2nd International Conference on Mathematics and Its Applications

August 19-20, 2025

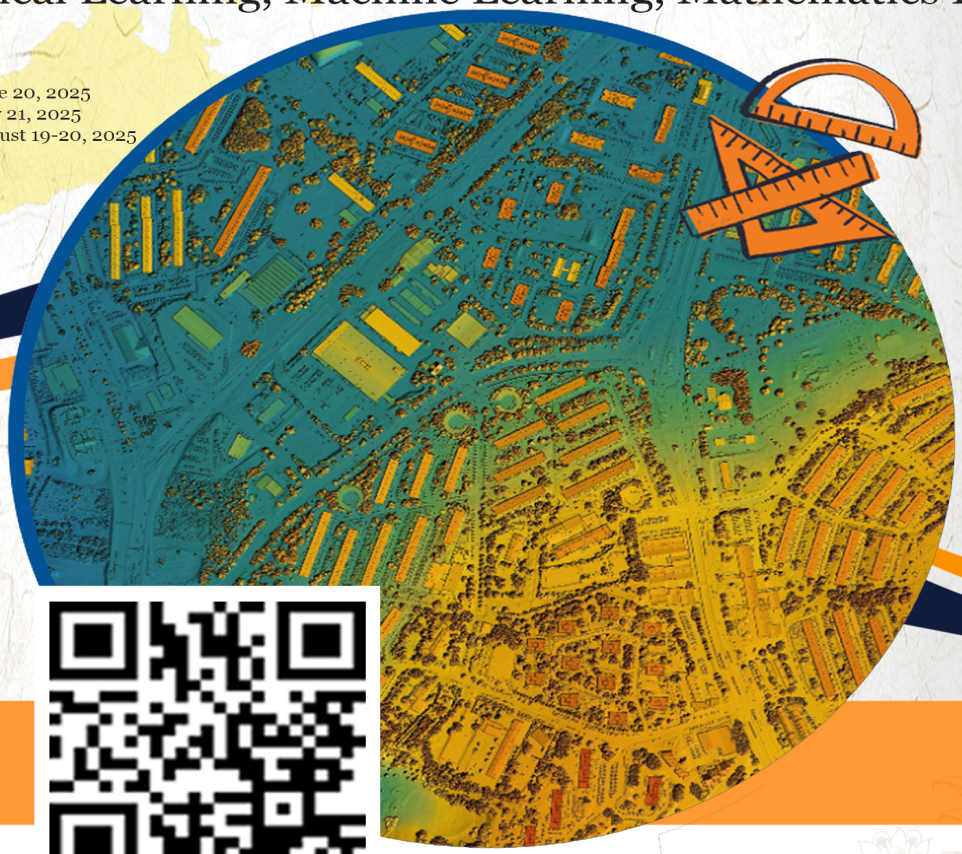
Shahid Chamran University of Ahvaz, Ahvaz, IRAN

Coinciding with the 70th anniversary of the establishment of Shahid Chamran University of Ahvaz

Conference Topics:

Pure and Applied Mathematics, Statistics and Probability, Computer Science, Statistical Learning, Machine Learning, Mathematics Education

Paper Submission Deadline: June 20, 2025
Final Paper Registeration Deadline: July 21, 2025
Confrence Dates: August 19-20, 2025



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In the Name of Allah



2nd *International Conference on Mathematics and Its Applications*

August 20-21,2025, Shahid Chamran University of Ahvaz, Ahvaz, IRAN



Proceeding of

The 2^{nd} International Conference on Mathematics and its Applications

August 20-21, 2025, Ahvaz, Iran

Faculty of Mathematical Sciences and Computer

Shahid Chamran University of Ahvaz

Ahvaz, Iran

August 20-21, 2025

Disclaimer

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Preface

The International Conference on Mathematics and its Applications is held biennially at Shahid Chamran University of Ahvaz, with the support of the Iranian Mathematical Society, the Iranian Statistical Society, and other supporting organizations. This event provides a platform for researchers, mathematicians, and graduate students from Iran and abroad to present their achievements in various mathematical fields.

In 2025, following this tradition, Shahid Chamran University of Ahvaz had the honor of hosting this event on August 20-21, 2025. The conference was organized by the University's virtual system, building upon the at least fifty-year legacy of the Statistics and Mathematics departments within the Faculty of Mathematical and Computer Sciences. It was hoped that this conference would provide a suitable platform for enhancing mathematical knowledge throughout the country.

To achieve this objective, the conference themes were carefully selected to align with the needs of the country's scientific community, covering relevant research areas while addressing the scientific community's concerns regarding mathematics education and statistics education. Additionally, the conference aimed to provide special coverage for research in computer science, which has recently garnered significant attention from the scientific community, particularly in the fields of artificial intelligence and statistical learning.

To this end, following the conference's approval by Shahid Chamran University of Ahvaz, the conference secretariat immediately began receiving submissions through digital advertising and the conference website starting from early February 2025. During the announced submission period, 325 papers were received through the conference website. All submissions underwent initial review by scientific committee members and were subsequently evaluated and peer-reviewed by the conference review committee.

Throughout the peer-review process, which continued until late July 2025, of the 325 submitted papers, 275 papers were accepted for oral and poster presentations, while 50 papers did not meet the required standards for conference presentation. The conference was ultimately held on August 20-21, 2025, with numerous participants including faculty members and graduate students from prestigious universities across Iran and several foreign countries such as Oman, Iraq, India, and Brazil, conducted through a virtual platform.

In conclusion, the secretariat of the 2nd International Conference on Mathematics and its Applications would like to express its utmost gratitude and appreciation to the members of the scientific committee, executive committee, and review committee. We also extend our appreciation to the Vice Chancellor for Research of Shahid Chamran University of Ahvaz, the Faculty of Mathematical and Computer Sciences, the esteemed Dean of the Faculty of Mathematical Sciences, the Mathematics Department, Statistics Department, Computer Science Department, and all staff and colleagues of the Faculty of Mathematical and Computer Sciences.

Finally, we acknowledge the spiritual support of the Iranian Mathematical Society, the Iranian Statistical Society, the Iranian Fuzzy Systems Association, the Iranian Intelligent Systems Association, the Ahvaz Mathematics House, and the Islamic World Science Citation Center. We pray to Almighty God for divine success and increasing achievements for all esteemed participants of this conference.

Rahim Chinipardaz
Scientific Chair of the 2nd International Conference on Mathematics and its Applications

Jalal Chachi
Chair of the 2nd International Conference on Mathematics and its Applications
October 2025

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Mathematics

- Analysis
- Geometry and Topology
- Graph Theory and Combinatorics
- Equations and Dynamical Systems
- Numerical Solution of Equations and Integrals
- Numerical Linear Algebra
- Control and Optimization
- Algebra
- Coding and Cryptography
- Mathematics Education
- Mathematical Logic
- Operations Research
- Biological Mathematics
- Financial Mathematics
- Number Theory
- Applications of Mathematics in Other Sciences

Statistics

- Theoretical Statistics
- Applied Statistics
- Probability and Stochastic Processes
- Biostatistics
- Fuzzy Statistics and Probability
- Data Mining and Data Science
- Statistical Learning
- Applications of Statistics in Other Sciences

Computer Science

- Algorithms and Computations, Genetic Algorithms
- Data Mining and Information Systems, Operational and Analytical Database Systems
- Machine Learning, Deep Learning and Neural Networks
- Artificial Intelligence and Robotics, Internet of Things
- Fuzzy Logic and Intelligent Systems, and Soft Computing
- Cognitive Modeling and Human-Computer Interaction, Knowledge Reasoning
- Machine Vision, Image Processing, Audio and Signal Processing
- Medical Image Processing and Imaging Systems
- Medical Equipment Processing
- Brain-Computer Interface Systems

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2nd International Conference on Mathematics and Its Applications

August 20-21,2025, Shahid Chamran University of Ahvaz, Ahvaz, IRAN

Shahid chanran University of Ahvaz

A new derivative concept for Uncertain-set valued functions

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Abstract

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain variables/processes. The main purpose of this paper is to investigate such the calculus for the differentiation and integration of uncertain set-valued functions. In addition, to demonstrate the efficiency of the method some various examples will be solved.
Keywords: Uncertain calculus, Integral and derivation, Uncertain-set valued function

1 Introduction

During the last decade, the interest of mathematicians in the theory of uncertainty theory [1] has been steadily increasing. Mathematical modeling of uncertain phenomenon requires gathering information and data from various sources. This type of data/variables are often uncertain due to rationally deal with belief degrees. Uncertainty theory is a well-known way for modeling systems with such the uncertainties. Over the past decade, with the development of the uncertainty theory, it is investigated that the proposed models were more consistent with reality and could express with a more comprehensive view. Therefore, the topic of differentiability and integrability of uncertain set-valued functions based on Hukuhara difference will be discussed in the following.

2 Fundamental definitions and theorems

In this section, we will explore various definitions and findings and introduce the notations that will be utilized throughout this paper. Our aim now is to examine some definitions and findings that are necessary for uncertainty calculus, utilizing definitions and results from [1].

Definition 1 ([1]). *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, then* $X : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathbb{R}$ *is an uncertain variable if*

$$\{\gamma \in \Gamma | X(\gamma) \in B\} \in \mathcal{L}$$

for any Borel set $B \in \mathcal{B}(\mathbb{R})$.

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Definition 2 (Hausdorff-metric [2]). *Let $\mathcal{K}(\mathbb{R}^d)$ be the family of all d-dimensional non-empty compact convex sets on $\mathbb{R}^d$. The Hausdorff-metric between sets $\mathbf{A}, \mathbf{B} \in \mathcal{K}(\mathbb{R}^d)$ is defined by*

$$\mathcal{H}(\mathbf{A}, \mathbf{B}) = \max\{\sup_{\mathbf{b}\in\mathbf{B}} \inf_{\mathbf{a}\in\mathbf{A}} \|\mathbf{a}-\mathbf{b}\|, \sup_{\mathbf{a}\in\mathbf{A}} \inf_{\mathbf{b}\in\mathbf{B}} \|\mathbf{a}-\mathbf{b}\|\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

Now, let us consider the real number space $\mathbb{R}$. In this case, the Hausdorff-metric between intervals becomes more convenient.

Proposition 2.1. *Let $I_1 = [a_1, a_2]$ and $I_2 = [b_1, b_2]$ be two intervals on $\mathbb{R}$. Then, the Hausdorff-metric between I_1 and I_2 is given by*

$$\begin{aligned} \mathcal{H}(I_1, I_2) &= \max\{|a_1 - b_1|, |a_2 - b_2|\} \\ &= |midI_1 - midI_2| + |sprI_1 - sprI_2|, \end{aligned}$$

where $midI_1 = \frac{a_1+a_2}{2}$ and $sprI_1 = \frac{a_2-a_1}{2}$.

Definition 3 (Uncertain set-valued function). *Let $\mathcal{K}(\mathbb{R})$ be a collection of all non-empty compact convex sets on $\mathbb{R}$, then $\vec{f} : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is called an uncertain set-valued function.*

Definition 4 (Uncertain set-valued continuous function). *An uncertain set-valued function $\vec{f} : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is said to be uncertain continuous at $\gamma_0 \in \Gamma$ if*

$$\forall \epsilon > 0,\ \exists \delta > 0 \ s.t.\ |\gamma - \gamma_0| < \delta \to \mathcal{H}(\vec{f}(\gamma), \vec{f}(\gamma_0)) < \epsilon,\ \forall \gamma \in \Gamma$$

Remark 2.2. We say that uncertain set-valued continuous function $\vec{f}$ is continues on Γ, if it is continuous for all $\gamma \in \Gamma$.

Definition 5 (Uncertain piecewise continuous set-valued function). *An uncertain set-valued function $\vec{f} : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is called an uncertain piecewise continuous set-valued function if $\vec{f}$ is continues on any subset Γ' of Γ except at a finite number of points $\gamma_1, \gamma_1, \ldots, \gamma_n$ in Γ'.*

Definition 6 (Uncertain set-valued variable (Weak condition)). *An uncertain set-valued function $\vec{X} : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is called uncertain set-valued variable if*

$$\left\{\gamma \in \Gamma | \vec{X}(\gamma) \cap B \neq \emptyset\right\} \in \mathcal{L}$$

for any Borel set $B \in \mathcal{B}(\mathbb{R})$.

Definition 7 (Uncertain set-valued variable (Strong condition) [1]). *An uncertain set-valued function $\vec{X} : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is called uncertain set-valued variable if*

$$\left\{\gamma \in \Gamma | \vec{X}(\gamma) \subseteq B\right\} \in \mathcal{L},$$
$$\left\{\gamma \in \Gamma | B \subseteq \vec{X}(\gamma)\right\} \in \mathcal{L},$$

for any Borel set $B \in \mathcal{B}(\mathbb{R})$.

Definition 8 (Uncertain process [1]). *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and T be a totally ordered set (e.g. time), then $X_t : T \times (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathbb{R}$ is an uncertain process such that for every $t \in T$*

$$\{\gamma \in \Gamma | X_t(\gamma) \in B\} \in \mathcal{L}$$

for any Borel set $B \in \mathcal{B}(\mathbb{R})$.

