = (V_1, V_2, E)$ into a bipartite graph G' by adding to it two sets K_1 and K_2 each have $k - 1$ new vertices inducing a graph with no edges. Join each vertex of V_1 to each vertex of K_2 and join each vertex of V_2 to each vertex of K_1. We can easily prove that $\gamma_k(G') \leq \gamma(G) + 2k - 2$. Now, suppose that there is a polynomial time approximation algorithm that computes a k-dominating set D' for G' such that $|D'| \leq (1 - \epsilon) \ln(|V(G')|)\gamma_k(G')$. It is easy t.o see that $D := D' \cap V(G)$ is a dominating set in G. So,

$$\begin{aligned} |D| &\leq |D'| \\ &\leq (1 - \epsilon)(\ln |V(G')|)\gamma_k(G') \text{ (suppose that } n := |V(G')|) \\ &\leq (1 - \epsilon)(\ln n)(1 + \epsilon + \epsilon^2)\gamma(G) \\ &= (1 - \epsilon')(\ln n)\gamma(G), \end{aligned}$$

. where $\epsilon' = \epsilon^3 > 0$. Therefore, the set D approximates a minimum dominating set in G within factor $(1 - \epsilon') \ln n$. But this contradicts Theorem 4.1. This completes the proof. □

5 Figure and Table

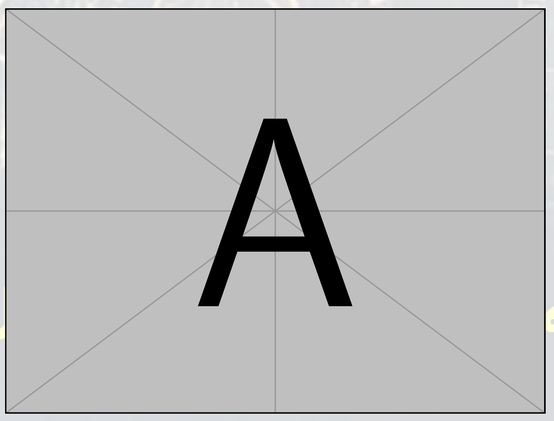


Figure 1: A sample figure caption

Here goes a table. Tables are "float" objects. It means that LaTeX does not generally place them on the same location in the source code as on the output. You can place them anywhere in the source code and then simply refer to them, like Table 1.

First column head	Second column head	Third column head
N/A	$x^2 + 1$	6
-20	y	11
-12	$x + y$	7

Table 1: A sample table caption

References

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