Definition 9 (Uncertain set-valued process)**.** *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and T be a totally ordered set (e.g. time), then $\vec{X}_t : T \times (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is an uncertain set-valued process such that for every $t \in T$*

$$\Big\{\gamma \in \Gamma | \vec{X}_t(\gamma) \cap B \neq \emptyset\Big\} \in \mathcal{L}$$

for any Borel set $B \in \mathcal{B}(\mathbb{R})$.

Definition 10 (Uncertain set-valued process [1])**.** *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and T be a totally ordered set (e.g. time), then $\vec{X}_t : T \times (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ is an uncertain set-valued process such that for every $t \in T$*

$$\Big\{\gamma \in \Gamma | \vec{X}_t(\gamma) \subseteq B\Big\} \in \mathcal{L},$$
$$\Big\{\gamma \in \Gamma | B \subseteq \vec{X}_t(\gamma)\Big\} \in \mathcal{L},$$

for any Borel set $B \in \mathcal{B}(\mathbb{R})$.

3 Uncertain Hukuhara derivative

Definition 11 (Hukuhara operator $\ominus_H$)**.** *Let $\mathfrak{I}(\mathbb{R})$ be the set of all intervals in the apace of $\mathbb{R}$. The Hukuhara difference operator $\ominus_H$ of two intervals $I_1 \in \mathfrak{I}(\mathbb{R})$, $I_2 \in \mathfrak{I}(\mathbb{R})$ is the interval $I_H \in \mathfrak{I}(\mathbb{R})$, (if it exists), such that*

$$I_1 \ominus_H I_2 = I_H \leftrightarrows \begin{cases} I_1 = I_2 \oplus I_H \\ or \\ I_2 = I_1 \oplus \big((-1) \otimes I_H\big) \end{cases}$$

Definition 12 (Uncertain Hukuhara derivative)**.** *The H-derivative of a uncertain set-valued function $\vec{f}$, also known as the uncertain Hukuhara derivative, is defined as follows:*

$$\vec{f}'_H(\gamma) = \lim_{h \to 0} \frac{\vec{f}(\gamma + h) \ominus_H \vec{f}(\gamma)}{h}, \quad \gamma \in \Gamma$$

We say that $\vec{f}$ is a H-differentiable Uncertain set-valued function provided that above limit exists and $\vec{f}'_H(\lambda)$ is an uncertain set at λ, we say that the uncertain Hukuhara derivative of $\vec{f}$ exists and it is equal to $\vec{f}'_H(\lambda)$, i.e. $\vec{f}'_H(\lambda)$ is the uncertain Hukuhara derivative of $\vec{f}$.

Definition 13 (Type 1 differentiability: Strictly increasing)**.** *Uncertain set-valued function $\vec{f}$ is strictly increasing on $I \subset \Gamma$ if*

1. *$\vec{f}'_H(\gamma)$ exists at any point $\gamma \in I \subset \Gamma$,*
2. *for $\gamma_1 < \gamma_2$, we have*
   - *$f'^l_H(\gamma_1) < f'^l_H(\gamma_2)$, and*
   - *$f'^r_H(\gamma_1) < f'^r_H(\gamma_2)$.*

Definition 14 (Type 2 differentiability: Strictly decreasing)**.** *Uncertain set-valued function $\vec{f}$ is strictly decreasing on $I \subset \Gamma$ if*

Table 1: H-derivative of $\omega \otimes \vec{f}$ as well as its type of derivative

Type of $\vec{f}'_H$	Domain of $\omega(\cdot)$	Domain of $\omega'(\cdot)$	$(\omega \otimes \vec{f})'_H(\cdot)$	Type of $(\omega \otimes \vec{f})'_H$
Type 1	$\mathbb{R}^+$	$\mathbb{R}^+$	$\left(\omega \otimes \vec{f}'_H\right) \oplus \left(\ \omega' \otimes \vec{f}\right)$	Type 1
Type 1	$\mathbb{R}^+$	$\mathbb{R}^-$	$\left(\omega \otimes \vec{f}'_H\right) \oplus \left(-\omega' \otimes \vec{f}\right)$	Type 1
Type 1	$\mathbb{R}^-$	$\mathbb{R}^+$	$\left(\omega \otimes \vec{f}'_H\right) \oplus \left(-\omega' \otimes \vec{f}\right)$	Type 2
Type 1	$\mathbb{R}^-$	$\mathbb{R}^-$	$\left(\omega \otimes \vec{f}'_H\right) \oplus \left(\ \omega' \otimes \vec{f}\right)$	Type 2
Type 2	$\mathbb{R}^+$	$\mathbb{R}^+$	$\left(\omega \otimes \vec{f}'_H\right) \ominus \left(-\omega' \otimes \vec{f}\right)$	Type 2
Type 2	$\mathbb{R}^+$	$\mathbb{R}^-$	$\left(\omega \otimes \vec{f}'_H\right) \oplus \left(\ \omega' \otimes \vec{f}\right)$	Type 2
Type 2	$\mathbb{R}^-$	$\mathbb{R}^+$	$\left(\omega \otimes \vec{f}'_H\right) \oplus \left(\ \omega' \otimes \vec{f}\right)$	Type 1
Type 2	$\mathbb{R}^-$	$\mathbb{R}^-$	$\left(\omega \otimes \vec{f}'_H\right) \ominus \left(-\omega' \otimes \vec{f}\right)$	Type 1

1. *$\vec{f}'_H(\gamma)$ exists at any point $\gamma \in I \subset \Gamma$,*
2. *for $\gamma_1 < \gamma_2$, we have*
   - *$f'^{l}_H(\gamma_1) > f'^{l}(\gamma_2)$, and*
   - *$f'^{r}_H(\gamma_1) > f'^{r}(\gamma_2)$.*

Theorem 3.1. *Let $\vec{f}$ be a H-differentiable Uncertain set-valued function. Then*

1. *if $\vec{f}$ is strictly increasing, then*
   - *$f^{l}_H(\gamma) = f'^{l}(\gamma)$, and*
   - *$f^{r}_H(\gamma) = f'^{r}(\gamma)$.*
2. *if $\vec{f}$ is strictly decreasing, then*
   - *$f'^{l}_H(\gamma) = f'^{r}(\gamma)$, and*
   - *$f'^{r}_H(\gamma) = f'^{l}(\gamma)$.*

Theorem 3.2. *Let $\vec{f}$ be a H-differentiable Uncertain set-valued function on $\Gamma' \subset \Gamma$. Suppose the type of H-differentiability of $\vec{f}$ doesn't change on Γ' and $\omega(\cdot)$ is a real-valued monotone continuous function. Then H-derivative of $\omega \otimes \vec{f}$ as well as its type of derivative are summarized in Table 1.*

Theorem 3.3. *Let $\vec{f} : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathcal{K}(\mathbb{R})$ be an H-differentiable Uncertain set-valued function and Let $\omega(\cdot) : \mathbb{R} \to \mathbb{R}$ be a strictly increasing real-valued continuous function. Then, we have*

1. *if $\vec{f}$ is Type 1 differentiable, then*

- $\omega \otimes \vec{f}$ *is Type 1 differentiable.*
- $\left(\omega \otimes \vec{f}\right)'_H(t) = \left(\omega(t) \otimes \vec{f}'_H(t)\right) \oplus \left(\omega'(t) \otimes \vec{f}(t)\right)$.

2. if $\vec{f}$ *is Type 1 differentiable, then*

- $\omega \otimes \vec{f}$ *is Type 2 differentiable.*
- $\left(\omega \otimes \vec{f}\right)'_H(t) = \left(\omega(t) \otimes \vec{f}'_H(t)\right) \ominus \left(-\omega'(t) \otimes \vec{f}(t)\right)$.

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A Density Based Index to Measure Spread of a Spatial Sample

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Abstract

In many disciplines, selecting spatially balanced or well-spread samples is crucial, as such samples often yield more accurate estimates for variables with spatial structure. However, quantifying the degree of spatial spread remains a challenge. Although several indices have been proposed for this purpose, each falls short in capturing all aspects of sample dispersion. This paper introduces a novel index based on the empirical density of population units, designed to address these limitations. The proposed index possesses desirable properties, such as invariance under any uniform spatial translation of the sample. Through illustrative examples and simulation studies, we demonstrate the superiority of this measure compared to existing alternatives.

Keywords: Well-Spread Sample, Balanced Sample, Inclusion Probabilities, Spatial data.

1 Introduction

In studies involving spatially referenced data, selecting well-spread samples is crucial, especially in environmental and ecological contexts characterized by significant spatial autocorrelation. A spatially balanced sample provides thorough coverage of the area of interest, thereby improving the precision and reliability of estimates for spatially distributed variables. Such samples typically exhibit balance with respect to auxiliary variables, a feature particularly advantageous when the target parameters depend nonlinearly on these variables [1].

Several indices have been developed to measure the spatial spread of a sample, including the Voronoi spatial balance index [3], Moran's index [4], and the local spatial balance measure [2], which extends the Voronoi-based approach to account for local structure in the spatial distribution of units.

In this paper, we introduce a novel index designed explicitly to measure the spatial spread of samples. This index exhibits spatial translation invariance, which means that when every unit in a sample is displaced by the same vector, the computed value of the index, and thus the verdict on the optimality of the sample, remains identical, confirming that the absolute location does not influence the assessment.

The rest of the paper is organized as follows. Section 2 introduces the main notation. In Section 3, we present the proposed spreadness index. Section 4 outlines complementary evaluation measures. Simulation results are shown in Section 5, and Section 6 concludes the paper.

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2 Notations and Basic Concepts

Let $U = \{1, \ldots, N\}$ be a population of size N, and define $\mathcal{S}$ as the set of all possible samples or subsets. The goal is to select a sample $S \in \mathcal{S}$ with first-order inclusion probabilities (FIP) $\boldsymbol{\pi} = \{\pi_k, k \in U\}$ and fixed size n_s, such that

$$\sum_{k \in U} \pi_k = E(n_s).$$

Let $\boldsymbol{p} = \{p_s = \Pr(S = s); s \in \mathcal{S}\}$ be a sampling design that assigns a probability to each sample in $\mathcal{S}$ such that

$$\sum_{s \in \mathcal{S}} p_s = 1, \quad \text{and} \quad \sum_{s \in \mathcal{S}, s \ni k} p_s = \pi_k \quad \text{for all } k \in U. \tag{2.1}$$

In this paper, we focus on fixed-size designs where $E(n_s) = n$ and $\mathrm{Var}(n_s) = 0$.

Let y_k and x_k be the main and auxiliary variables associated with unit $k \in U$, respectively. For a subset $A \subset U$, define the indicator vector $\boldsymbol{a}(A) = \{a_k(A), k \in U\}$ where $a_k = 1$ if $k \in A$ and 0 otherwise. A common goal in sampling is to estimate population-level parameters, such as the total:

$$Y(A) = \sum_{k \in U} y_k a_k(A).$$

Estimation of $Y(A)$ and its variance relies on the design, FIPs, and second-order inclusion probabilities (SIP). The Narain-Horvitz-Thompson (NHT) estimator [**?**, **?**] provides an unbiased estimate:

$$\hat{Y}(A) = \sum_{k \in U} \frac{y_k}{\pi_k} a_k(A \cap s),$$

Each unit $k \in U$ is associated with a spatial coordinate $\boldsymbol{c}_k$. The spatial position of unit k in d-dimensional space is represented by the matrix

$$\boldsymbol{C} = (\boldsymbol{c}_k, k \in U), \quad \text{where} \quad \boldsymbol{c}_k = (c_{kj}, j = 1, 2, \ldots, d),$$

with c_{kj} being the coordinate of unit k along the jth dimension, and d denoting the spatial dimensionality.

3 Spatial Balance Indices

3.1 Voronoi Index [3]

Let $\{U_k\}_{k \in s}$ denote the Voronoi partitioning of the population U around the sample units s, such that U_k is the set of population units closer to unit $k \in S$ than any other sample unit. The Voronoi Spatial Index $Vo(s)$ is a measure of how well the sample S is distributed across the population U. It is defined as:

$$Vo(s) = n \sum_{k \in s} \left(\hat{G}_\pi(U_k) - G_\pi(U_k) \right)^2,$$

where

$$G_x(U^*) = F_x(U^*), \quad \hat{G}_x(U^*) = \frac{1}{X(U)} \sum_{k \in U} \frac{x_k}{\pi_k} a_k(U^*), \quad \text{and} \quad F_x(U^*) = \frac{X(U^*)}{X(U)}.$$

This index reflects the deviation of the observed inclusion probabilities within each Voronoi cell from their expected values under uniform distribution.

3.2 Moran's Index [4]

Moran's Index $Mo(s)$ is a measure of spatial autocorrelation within the sample S, indicating whether similar values are clustered together or dispersed. It is defined as:

$$Mo(s) = \frac{[\boldsymbol{a}(s) - \bar{\boldsymbol{a}}(s)]^\top W[\boldsymbol{a}(s) - \bar{\boldsymbol{a}}(s)]}{(1^\top W 1)[\boldsymbol{a}(s) - \bar{\boldsymbol{a}}(s)]^\top[\boldsymbol{a}(s) - \bar{\boldsymbol{a}}(s)]},$$

where $W = [w_{k\ell}]_{N\times N}$, and $w_{k\ell}$ is the spatial weight between sample units k and ℓ, typically defined as $w_{k\ell} = \frac{1}{d_{k\ell}}$ for nearby units, and 0 otherwise, where $d_{k\ell}$ is a distance between units k and ℓ. Also,

$$\bar{\boldsymbol{a}}(s) = (\bar{a}(s), \bar{a}(s), \ldots, \bar{a}(s)), \qquad \bar{a}(s) = \frac{1}{N}\sum_{k\in U} a_k(s),$$

A positive value of $Mo(s)$ suggests positive spatial autocorrelation (similar values cluster together), while a negative value suggests negative spatial autocorrelation (dissimilar values are adjacent).

3.3 Local Spatial Index [2]

The local spatial balance measure extends the Voronoi spatial balance measure to account for the balance of auxiliary variables within each Voronoi region. It is calculated as a weighted sum of squared discrepancies between expected and observed values for these auxiliary variables.

The Local Spatial Index $L(s)$ extends the Voronoi Spatial Index by incorporating the distribution of auxiliary variables within each Voronoi cell. It is defined as:

$$Lb(s) = \sqrt{\frac{1}{N}\sum_{k\in s} \boldsymbol{d}_x(\mathbf{U}_k)^\top \mathbf{Q}^{-1} \boldsymbol{d}_x(\mathbf{U}_k)},$$

where

$$\boldsymbol{d_x}(U^*) = X(U)[\hat{G}_\pi(U^*) - G_\pi(U^*)]$$

This index measures how well the auxiliary variables are balanced within each Voronoi cell, with a lower value indicating better local balance.

Although these indices, offer valuable insight into spatial balance and distribution, each have notable limitations. These metrics typically emphasize specific spatial characteristics, such as clustering, dispersion, or autocorrelation, but often fall short of fully capturing the overall "spreadness" of a sample. They may be sensitive to edge effects, scale dependencies, or inconsistencies in coverage across the study area. As our simulations will demonstrate, such limitations can lead to suboptimal evaluations of spatial representativeness in certain contexts.

To address these shortcomings, we introduce a novel index that approaches the notion of spreadness from a new perspective.

4 A New Measure for Spread of Sample

We introduce a spatial translation-invariant index for determining optimal spatial spread patterns of samples drawn from a given population. The core idea is that if a sample is optimal, then translating the spatial arrangement of all sample units by the same vector yields another equivalent optimal sample, since the index returns exactly the same value under such a translation. To implement this, the population is first partitioned into clusters so that each cluster is balanced (the sum of inclusion probabilities equals one). We

achieve this balance using the algorithm of [5], modified to target inclusion-probability balance rather than equal unit counts in each cluster.

Within this clustering framework, we take the arrangement of the centroids of the resulting clusters as the reference for the optimal sample arrangement. Accordingly, any sample arrangement that replicates the spatial arrangement of these centroids can be considered optimally spread under the index. We assign each sample unit to its nearest cluster and then assess how closely each sample unit aligns with its corresponding centroid. However, merely calculating the distance from each sample unit to the centroid of its assigned cluster does not adequately account for the relative arrangement among sample units.

To address this, we assess how the population's spatial arrangement changes when each cluster is translated so that its centroid aligns with the spatial position of the assigned sample unit. We use Multivariate Kernel Density Estimation (MKDE) to model the original spatial distribution of the population as a reference. For each sample unit, we translate the position of all units in its cluster so that the sample unit occupies the cluster's centroid position, then compute the MKDE for this translated arrangement of the population. Finally, we quantify the divergence between the original and translated MKDE distributions to define our new index for optimal spatial spread.

We formally define a spatial translation-invariant index to quantify optimal spatial spread in multivariate form. The following notation describes the coordinates of cluster centroids and sample units:

$$\{\boldsymbol{o}_i; i \in o\}, \quad \{\boldsymbol{s}_i; i \in s\},$$

where o represents the set of centroids.

Define the translated coordinates based on sample units as:

$$\boldsymbol{C}^{\delta} = (\boldsymbol{c}_k^{\delta}, k \in U), \quad \text{with} \quad \boldsymbol{c}_k^{\delta} = \boldsymbol{c}_k + (\boldsymbol{o}_i - \boldsymbol{\delta}_i), \quad k \in U_i, i \in s, \delta = o, s.$$

Our proposed index is derived from the difference between two density representations: one reflecting the original population distribution based on centroids ($\delta = o$) and the other representing the translated distribution based on selected sample units ($\delta = s$):

$$\Delta_s(U) = \|f(\boldsymbol{C}^o) - f(\boldsymbol{C}^s)\|_2^2, \tag{4.1}$$

where each density function $f(\cdot)$ is obtained via MKDE and is defined as:

$$f(\boldsymbol{C}^{\delta}) = \left(f(\boldsymbol{c}_k^{\delta}), k \in U\right)^T, \quad \delta = o, s. \tag{4.2}$$

The MKDE for a point $\boldsymbol{c}_k^{\delta}$ is computed as follows:

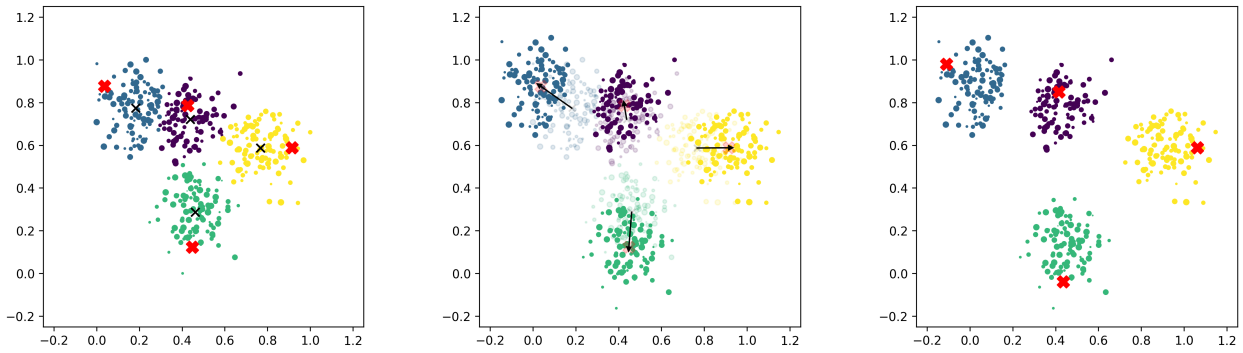
$$f(\boldsymbol{c}^{\delta}) = \frac{1}{N} \sum_{i \in s} \sum_{k \in U_i} K_{\mathbf{H}}\left(\boldsymbol{c} - \boldsymbol{c}_k^{\delta}\right), \tag{4.3}$$

with:

- The multivariate kernel function $K_{\mathbf{H}}(\cdot)$ defined by

  $$K_{\mathbf{H}}(\boldsymbol{c}) = |\mathbf{H}|^{-1/2} K\left(\mathbf{H}^{-1/2}\boldsymbol{c}\right),$$

  where $K(\cdot)$ is a symmetric multivariate density.
- The bandwidth matrix $\mathbf{H}$, which is symmetric and positive definite, regulating the smoothing.
- The set U_i containing all units of the i-th cluster.



(a) Original Arrangement (b) Translation Process (c) Translated Arrangement

Figure 1: Coordinate rearrangement example

We adopt a uniform kernel defined as

$$K(\boldsymbol{c}) = \frac{1}{2^d}\mathbf{1}\left\{\|\boldsymbol{c}\|_\infty \leq 1\right\},$$

so that the kernel becomes

$$K_{\mathbf{H}}(\boldsymbol{c}) = |\mathbf{H}|^{-1/2}\frac{1}{2^d}\mathbf{1}\left\{\|\mathbf{H}^{-1/2}\boldsymbol{c}\|_\infty \leq 1\right\}.$$

The bandwidth matrix $\mathbf{H}$ significantly affects the accuracy of MKDE, determining the intensity and direction of the smoothing. In our approach, we adapt the 'Scott' method to select $\mathbf{H}$, thereby ensuring that the smoothing is both balanced and effective.

To scale this measure appropriately, we define the normalized dispersion index as:

$$D_s(U) = 1 - \left(1 - \frac{\Delta_s(U)}{\max\limits_{s\in\mathcal{S}}\Delta_s(U)}\right)^n, \tag{4.4}$$

where $\mathcal{S}$ includes all possible sample sets. This index scales the deviation $\Delta_s(U)$ within $[0,1]$, with higher values indicate greater divergence from the original spatial distribution, meaning the spatial arrangement of the sample deviates more from the optimal arrangement defined by the cluster centroids.

5 Examples and Simulation

In this section, we visually demonstrate key aspects of the index and compare its performance with other spatial balance indices across some populations.

5.1 Clusters Translation Example

Figure 1 illustrates the clusters translation process described previously in three stages: the original arrangement of population, the translation process, and the translated arrangement of population.

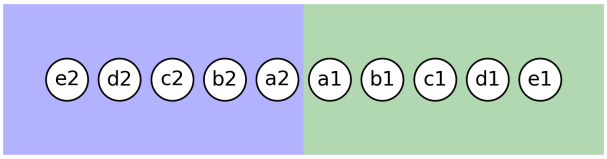


Figure 2: Example fixed size with equal probabilities

Table 1: Statistics for various sample types including density, spatial metrics, and local balance.

Sample	#	Density	Voronoi	Moran	Local Balance
c_ic_j	1	0.00	0.00	-1.00	0.00
b_id_j	2	0.00	0.04	-1.00	0.28
b_ic_j	2	0.60	0.01	-0.84	0.18
b_ib_j	1	0.88	0.00	-0.79	0.25
a_ic_j	2	0.88	0.04	-0.79	0.36
e_ie_j	1	0.94	0.00	-0.41	0.49
a_ia_j	1	1.00	0.00	-0.25	0.49
a_ib_i	2	1.00	0.04	-0.06	0.55
d_ie_i	2	1.00	0.64	0.37	1.14
Mean	45	0.78	0.15	-0.37	0.54

5.2 Comparison of Sample Arrangements

Figure 2 illustrates a fixed-size sample with equal probabilities, providing the basis for our sample arrangement. Table 1 summarizes key spatial metrics—density, Voronoi measure, Moran's I, and local balance—where the "Mean" row averages all 45 arrangements, and the other rows highlight select examples.

We highlight three pairs of sample arrangement to demonstrate the robustness of the density measure relative to other spatial metrics. First, comparing the arrangement c_ic_j and b_id_j, the density measure remains constant, illustrating its invariance under uniform spatial translations, whereas both the Voronoi measure and the local balance metric report substantial changes. Second, examining a_ia_j and d_ie_i, the density measure assigns identical poor scores, reflecting their equivalent compactness under translation; by contrast, the Voronoi measure increases from 0.00 to 0.64 and Moran's I shifts from -0.25 to $+0.37$ (a difference exceeding 0.5 with a sign reversal), suggesting contradictory interpretations of spatial regularity. Finally, comparing e_ie_j and a_ia_j, the density measure more sensitively penalizes the tighter clustering in a_ia_j, thereby favoring the more dispersed arrangement e_ie_j, whereas both the Voronoi measure and local balance remain unchanged. These examples underscore the advantages of density as a consistent, translation-invariant measure of spatial regularity.

5.3 Spatial Translation Invariance of the Proposed Index

One desirable property of a good spatial spread measure is spatial translation invariance, meaning that the index remains unchanged under any uniform spatial translation of the sample. In this subsection, we demonstrate that our proposed index is spatial translation-invariant and compare it with Moran's I, the Voronoi-based index, and Local Balance to confirm consistent behavior across all four.

As illustrated in Figure 3, different sampling arrangements influence the evaluation metrics in distinct ways. Samples (a) and (b) demonstrate that the Density metric remains relatively stable, even as other

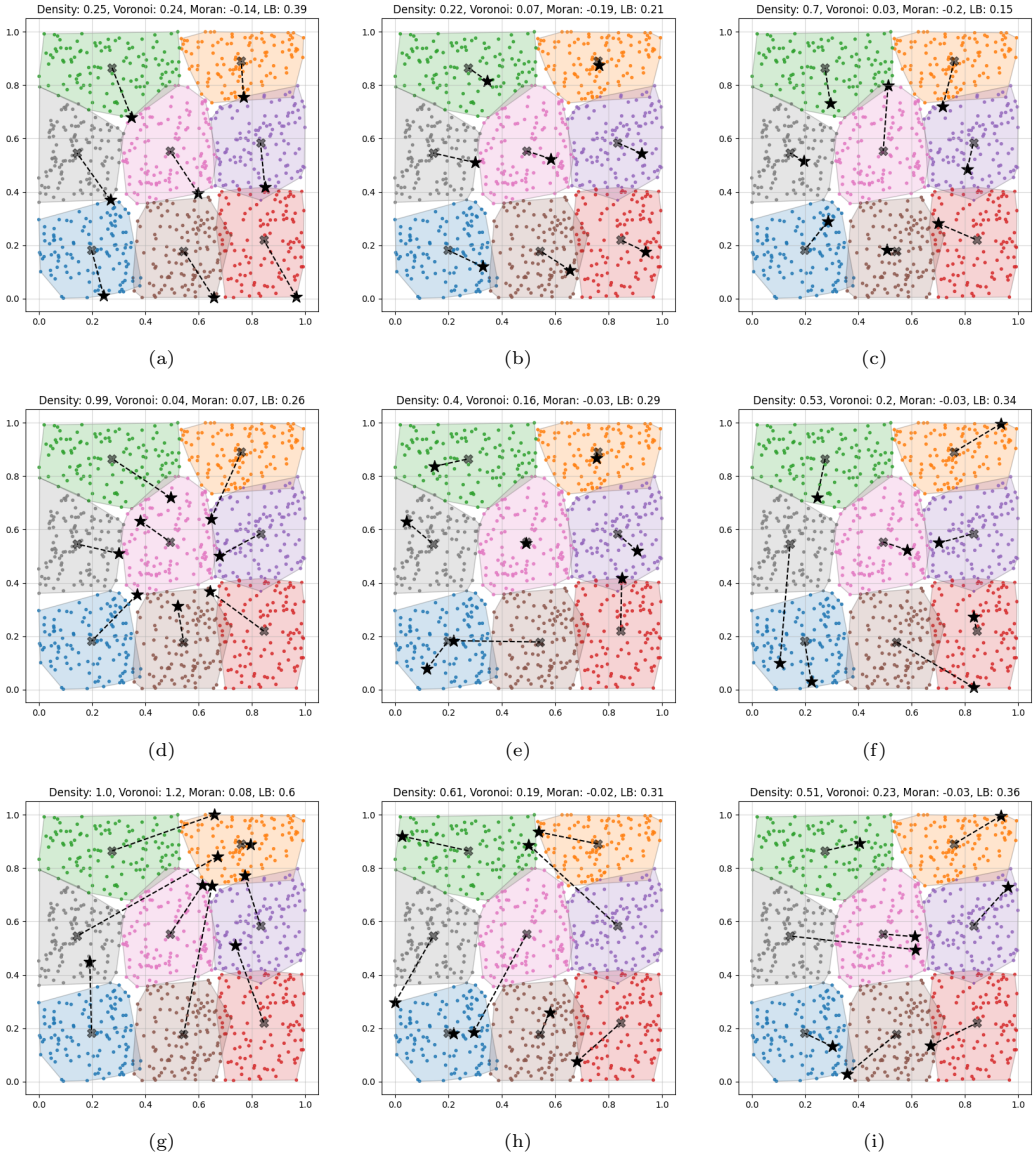


Figure 3: Various sampling arrangements and their effects on different evaluation metrics.

metrics such as Voronoi and Moran's I vary. This indicates the robustness of the Density metric in capturing core distribution characteristics. Samples (c) and (d) highlight a scenario where scaling the sample around the center changes the Density score while the Voronoi metric remains nearly constant. Finally, Samples (e) through (i) represent random arrangements. In these examples, the Local Balance (LB) metric remains relatively consistent, while the Density score exhibits significant variation, reflecting its sensitivity to underlying point distributions.

6 Summary

We propose a novel index of spatial spread, invariant to uniform spatial translations and grounded in balanced clustering. By translating each cluster so that its centroid aligns with the position of its assigned sampled unit and comparing the resulting multivariate kernel density estimate to the original distribution, the index quantifies deviation from the optimal arrangement. This dispersion index is then normalized onto [0, 1] for intuitive interpretation. Simulations reveal its distinctive behavior under translations and structural variations, and comparative analyses demonstrate its superiority over existing spatial balance indices.

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Design Strategies for Training Samples in Regression

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Abstract

Selecting a representative subset of a large dataset is a critical step in many machine-learning workflows, and it naturally parallels the framework of finite-population sampling. Recent work shows that unequal-probability sampling often yields more efficient estimators than equal-probability sampling, yet the repercussions of different sampling designs remain underexplored. In this study we investigate how alternative designs, each calibrated to identical first-order inclusion probabilities, affect model performance. We adopt two complementary modeling paradigms: traditional statistical inference and finite-population inference. Simulations on real-world datasets reveal that the sampling design itself can materially influence both predictive accuracy and estimator stability.and deserves to be more explored.
Keywords: Inclusion Probabilities, Test-Train Sampling, Regression Models, Sampling Design.

1 Introduction

The rapid growth of data presents significant analytical challenges due to its sheer volume and complexity. Traditional statistical methods frequently struggle to handle large-scale datasets efficiently. Sampling techniques, particularly subsampling methods, have become increasingly valuable, offering an effective approach to reduce computational burdens by analyzing representative subsets of data [1, 2].

Among various sampling methods, sampling approaches such as statistical leverage–based techniques strategically select influential data points to enhance both estimation accuracy and computational efficiency [3, 4]. Recent studies highlight that optimal sampling designs, such as those applied in logistic regression contexts, substantially improve model performance by efficiently reducing data without compromising accuracy [5]. Additionally, carefully designed data selection methodologies guided by statistical considerations significantly improve model generalizability [6].

This study investigates how different sampling designs, all utilizing identical first-order inclusion probabilities, impact regression model results. Our goal is to illustrate that the choice of sampling design, despite having fixed inclusion probabilities, can significantly influence model accuracy and efficiency.

The paper is structured as follows: Section 2 introduces essential notation. Section 3 provides an overview of simple linear regression. Section 4 discusses the estimation of model parameters α and β. Section 5 outlines

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criteria for determining inclusion probabilities. Section 6 presents simulation studies and empirical analyses. Finally, Section 7 summarizes our findings and offers practical recommendations.

2 Notation

Let $U = \{1, \ldots, N\}$ be a population of size N, and define $\mathcal{S}$ as the set of all possible samples or subsets. The aim is to select a sample S of $\mathcal{S}$ with first-order inclusion probabilities (FIP) $\boldsymbol{\pi} = \{\pi_k, k \in U\}$, and size n_s where

$$\sum_{k \in U} \pi_k = E(n_s).$$

Let $\boldsymbol{p} = \{p_s = Pr(S = s); s \in \mathcal{S}\}$ be a sampling design that assigns a probability to each possible sample such that

$$\sum_{s \in \mathcal{S}} p_s = 1, \text{ and } \sum_{s \in \mathcal{S}, s \ni k} p_s = \pi_k \text{ for all } k \in U.$$

Consider $y_k, x_k, z_k, k \in U$ as the main and auxiliary variables respectively. Also $\boldsymbol{a} = \{a_k, k \in U\}$ stands for a vector of indicators in which $a_k = 1$ if $k \in S$. the main objective of sampling is typically to estimate parameters such as the population total, denoted by,

$$Y = \sum_{k \in U} y_k.$$

Estimation of parameters involving total estimation and its variance is based on designs, FIPs, and second-order inclusion probabilities. We can estimate Y unbiasedly by the Narain-Horvitz-Thompson estimator (NHT),

$$\hat{Y} = \sum_{k \in U} \frac{y_k}{\pi_k},$$

with

$$var(\hat{Y}) = \sum_{k \in U} \sum_{\ell \in U} \frac{y_k}{\pi_k} \frac{y_\ell}{\pi_\ell} (\pi_{k\ell} - \pi_k \pi_\ell) a_k a_\ell,$$

and an unbiased estimator

$$\hat{var}(\hat{Y}) = \sum_{k \in U} \sum_{\ell \in U} \frac{y_k}{\pi_k} \frac{y_\ell}{\pi_\ell} \frac{\pi_{k\ell} - \pi_k \pi_\ell}{\pi_{k\ell}} a_k a_\ell.$$

3 Simple Linear Regression

In a simple linear regression model, we assume there is a single explanatory variable x and a response variable y. We aim to model the relationship

$$y_k = \alpha + \beta x_k + \varepsilon_k,$$

where α is the intercept, β is the coefficient (slope), and ε_k is the random error term (or noise) associated with observation k. Our goal is to estimate α and β in such a way that the fitted line

$$\hat{y}_k = \hat{\alpha} + \hat{\beta} x_k$$

is as close as possible to the observed data.

4 Estimating α and β

To find $\hat{\alpha}$ and $\hat{\beta}$, we typically use the method of least squares. For this purpose, two main approaches can be considered: *(i)* Statistical Inference, and *(ii)* Finite Population methods.

4.1 Statistical Inference (STI)

We define the residuals $e_k = y_k - \hat{y}_k$ for each data point and minimize the sum of squared residuals:

$$S(\alpha, \beta) \;=\; \sum_{k \in S} \big(y_k - (\alpha + \beta\, x_k)\big)^2.$$

By setting the partial derivatives of $S(\alpha, \beta)$ with respect to α and β to zero, we obtain the normal equations. In closed form, for a single predictor x,

$$\hat{\beta} = \frac{\sum_{k \in S}(x_k - \bar{x})(y_k - \bar{y})}{\sum_{k \in S}(x_k - \bar{x})^2} \quad \text{and} \quad \hat{\alpha} = \bar{y} \;-\; \hat{\beta}\, \bar{x},$$

where $\bar{x} = \frac{1}{n}\sum_{k \in S} x_k$ and $\bar{y} = \frac{1}{n}\sum_{k \in S} y_k$ are the sample means.

4.2 Finite Population Approach (FPS)

In sampling contexts, each observation k is drawn with inclusion probability π_k. Let

$$\hat{N}_{\text{NHT}} \;=\; \sum_{k \in S} \frac{1}{\pi_k}$$

approximate the total population size, where S denotes the sampled set. For finite population inference, we can estimate population means of x and y using the NHT estimator:

$$\bar{x}_{\text{NHT}} = \frac{1}{\hat{N}_{\text{NHT}}} \sum_{k \in S} \frac{x_k}{\pi_k} \quad \text{and} \quad \bar{y}_{\text{NHT}} = \frac{1}{\hat{N}_{\text{NHT}}} \sum_{k \in S} \frac{y_k}{\pi_k}.$$

Then, the NHT-based slope and intercept estimators for the simple linear regression become

$$\hat{\beta}_{\text{NHT}} = \frac{\sum_{k \in S} \frac{1}{\pi_k} \left(x_k - \bar{x}_{\text{NHT}}\right) \left(y_k - \bar{y}_{\text{NHT}}\right)}{\sum_{k \in S} \frac{1}{\pi_k} \left(x_k - \bar{x}_{\text{NHT}}\right)^2} \quad \text{and} \quad \hat{\alpha}_{\text{NHT}} = \bar{y}_{\text{NHT}} \;-\; \hat{\beta}_{\text{NHT}}\, \bar{x}_{\text{NHT}}.$$

This finite population approach accounts for the sampling design by weignhting each observation inversely proportional to its inclusion probability.

5 Criteria for FIP

To construct a sampling design that focuses computational effort on the most informative observations, we assign each unit k a non-negative importance score z_k. Once the scores are computed, the first-order inclusion probabilities (FIP) are defined by

$$\pi_k \;=\; \min\Big(\frac{n\, z_k}{Z},\, 1\Big), \qquad Z \;=\; \sum_{k \in U} z_k, \quad \text{so that} \quad \sum_{k \in U} \pi_k = n.$$

We consider four alternative criteria for z_k:

5.1 Residuals

After fitting the model $\hat{y}_k = \hat{\alpha} + \hat{\beta} x_k$, the residual is

$$z_k \;=\; y_k - \hat{y}_k.$$

Large absolute residuals highlight lack of fit, so choosing $z_k = |y_k - \hat{y}_k|$ oversamples observations that are poorly explained by the model and may reveal misspecification or outliers.

5.2 Influence

Observation k is influential if removing it appreciably changes the parameter vector $\hat{\theta} = (\hat{\alpha}, \hat{\beta})^\top$. A first-order approximation avoids n costly refits. With design matrix

$$X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \qquad \boldsymbol{x}_k = \begin{pmatrix} 1 \\ x_k \end{pmatrix},$$

set

$$\psi_k = \left(y_k - x_k^\top \hat{\theta}\right) (X^\top X)^{-1} \boldsymbol{x}_k, \qquad z_k = \|\psi_k\|_2.$$

Here $(X^\top X)^{-1}$ is 2×2 and x_k is 2×1, so the product is well-defined. The norm z_k summarises the potential shift in $\hat{\theta}$ were point k down-weighted.

5.3 Leverage

Leverage measures how extreme an x-value is relative to the data cloud. For simple regression the hat matrix $H = X(X^\top X)^{-1} X^\top$ has diagonal elements

$$H_{kk} \;=\; \frac{1}{n} + \frac{(x_k - \bar{x})^2}{\sum_{k \in U} (x_l - \bar{x})^2},$$

where $\bar{x} = n^{-1} \sum_{k \in U} x_l$ is the sample mean of the predictor. Setting $z_k = H_{kk}$ inflates inclusion probabilities for observations lying far from $\bar{x}$.

5.4 Gradient Weights

A complementary metric combines the size of the residual with the magnitude of the predictor value:

$$\varepsilon_k = y_k - \hat{\alpha} - \hat{\beta} x_k, \qquad r_k = |x_k|, \qquad z_k = |\varepsilon_k|\, r_k.$$

Points that deviate substantially from the fitted line and lie far from the predictor mean receive larger z_k, thereby entering the sample more frequently under the FIP scheme.

6 Simulation

To evaluate the four sampling designs and their associated criteria under both statistical–inference and finite-population frameworks, we use the *Online News Popularity* data set[1] from the UCI Machine Learning

[1] https://archive.ics.uci.edu/ml/datasets/online+news+popularity

Repository. Each row corresponds to a news article originally published on the `mashable.com` platform, and the response variable records the total number of social-media shares. The covariates—58 in total, spanning 39 797 articles—cover a heterogeneous mix of content, social, and temporal signals (e.g. keyword frequencies, publication weekday, global sentiment polarity, and channel type). The data set is completely free of missing values, is already split into meaningful feature groups, and has become a benchmark for popularity-prediction tasks.

Its moderate size and clean structure make it an ideal test bed for our study: the sample is large enough for unequal-probability designs to matter, yet small enough to allow exhaustive evaluation of inference accuracy under different designs and criteria.

Because the training subset is chosen with unequal–probability sampling, a deterministic 70 / 30 split would distort the evaluation: high-weight observations would be over-represented in the training set yet systematically under-represented in the hold-out data. Instead, after drawing the weighted sample that forms the training set, we select the test set *uniformly at random* from the entire population. This ensures that the hold-out data remain independent of the sampling weights and thus provide an unbiased benchmark for assessing out-of-sample performance.

We focus on three variables: one response and two predictors.

- **`n_unique_tokens` (y):** share of unique words in the article; a proxy for lexical diversity.
- **`n_tokens_content` (x):** total word count, capturing document length.
- **`n_non_stop_unique_tokens` (x):** share of unique *non-stop* words, isolating meaningful vocabulary richness.

This simulation study is intended to supplement the theoretical discussions presented earlier and to demonstrate the efficacy of different sampling designs and evaluation criteria in a practical setting.

The inclusion probabilities based on the four criteria discussed in Section 5 were constructed using a training sample size equal to 70% of the entire population on four designs [7]:

1. **SRS (simple random sampling without replacement):** Baseline design; fixed sample size with equal π_k,
2. **Systematic:** A very low entropy while preserving the desired unequal probabilities and the fixed sample size,
3. **Sampford:** A fixed-size design whose inclusion probabilities exactly match the π_k, with intermediate entropy,
4. **Poisson:** Each unit is selected independently with probability π_k, A random size with *high* entropy, respecting the unequal probabilityies.

The efficiency of each sampling design was assessed using the following criteria:

$$\text{MSE Efficiency} = \frac{E_M(\text{MSE}_{\text{SRS}})}{E_M(\text{MSE}_{\cdot})}, \tag{6.1}$$

$$\hat{\alpha}\,\text{Variation Efficiency} = \frac{\text{VAR}_M(\hat{\alpha}_{\text{SRS}})}{\text{VAR}_M(\hat{\alpha}_{\cdot})}, \tag{6.2}$$

$$\hat{\beta}\,\text{Variation Efficiency} = \frac{\text{VAR}_M(\hat{\beta}_{\text{SRS}})}{\text{VAR}_M(\hat{\beta}_{\cdot})}, \tag{6.3}$$

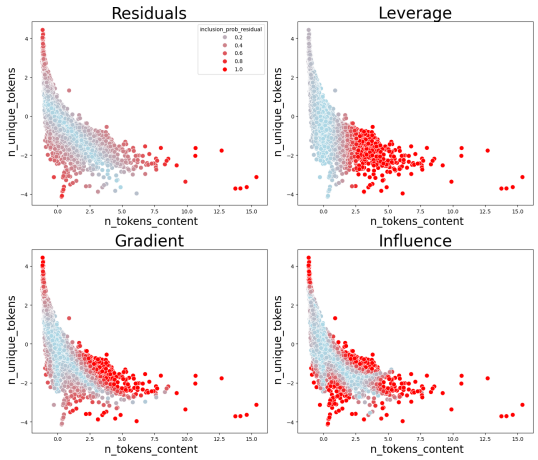


Figure 1: Model diagnostic plots showing the distribution of sampling inclusion probabilities based on criteria over the feature space defined by `n_unique_tokens` and `n_tokens_content`.

$$\text{MSE VAR Efficiency} = \frac{\text{VAR}_M(\text{MSE}_{\text{SRS}})}{\text{VAR}_M(\text{MSE}_\cdot)}, \tag{6.4}$$

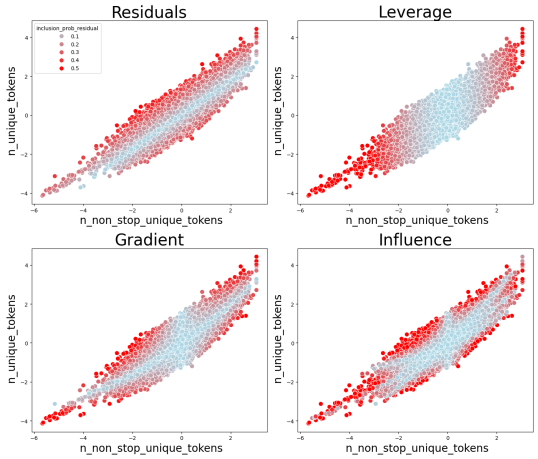


Figure 2: Model diagnostic plots showing the distribution of sampling inclusion probabilities based on criteria over the feature space defined by `n_unique_tokens` and `n_non_stop_unique_tokens`.

Where:

- $E_M(\cdot)$ denotes the expected value under Monte Carlo iterations.
- $\text{VAR}_M(\cdot)$ denotes the variance under Monte Carlo iterations.
- The subscript "SRS" refers to the Simple Random Sampling without replacement design.

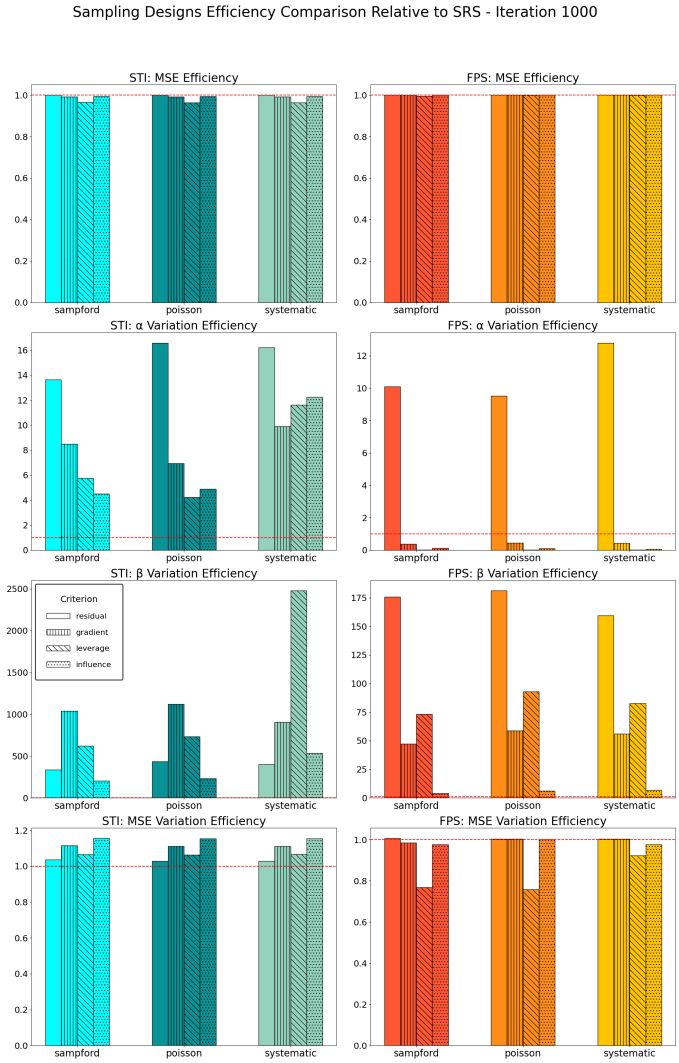


Figure 3: Comparison of sampling design efficiencies relative to SRS under the frameworks of STI and FPS across different criteria over the feature space defined by `n_unique_tokens` and `n_tokens_content`

- The "$\cdot$" symbol in the denominator signifies that the measure is calculated for all combinations of criteria and sampling designs.
- $\hat{\alpha}$ and $\hat{\beta}$ denote the estimators for the parameters α and β.

Figures 1 and 2 each display four diagnostic scatter plots—Residuals, Leverage, Gradient, and Influence—applied to the Online News Popularity data. In Figure 1, points are positioned by total word count (`n_tokens_content`, x-axis) versus unique token count (`n_unique_tokens`, y-axis); in Figure 2, the x-axis is non-stop unique token share (`n_non_stop_unique_tokens`) against the same response. Colour denotes the FIP, from blue (low) to red (high).

Figures 3 and 4 compare the efficiency of three sampling designs—Sampford, Poisson, and

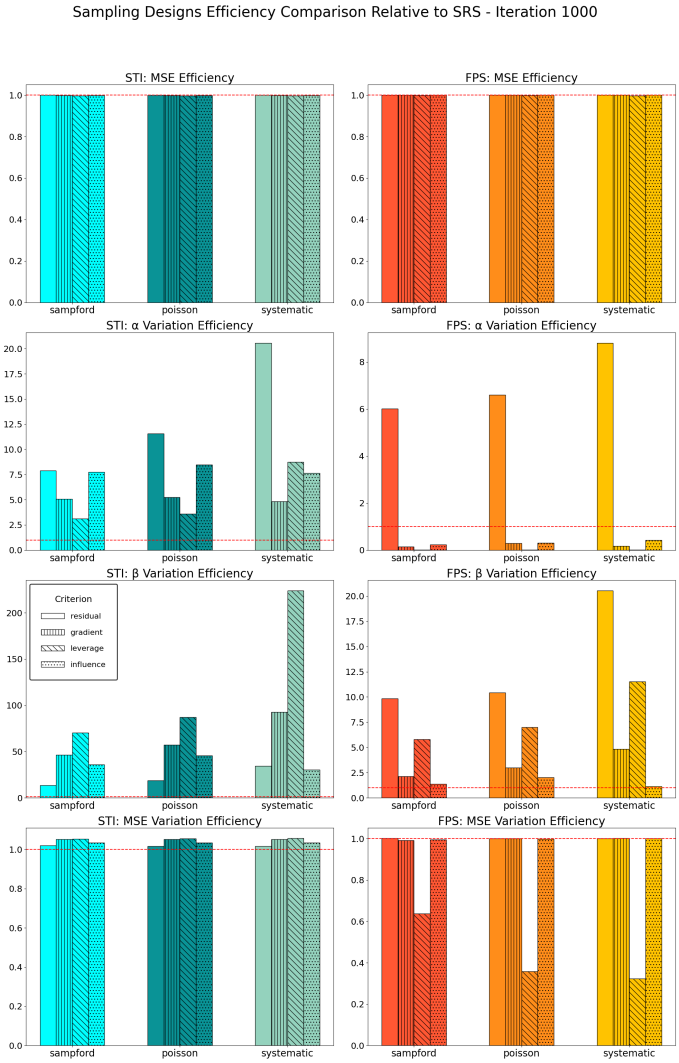


Figure 4: Comparison of sampling design efficiencies relative to SRS under the frameworks of STI and FPS across different criteria over the feature space defined by `n_unique_tokens` and `n_non_stop_unique_tokens`.

Systematic—under two estimation frameworks: STI and FPS, evaluated on test data. Each bar in the plots represents a distinct FIP criterion: residual, gradient, leverage, and influence. Efficiency metrics greater than 1 indicate superior performance of the considered design–criterion combination relative to SRS.

The first row plots presents MSE efficiency, while the second and third rows show the variance efficiency of the estimated regression parameters ($\hat{\alpha}$ and $\hat{\beta}$, respectively). The fourth row displays the variance efficiency of the MSE estimates. After 1000 iterations, we observe distinct behaviors across both criteria and sampling designs. Although previous theoretical considerations indicated that different criteria would influence estimator performance, these simulations highlight that the choice of sampling design also significantly impacts outcomes.

For the nonlinear model (Figure 3), the FPS approach yields reliable MSE efficiency comparable to SRS,

along with acceptable gains in variance efficiency of α and $\hat{\beta}$. Meanwhile, the STI approach demonstrates consistently strong performance across variance efficiencies, despite slightly lower MSE efficiency. For the linear model (Figure 4), FPS and STI perform similarly in terms of MSE, though STI again offers notably improved variance efficiencies.

Also, interestingly, the residual criterion demonstrated better overall performance compared to the other criteria, and among sampling designs, the one with the lowest entropy (systematic sampling) generally outperformed designs with higher entropy.

As the last point, overall, unequal-probability sampling strategies provide greater stability and efficiency compared to SRS, with the optimal combination dependent on the specific criterion and estimation approach.

7 Summary and Conclusion

This paper demonstrated that the choice of sampling design—beyond merely specifying the first-order inclusion probabilities—plays a crucial role in the selection of training samples and the subsequent performance of regression models. Our simulation results clearly illustrated that sampling designs exhibit distinct behaviors and yield varying efficiencies, even when other conditions such as inclusion probabilities and estimation approaches are identical. Future research could further explore the unique contributions of different sampling designs, alongside the criteria used for constructing inclusion probabilities, to optimize model accuracy and stability.

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Memory-Driven Portfolio Optimization in Multi-Period Regime-Switching Markets

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Abstract

We propose a novel multi-period portfolio optimization framework for regime-switching markets, where market memory–the dependence of asset returns and regime transitions on historical market states–explicitly influences optimal strategies. Unlike classical Markovian models that ignore path dependence, our approach incorporates memory effects through state transitions conditioned on past regimes. Using a dynamic programming formulation under the mean-variance criterion, we derive allocation rules that adapt to market memory patterns.

Keywords: Mean-variance portfolio selection, Memory effects, Regime-switching, Dynamic programming.

1 Introduction

Portfolio optimization has been one of the central problems in mathematical finance ever since the pioneering work of Markowitz [6]. His mean-variance frame work laid the foundation for a large body of literature, especially in dynamic settings. Over the decades, this classical model has been extended to incorporate various market complexities such as transaction costs, stochastic interest rates, and regime-switching behaviors. In particular, regime-switching models have attracted considerable attention due to their ability to capture sudden shifts in market dynamics. These models typically assume that the market switches between different regimes, each characterized by distinct statistical properties. A common assumption is that the regime-switching mechanism follows a finite-state Markov chain, with the current regime determining the future transition probabilities and asset returns (for example see Cakmak and Ozekici [1] and Zhou and Yin [9]).

Despite their analytical tractability, Markovian regime-switching models may fall short in capturing complex real-world phenomena where the future market behavior depends not only on the current regime but also on the historical trajectory. For instance, investors' expectations and market risk premiums often reflect accumulated experiences over time, not just the current market state. While incorporating path-dependence significantly increases model complexity and challenges analytical tractability, recent studies have begun to address this gap by proposing non-Markovian models where asset returns exhibit path-dependence. Specifically, these models generalize state-dependent coefficients by conditioning them on the filtration generated by a Markov chain [2, 7, 8, 9, 10]. However, it should be noted that these studies have focused on continuous-time settings.

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We propose a novel multi-period portfolio optimization framework for regime-switching markets, where market memory–the dependence of asset returns and regime transitions on historical market states–explicitly influences optimal strategies. Unlike classical Markovian models that ignore path dependence, our approach incorporates memory effects through state transitions conditioned on past regimes. Using a dynamic programming formulation under the mean-variance criterion, we derive allocation rules that adapt to market memory patterns.

2 Market Model

We consider a discrete-time financial market comprising d risky assets and a risk-free asset over a finite investment horizon of T periods. The market environment at time n is characterized by a state variable X_n taking values in a finite set $\mathcal{X} = \{1, \ldots, M\}$. The sequence $\{X_n\}_{n=0}^{T}$ represents the regime process, which evolves according to non-Markovian dynamics; that is, the transition probability of X_{n+1} may depend on the entire history $(X_0, \ldots, X_n)$ rather than solely on the current state X_n. For a given market path $(X_0, \ldots, X_{n-1}, X_n) = (x_0, \ldots, x_{n-1}, x_n)$, we define

$$Q_n(x_0, \ldots, x_{n-1}, x_n) := \mathrm{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0),$$

where P(.) is the probability measure and

$$\sum_{x_n=1}^{M} Q_n(x_0, \ldots, x_{n-1}, x_n) = 1.$$

The returns of all assets at each time step are allowed to depend on the full path of the regime process up to that time. This formulation accommodates a wide range of path-dependent dynamics beyond the classical Markov regime-switching models. At each time $n = 0, \ldots, T$, let $R_n = (R_{1,n}, \ldots, R_{d,n})'$ denote the return vector of the d risky assets, and let r_n be the return of the risk-free asset. We assume that both R_n and r_n depend on the entire history of the market regime up to time n. That is,

$$R_n = R_n(X_0, \ldots, X_n), \quad r_n = r_n(X_0, \ldots, X_n).$$

Here, each $R_{i,n}(X_0, \ldots, X_n)$ represents the return of the i-th risky asset conditional on the observed regime path, while $r_n(X_0, \ldots, X_n)$ denotes the corresponding path-dependent return of the risk-free asset.

By allowing both risky and risk-free returns to depend on the regime history, the model captures a fully path-dependent market structure, incorporating memory effects and richer transition dynamics beyond the Markovian setting.

We now describe the evolution of the investor's total wealth over time. Let W_n denote the total wealth at time n, and let $\pi_n = (\pi_{1,n}, \ldots, \pi_{d,n})' \in \mathbb{R}^d$ represent the allocation of wealth among the d risky assets at time n, where $\pi_{m,n}$ is the monetary amount invested in asset m. The amount allocated to the risk-free asset at time n is given by $W_n - \sum_{m=1}^{d} \pi_{m,n}$.

To simplify notation and improve readability–similarly to the conventions adopted for tensor notation and operations–we suppress the explicit dependence on the path $(X_0, \ldots, X_n)$ for parameters that are path-dependent, and factor out the path unless clarity requires otherwise. All algebraic operations involving such quantities are understood to be applied pointwise over the relevant regime path. For example, the wealth dynamics can be expressed as

$$\begin{aligned} W_{n+1} &= \left(\left(W_n - \sum_{m=1}^{d} \pi_{m,n} \right) r_n + \sum_{m=1}^{d} \pi_{m,n} R_{m,n} \right) (X_0, \ldots, X_n) \\ &= (W_n r_n + \pi_n' R_n^e)(X_0, \ldots, X_n), \end{aligned}$$

where $R_n^e = (R_{1,n} - r_n, \ldots, R_{d,n} - r_n)'$ denotes the excess return vector. For a given regime path $(X_0, \ldots, X_n) = (x_0, \ldots, x_n)$, we define

$$V_n(x_0, \ldots, x_n) = \mathbb{E}\left[R_n^e \left(R_n^e\right)'\right](x_0, \ldots, x_n),$$
$$\bar{R}_n^e(x_0, \ldots, x_n) = \mathbb{E}\left[R_n^e\right](x_0, \ldots, x_n).$$

Using these quantities, we further define the following path-dependent scalar measures:

$$H_n = \left(\bar{R}_n^e\right)' V_n^{-1} \bar{R}_n^e,$$
$$G_n = r_n \left(1 - H_n\right),$$
$$F_n = r_n^2 \left(1 - H_n\right).$$

Under the assumption that the covariance matrices are positive definite, we establish the following result. The proof can be derived from Lemmas 1 and 2 in [1]. This lemma guarantees the invertibility of the matrix $V_n(x_0, \ldots, x_n)$.

Lemma 2.1. *The matrix $V_n(x_0, \ldots, x_n)$ is positive definite. Furthermore, the scalar quantities $F_n(x_0, \ldots, x_n)$ and $G_n(x_0, \ldots, x_n)$ are strictly positive, while H_n satisfies the inequality $0 < H_n(x_0, \ldots, x_n) < 1$.*

We now formulate the multi-period mean-variance portfolio selection problem under the path-dependent regime-switching market described above. Let the initial market state be $X_0 = x_0$ and the initial wealth be $W_0 = w_0$. The investor selects a sequence of portfolio allocations $\pi = \{\pi_n\}_{n=0}^{T-1}$, where each $\pi_n = (\pi_{1,n}, \ldots, \pi_{d,n})'$ represents the monetary amounts invested in the d risky assets at time n, based on the observed market history up to that time. The objective is to find an admissible investment strategy that minimizes the variance of the terminal wealth, while ensuring that the expected terminal wealth is equal to κ, given the initial market information. Formally, the *Path-Dependent Mean-Variance Problem (PDMV)* is stated as:

$$\begin{aligned} \min_{\pi_0, \ldots, \pi_{T-1}} \quad & \mathbb{V}\mathrm{ar}_0(W_T), \\ \text{s.t.} \quad & \mathbb{E}_0[W_T] = \kappa, \\ & W_{n+1} = \left(W_n r_n + \pi_n' R_n^e\right)(X_0, \ldots, X_n), \quad n = 0, \ldots, T-1, \end{aligned}$$

where $\mathbb{E}_0[\cdot]$ and $\mathbb{V}\mathrm{ar}_0(\cdot)$ denote the conditional expectation and variance given $X_0 = x_0$ and $W_0 = w_0$.

3 Solution via Lagrangian Duality and Dynamic Programming

In this section, we develop a recursive solution method for problem (PDMV) formulated in the previous section. Our approach relies on Lagrangian duality and dynamic programming techniques. To solve problem (PDMV), we first rewrite the variance term in the objective function using its definition, and transform the problem into an equivalent constrained quadratic optimization problem:

$$\begin{aligned} \min_{\pi_0, \ldots, \pi_{T-1}} \quad & \mathbb{E}_0[(W_T - \kappa)^2], \\ \text{s.t.} \quad & \mathbb{E}_0[W_T - \kappa] = 0, \\ & W_{n+1} = \left(W_n r_n + \pi_n' R_n^e\right)(X_0, \ldots, X_n), \quad n = 0, \ldots, T-1, \end{aligned}$$

By introducing the Lagrange multiplier $2\lambda \in \mathbb{R}$, we convert the constrained problem into an unconstrained one:

$$\begin{aligned} \min_{\pi_0,\dots,\pi_{T-1}} \quad & \mathbb{E}_0[(W_T-\kappa)^2] + 2\lambda \mathbb{E}_0[W_T-\kappa], \\ \text{s.t.} \quad & W_{n+1} = (W_n r_n + \pi_n' R_n^e)\,(X_0,\dots,X_n), \quad n=0,\dots,T-1. \end{aligned}$$

Introducing the substitutions $d_1 = 2(\lambda-\kappa)$ and $d_0 = \kappa^2 - 2\lambda\kappa$, the problem becomes:

$$\begin{aligned} \min_{\pi_0,\dots,\pi_{T-1}} \quad & \mathbb{E}_0[W_T^2 + d_1 W_T + d_0], \\ \text{s.t.} \quad & W_{n+1} = (W_n r_n + \pi_n' R_n^e)\,(X_0,\dots,X_n), \quad n=0,\dots,T-1. \end{aligned} \tag{3.1}$$

This reformulation allows us to apply dynamic programming techniques to solve the unconstrained quadratic problem (3.1), as described in the following sections.

3.1 Dynamic Programming Approach

We now solve the problem (3.1) using a backward dynamic programming method. Define the value function at each time n as the minimal expected value of the cost-to-go, given the current wealth and the past path of the market states:

$$J_n(x_0,\dots,x_n;w_n) := \min_{\pi_n,\dots,\pi_{T-1}} \mathbb{E}\left[W_T^2 + d_1 W_T + d_0 \,\middle|\, X_0=x_0,\dots,X_n=x_n, W_n=w_n\right].$$

Clearly, the terminal condition is given by:

$$J_T(x_0,\dots,x_T;w_T) = w_T^2 + d_1 w_T + d_0.$$

The Bellman principle implies that the value function satisfies the recursive equation:

$$J_n(x_0,\dots,x_n;w_n) = \min_{\pi_n} \mathbb{E}\left[J_{n+1}\left(x_0,\dots,x_n,X_{n+1};W_{n+1}(\pi_n)\right) \,\middle|\, X_0=x_0,\dots,X_n=x_n,W_n=w_n\right].$$

By conditioning on the next state X_{n+1} an applying the model's assumptions we obtain

$$J_n(x_0,\dots,x_n;w_n) = \min_{\pi_n} \sum_{x_{n+1}=1}^{M} Q_{n+1}(x_0,\dots,x_{n+1}) \mathbb{E}\left[J_{n+1}\left(x_0,\dots,x_{n+1};(w_n r_n + \pi_n' R_n^e)\,(x_0,\dots,x_n)\right)\right].$$

The Bellman equation is solved recursively, starting from the terminal condition at T and proceeding backward to $n=0$,

Theorem 3.1. *The value function of problem (3.1) admits the following quadratic form:*

$$J_n(x_0,\dots,x_n;w_n) = (a_n w_n^2 + b_n w_n + c_n)(x_0,\dots,x_n),$$

3.2 Solution to Problem (PDMV)

To derive the solution for problem (PDMV) under the initial conditions $X_0 = x_0$ and $W_0 = w_0$, we reformulate the optimal value function of problem (3.1), given by $J_0(x_0;w_0)$ in terms of the Lagrange multiplier λ. Since $J_0(x_0;w_0)$ depends on λ, we define $L(\lambda) := J_0(x_0;w_0)$. By the Lagrange duality theorem

(see [5]), maximizing $L(\lambda)$ over $\lambda \in \mathbb{R}$ provides the optimal value for problem (PDMV), denoted by $\mathbb{V}\mathrm{ar}_0^*(\kappa)$, i.e.,

$$\mathbb{V}\mathrm{ar}_0^*(\kappa) = \max_{\lambda \in \mathbb{R}} L(\lambda).$$

Replacing

$$\lambda^* = \arg\max_{\lambda \in \mathbb{R}} L(\lambda)$$

in $J_0(x_0; w_0)$ yields the minimum variance associated with problem (PDMV).

4 Finite-Memory Models

In this section, we extend the path-dependent mean-variance framework developed earlier by introducing memory-based dependencies of finite length. While the original model assumes that both regime transitions and asset returns depend on the full market path $(x_0, \ldots, x_n)$, such full-path dependence may be impractical in empirical settings due to computational complexity and risk of overfitting.

In contrast, many real-world financial systems exhibit behavior where the influence of past information decays beyond a certain lag. To capture this, we propose a generalized formulation in which each component of the model–the regime transition probabilities and the asset return processes–is allowed to depend only on a finite segment of the past market states. Importantly, these memory lengths may vary across components and may optionally depend on the current regime, allowing for a flexible specification of state-dependent memory horizons.

This modeling choice is supported by both theoretical and empirical considerations. For example, some assets are driven by short-term fluctuations, while others respond to long-term market trends. Similarly, regime indicators commonly used in practice, such as moving averages or recent volatility, rely only on recent data. The proposed framework thus encompasses both classical Markovian models and the fully path-dependent model as special cases, providing a unified and tractable approach to modeling regime-switching behavior with heterogeneous memory structure.

4.1 Memory Length Functions

To formalize this idea, we first define the memory length function $\alpha(n, x_n)$ for the regime transition probabilities as follows:

$$\alpha(n, x_n) = \begin{cases} n, & n < t_{x_n}, \\ t_{x_n}, & t_{x_n} \le n \le T, \end{cases}$$

where $t_{x_n} \in \mathbb{N}$ denotes the memory length assigned to regime x_n. Then, the path-dependent transition probabilities can be written as:

$$\begin{aligned} Q_n(x_0, \ldots, x_n) &= Q_n^{(\alpha)}(x_{n-\alpha}, \ldots, x_n) \\ &= \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_{n-\alpha} = x_{n-\alpha}). \end{aligned}$$

Next, we define memory functions for the asset returns. For each risky asset $i = 1, \ldots, d$ and the riskless asset $i = 0$, we define:

$$\beta_i(n, x_n) = \begin{cases} n, & n < t_{i,x_n}, \\ t_{i,x_n}, & t_{i,x_n} \le n \le T-1, \end{cases}$$

where $t_{i,x_n} \in \mathbb{N} \cup \{0\}$ is the memory length for asset i in regime x_n. Then

$$\begin{aligned} R_{i,n}(x_0, \ldots, x_n) &= R_{i,n}^{(\beta_i)}(x_{n-\beta_i}, \ldots, x_n), \quad i = 1, \ldots, d, \\ r_n(x_0, \ldots, x_n) &= r_n^{(\beta_0)}(x_{n-\beta_0}, \ldots, x_n). \end{aligned}$$

4.2 Adapting Solutions to Finite-Memory Models

An important advantage of the finite-memory formulation is that the core components of the solution — including the value function, the optimal strategy, and the efficient frontier — can be directly adapted from the fully path-dependent framework developed earlier. In that setting, all results were formulated as deterministic functions of the full market path $(x_0, \ldots, x_n)$. In the finite-memory model, each component depends only on a truncated effective subpath $(x_{n-\gamma}, \ldots, x_n)$, with γ varying across components. Accordingly, all analytical expressions remain valid under this substitution, allowing the solution structure to be retained without modification. This preserves analytical tractability while significantly enhancing the model's generality and flexibility.

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Portfolio Selection Under Regime-Switching with Path-Dependent Transitions

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Abstract

Modern portfolio theory was first introduced by Harry Markowitz, who proposed a single-period static model where the mean and variance of portfolio returns serve as measures of reward and risk, respectively. Subsequent research extended this framework to dynamic settings, including multi-period and continuous-time models. However, these early models overlooked the intertemporal dependencies between asset returns. Recent advances address this limitation by employing regime-switching models, where asset returns in each period depend on the market state, assumed to follow a continuous- or discrete-time Markov process. This implies that transition probabilities between states depend solely on the current state, disregarding the market's historical trajectory–an assumption often unrealistic, as historical data critically influence future market trends. This paper proposes a novel portfolio optimization model that explicitly incorporates market history into transition probabilities. We formulate the problem under a mean-variance framework and adapt established portfolio optimization techniques to solve it.

Keywords: Mean-variance portfolio selection, Non-Markovian models, Regime-switching, Dynamic programming.

1 Introduction

The dynamic mean-variance portfolio selection problem extends the single-period framework pioneered by Markowitz [6], in which investors seek to minimize the variance of final wealth subject to achieving a specified expected return. Within this multi-stage context, portfolio rebalancing occurs periodically throughout the investment horizon based on a predefined optimal policy. Although dynamic programming serves as the conventional method for stochastic optimization, the non-linear nature of the variance criterion in mean-variance analysis prevents its straightforward application. To overcome this issue, Li and Ng [3] transformed the problem into an auxiliary form amenable to dynamic programming, resulting in closed-form solutions. A significant drawback of their method, nonetheless, is the presumption of intertemporally independent asset returns—an assumption starkly at odds with empirical evidence, which consistently demonstrates return dependence across time. Recent advancements by Cakmak and Ozekici [1] and other scholars have generalized the model to accommodate regime-switching environments, wherein returns are influenced by the prevailing market regime, represented as a Markov process. This approach effectively incorporates temporal dependencies while maintaining solvability.

The Markovian assumption for market state processes in prior literature inherently ignores the historical path and evolution of market dynamics. This simplification is economically unrealistic, as market

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history and memory invariably influence current market behavior and, consequently, investor decisions. While incorporating path-dependence significantly increases model complexity and challenges analytical tractability, recent studies have begun to address this gap by proposing non-Markovian models where asset returns exhibit path-dependence. Specifically, these models generalize state-dependent coefficients by conditioning them on the filtration generated by a Markov chain [2, 7, 8, 9, 10]. However, it should be noted that these studies have focused on continuous-time settings. A fundamental limitation persists: these models still assume the underlying market state follows a Markov process (i.e., transition probabilities remain path-independent). This contradicts empirical evidence where market history exerts persistent effects-exemplified by momentum (trend persistence) or regime shifts after prolonged bull/bear markets. Such phenomena demonstrate that market states are inherently non-Markovian.

The multi-period mean-variance portfolio optimization problem has not been studied in regime-switching markets where the state process is non-Markovian. In this work, we examine this scenario where asset returns depend on current market states while transition probabilities exhibit path-dependence rather than following Markovian properties.

2 Problem Formulation

We consider a discrete-time, multi-period investment horizon with T periods, where the financial market consists of $N+1$ assets. Among these assets, one is risk-free, and the remaining N are risky assets. Let $\{\Theta_n\}_{n=0}^{T}$ denote the sequence of market states (or regimes) over the investment horizon, where $\Theta_n \in \{1, 2, \ldots, M\}$ represents the regime of the market at time n. The market exhibits regime-switching dynamics, but the transitions between states are not governed by a Markov process. Instead, the transition probabilities depend on the entire past trajectory of the market states, encapsulating a path-dependent behavior. The return of the i-th asset at time n, denoted by $R_n^i(\Theta_n)$, depends on the market state Θ_n, where $i = 0, 1, \ldots, N$, with $i = 0$ representing the risk-free asset. The transition probabilities between market states depend not only on the current state Θ_{n-1} but also on the past trajectory of the market, represented by $(\Theta_0, \Theta_1, \ldots, \Theta_{n-1}) = (\theta_0, \theta_1, \ldots, \theta_{n-1})$. The probability of transitioning from state θ_{n-1} at time $n-1$ to state θ_n at time n under the history of market states $(\theta_0, \theta_1, \ldots, \theta_{n-1})$ is defined as

$$Q_n(\theta_0, \ldots, \theta_{n-1}, \theta_n) = \mathrm{P}(\Theta_n = \theta_n \mid \Theta_{n-1} = \theta_{n-1}, \ldots, \Theta_1 = \theta_1, \Theta_0 = \theta_0),$$

where P(.) is the probability measure and

$$\sum_{\theta_n=1}^{M} Q_n(\theta_0, \ldots, \theta_{n-1}, \theta_n) = 1.$$

This non-Markovian model allows for a richer and more realistic representation of market dynamics, accommodating situations where the probability of state transitions is influenced by prior trends, volatility clusters, or other historical features.

At each time $n \in \{0, 1, \ldots, T-1\}$, the investor allocates their wealth among the $N+1$ assets in the market. Let $\pi_n = (\pi_n^1, \pi_n^2, \ldots, \pi_n^N)' \in \mathbb{R}^N$ (where $'$ denotes transpose) denote the portfolio assigned to the N risky assets at time n, where π_n^i represents the wealth allocated to the i-th risky asset. The remaining wealth

$$W_n - \sum_{i=1}^{N} \pi_n^i$$

is allocated to the risk-free asset, where W_n denote the total investor's wealth at time n. Here, $\pi = \{\pi_0, \pi_1, \ldots, \pi_{T-1}\}$ denote the overall investment strategy over the entire investment horizon. The evolution

of the investor's wealth is driven by the returns of the assets. Assuming that the returns are expressed relative to the risk-free asset, the excess return of the i-th risky asset at time n in regime Θ_n is given by

$$R_n^{e,i}(\Theta_n) = R_n^i(\Theta_n) - R_n^0(\Theta_n).$$

The wealth dynamics over time can then be expressed as

$$W_{n+1} = (W_n - \sum_{i=1}^{N} \pi_n^i) R_n^0(\Theta_n) + \sum_{i=1}^{N} \pi_n^i R_n^i(\Theta_n) = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n),$$

where $R_n^e(\Theta_n) = \left(R_n^{e,1}(\Theta_n), \ldots, R_n^{e,N}(\Theta_n)\right)'$.

We assume the following regarding the returns of the assets and market states. Let $R_n(\Theta_n) = \left(R_n^0(\Theta_n), \ldots, R_n^N(\Theta_n)\right)'$ represent the vector of asset returns under the market state Θ_n. For different time points $m \neq n$, given the market states $\Theta_n = \theta_n$ and $\Theta_m = \theta_m$, the random vectors $R_n(\theta_n)$ and $R_m(\theta_m)$ are independent. Additionally, the future market state Θ_{n+1} is assumed to be independent of the given current wealth $W_n = w_n$. Also, we assume that the covariance matrix corresponding to the returns of the assets for a given market state is positive definite. Finally, we assume frictionless trading and exclude transaction costs from the model for analytical tractability.

The objective of the investor in this framework is to construct a portfolio strategy π^* that minimizes the variance of the terminal wealth W_T while achieving a predefined level of expected terminal wealth, E_T. The multi-period Markowitz's mean-variance optimization problem can be formulated as

$$P(MV): \begin{cases} \min\limits_{\pi} \mathbb{V}\mathrm{ar}_0 [W_T] \\ \text{s.t. } \mathbb{E}_0 [W_T] = E_T, \\ \quad W_{n+1} = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n), \end{cases}$$

where $\mathbb{E}_0$ and $\mathbb{V}\mathrm{ar}_0$ represent the expectation and variance operators under the initial market condition $\Theta_0 = \theta_0$.

In this framework, the path-dependence of the market regimes Θ_n introduces additional complexity. The optimal strategy π^* is inherently influenced by the entire history of market states $(\Theta_0, \Theta_1, \ldots, \Theta_n)$, reflecting the non-Markovian nature of the regime-switching dynamics. To analyze and solve this problem, we employ dynamic programming principles, incorporating the path-dependent transition probabilities and the regime-dependent asset returns into the optimization framework.

Before addressing the solution to the optimization problem, we introduce some certain matrix notations that will play a fundamental role in simplifying the subsequent computations.

For the given market state $\Theta_n = \theta_n$ we define

$$\begin{aligned} h_n(\theta_n) &= \bar{R}_n^e(\theta_n)' V_n(\theta_n)^{-1} \bar{R}_n^e(\theta_n), \\ g_n(\theta_n) &= R_n^0(\theta_n) \left(1 - h_n(\theta_n)\right), \\ f_n(\theta_n) &= R_n^0(\theta_n)^2 \left(1 - h_n(\theta_n)\right), \end{aligned}$$

where

$$\begin{aligned} V_n(\theta_n) &= \mathbb{E}\left[R_n^e(\theta_n) R_n^e(\theta_n)'\right], \\ \bar{R}_n^e(\theta_n) &= \mathbb{E}\left[R_n^e(\theta_n)\right]. \end{aligned}$$

Assuming the positive definiteness of the covariance matrices, we state the following lemma. For the proof, please refer to Lemmas 1 and 2 in [1]. Note that the following lemma guarantees the invertibility of the matrix $V_n(\theta_n)$ in the above notations.

Lemma 2.1. *The matrix $V_n(\theta_n)$ is positive definite. Additionally, the scalars $f_n(\theta_n)$ and $g_n(\theta_n)$ are strictly positive, and $h_n(\theta_n)$ satisfies $0 < h_n(\theta_n) < 1$.*

3 Dynamic Programming Formulation

The dynamic programming method serves as a powerful tool for solving stochastic optimization problems, particularly in multi-stage decision-making scenarios. This approach facilitates breaking down complex problems into smaller, more manageable subproblems. However, when applied to the mean-variance portfolio selection problem, dynamic programming faces challenges due to the non-separability of variance in the dynamic programming framework. To address this issue, Li and Ng [3] introduced an embedding technique that redefines the problem, allowing the variance to be addressed indirectly. By constructing an auxiliary problem, they derived solutions for the classical mean-variance problem. While effective, their approach can be computationally intricate and is not always intuitive. In subsequent work, Li et al. [4] proposed a more practical method by employing the Lagrange duality technique to derive optimal solutions. This approach reduces computational complexity and simplifies the solution process. In this study, we adopt a similar method to solve the problem $P(MV)$.

First, we reformulate $P(MV)$ using the variance definition, resulting in the following formulation:

$$P(MV): \begin{cases} \min\limits_{\pi} \mathbb{E}_0\left[(W_T - E_T)^2\right] \\ \text{s.t. } \mathbb{E}_0\left[W_T - E_T\right] = 0, \\ \quad W_{n+1} = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n). \end{cases}$$

By introducing the Lagrange multiplier $2\lambda \in \mathbb{R}$, the constrained problem can be transformed into the following unconstrained formulation:

$$\tilde{P}(MV): \begin{cases} \min\limits_{\pi} \mathbb{E}_0\left[(W_T - E_T)^2\right] + 2\lambda \mathbb{E}_0\left[W_T - E_T\right] \\ \text{s.t. } W_{n+1} = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n). \end{cases}$$

Introducing the substitutions $d_1 = 2(\lambda - E_T)$ and $d_0 = E_T^2 - 2\lambda E_T$, we obtain:

$$\tilde{P}(MV): \begin{cases} \min\limits_{\pi} \mathbb{E}_0\left[W_T^2 + d_1 W_T + d_0\right] \\ \text{s.t. } W_{n+1} = W_n R_n^0(\Theta_n) + \pi_n' R_n^e(\Theta_n). \end{cases}$$

3.1 Solution to Problem $\tilde{P}(MV)$

To determine the optimal solution for problem $\tilde{P}(MV)$, the approach involves minimizing the expected cost function of the terminal wealth, i.e.,

$$\min_{\pi} \ \mathbb{E}_0\left[g(W_T)\right],$$

using the dynamic programming approach, where the cost function is defined as

$$g(W_T) = W_T^2 + d_1 W_T + d_0.$$

Define $J_n(\theta_0, \ldots, \theta_n; w_n; \pi_n)$ as the expected cost incurred when employing the investment policy π_n at time n, followed by optimal strategies from time $n+1$ to T. This is conditional on the market path $(\theta_0, \ldots, \theta_n)$ and the wealth w_n available at time n. Accordingly,

$$v_n(\theta_0, \ldots, \theta_n; w_n) = \min_{\pi_n} J_n(\theta_0, \ldots, \theta_n; w_n; \pi_n),$$

represents the optimal expected cost under the given market path $(\theta_0, \ldots, \theta_n)$ and the available wealth w_n at stage n. Using the dynamic programming principle, the relationship between J_n and v_{n+1} is given by

$$\begin{aligned} &J_n(\theta_0, \ldots, \theta_n; w_n; \pi_n) \\ &= \mathbb{E}\left[v_{n+1}\left(\theta_0, \ldots, \theta_n, \Theta_{n+1}; W_{n+1}(\pi_n)\right) \mid \Theta_0 = \theta_0, \ldots, \Theta_n = \theta_n, W_n = w_n\right], \end{aligned}$$

where $W_{n+1}(\pi_n)$ is the wealth at time $n+1$ resulting from applying policy π_n. The dynamic programming equation (DPE) for this problem can thus be expressed as

$$\begin{aligned} &v_n(\theta_0, \ldots, \theta_n; w_n) \\ &= \min_{\pi_n} \mathbb{E}\left[v_{n+1}\left(\theta_0, \ldots, \theta_n, \Theta_{n+1}; W_{n+1}(\pi_n)\right) \mid \Theta_0 = \theta_0, \ldots, \Theta_n = \theta_n, W_n = w_n\right]. \end{aligned}$$

Rewriting this, the DPE becomes

$$\begin{aligned} v_n(\theta_0, \ldots, \theta_n; w_n) = \min_{\pi_n} \Bigg\{ &\sum_{\theta_{n+1}=1}^{M} Q_{n+1}(\theta_0, \ldots, \theta_n, \theta_{n+1}) \\ &\times \mathbb{E}\left[v_{n+1}\left(\theta_0, \ldots, \theta_{n+1}; w_n R_n^0(\theta_n) + \pi_n' R_n^e(\theta_n)\right)\right] \Bigg\}, \end{aligned}$$

with the boundary condition

$$v_T(\theta_0, \ldots, \theta_T; w_T) = w_T^2 + d_1 w_T + d_0.$$

The DPE is solved recursively, starting from the terminal condition at T and proceeding backward to $n = 0$, to determine the optimal strategy.

Theorem 3.1. *$v_n(\theta_0, \ldots, \theta_n; w_n)$ has a quadratic form, i.e.,*

$$v_n(\theta_0, \ldots, \theta_n; w_n) = a_n(\theta_0, \ldots, \theta_n) w_n^2 + b_n(\theta_0, \ldots, \theta_n) w_n + c_n(\theta_0, \ldots, \theta_n),$$

3.2 Solution to Problem $P(MV)$

To derive the solution for problem $P(MV)$ under the initial conditions $\Theta_0 = \theta_0$ and $W_0 = w_0$, we reformulate the optimal value function of $\tilde{P}(MV)$, given by

$$v_0(\theta_0; w_0)$$

in terms of the Lagrange multiplier λ. Since $v_0(\theta_0; w_0)$ depends on λ, we define

$$L(\lambda) := v_0(\theta_0; w_0).$$

By the Lagrange duality theorem (see [5]), maximizing $L(\lambda)$ over $\lambda \in \mathbb{R}$ provides the optimal value for problem $P(MV)$, denoted by $\mathbb{V}\mathrm{ar}_0^*(E_T)$, i.e.,

$$\mathbb{V}\mathrm{ar}_0^*(E_T) = \max_{\lambda \in \mathbb{R}} L(\lambda).$$

Replacing

$$\lambda^* = \arg\max_{\lambda \in \mathbb{R}} L(\lambda)$$

in $v_0(\theta_0; w_0)$ yields the minimum variance associated with problem $P(MV)$.

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Total Time on Test Transform Ageing Intensity Function

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Abstract

In this article introduces the total time on test ageing intensity (TTTAI) function to review behaviour of lifetime and aging of components in a system. Afterward, the properties of the TTTAI function is studed. In the following , with introduction a stochastic order based on TTTAI interesting results is provided in the comparison of life span variables. In the end, the presented results are explained with examples and applications of this concept are stated.
Keywords: Total Time on Test Transform, Quantile Function, Ageing Concepts, Economic Inequality.

1 Introduction

Total time in test (TTT) transform is also one of the most widely used and useful reliability tools that has many applications in other fields such as economics, insurance and aging concepts. This transform was introduced by [1, 2] and was further studied by researchers. One of the important applications of this transform is the characterization of distribution functions. To read more about the features and applications of TTT transform, can be refer to [8, 3, 4, 9]. In recent studies, [7] have presented new applications of the TTT plot. [4] has also expressed applications of this transform in reliability and aging concepts. Also, [6] used the TTT plot as an index of economic inequality and by comparing it with other known indexes, they presented interesting results.

2 Preliminaries

In this section, we first discuss the definition of the TTT transform. In the end, we briefly state some definitions and concepts needed in the article.

Considering that TTT transform is also used to compare random variables and its distribution functions, a stochastic order is made based on it, we assume that X and Y are two random variables with the corresponding distribution functions $F(\cdot)$ and $G(\cdot)$ and probability density functions $f(\cdot)$ and $g(\cdot)$, respectively. In addition, let $\bar{F}(\cdot) = 1 - F(\cdot)$ $\left(\bar{G}(\cdot) = 1 - G(\cdot)\right)$ be the survival function, and also let $Q_X(p)$ and $Q_Y(p)$, are the corresponding quantile functions af X and Y.

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Definition 1. *For the lifetime continuous random variable X, the TTT transform is, defined as*

$$T_X(p) = \int_0^{Q(p)} \bar{F}(x)dx, \quad p \in (0,1), \tag{2.1}$$

It can be easily concluded that $T_X(p)$ for $p = 1$ is the mathematical expectation of random variable X which can be finite or infinite.

For finite mathematical expectation $\mu = E(X)$, the scale TTT transform is $\varphi_X(p) = \frac{T_X(p)}{\mu}, \quad 0 < p < 1.$

Definition 2. *Let X and Y be two random variables with distribution functions $F(\cdot)$ and $G(\cdot)$ that have finite means. Then for $p \in (0,1)$, X is said to be smaller than Y in the TTT transform order, denoted by $X \leq_{ttt} Y$, if $T_X(p) \leq T_Y(p)$.*

For the first time [8] used the TTT transform order to compare two nonnegative random variables and also presented various applications for them.

Throughout this study, we have used notations similar to [10]. On the other hand, stochastic orders are relationships between probability distributions.

Definition 3. *Let X and Y be two random variables with distribution functions $F(\cdot)$ and $G(\cdot)$.*

- *X is stochastically smaller than Y ($X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x values.*
- *X is smaller than Y in the likelihood ratio order ($X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ be increasing.*
- *X is smaller than Y in the hazard rate order ($X \leq_{hr} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing.*
- *X is smaller than Y in the convex transform order ($X \leq_{c} Y$) if $Q_Y(F(x))$ be convex.*
- *X is smaller than Y in the dispersive order ($X \leq_{disp} Y$) if $Q_X(b) - Q_X(a) \leq Q_Y(b) - Q_Y(a)$ whenever $0 < a \leq b < 1$.*

It can be easily concluded that;

$$X \leq_{lr} Y \Longrightarrow X \leq_{hr} Y \Longrightarrow X \leq_{st} Y,$$

also $X \leq_{st} Y \Longrightarrow X \leq_{ttt} Y$. Therefore stochastic orders $\leq_{lr}$ and $\leq_{hr}$ are also preserved the TTT transform order.

[10] is a complete reference for studying the conditions for establishing the opposite of the above relations.

3 Total Time on Test Ageing Intensity (TTTAI) Function

In this section, the total time on test ageing intensity (TTTAI) function is introduced and its properties are examined

Definition 4. *Let X be a non-negative lifetime random variable with distribution function $F(\cdot)$. The Total Time on Test Ageing Intensity (TTTAI) Function of X is defined as*

$$L_X^T(p) = \frac{T_X(p)}{\frac{1}{p}\int_0^p T_X(u)du}, \quad p \in (0,1), \tag{3.1}$$

where in $T_X(p)$ is the total time on test transform of X and denoted in (1).

The TTTAI functuin denited that ageing intensity of tolal time on test transform. As stated by the total time on test concept, the range of TTTAI function will be $(0,1)$. According to equation (1), TTTAI Function can be written as follows;

$$L_X^T(p) = \frac{pT_X(p)}{\int_0^p (\int_0^{Q(u)} \bar{F}(z)dz)du}, \quad p \in (0,1),$$

Example 3.1. Let X be an exponential random variable with survival function $\bar{F}(x) = \exp\{-\theta x\}, \quad x, \theta > 0$, then $T_X(p) = \frac{p}{\theta}$ and $L_X^T(p) = 2$. According to the value obtained for TTTAI, it can be concluded that ageing intensity of total time on test in exponential family is constant over time.

Example 3.2. a) Let X be a random variable with Pareto (I), or classical Pareto distribution function with survival function $\bar{F}(x) = (\frac{x}{\sigma})^{-\alpha}, \quad x > \sigma, \quad \alpha, \sigma > 0$. If $\alpha > 1$, the mean of X exists and $E(X) = \frac{\sigma\alpha}{\alpha-1}$. Now for X the total time on test transform and TTTAI function are obtained as follows;

$$T_X(p) = \frac{\sigma(\alpha - (1-p)^{\frac{\alpha-1}{\alpha}})}{\alpha - 1},$$

$$L_X^T(p) = \frac{\alpha p - p(1-p)^{\frac{\alpha-1}{\alpha}}}{\alpha p - \frac{\alpha}{2\alpha-1}(1-(1-p)^{\frac{2\alpha-1}{\alpha}})}$$

for $\alpha > 1, \sigma > 0$ and $p \in (0,1)$.

b) If $X \sim Pareto(II)$ with survival function $\bar{F}(x) = (1 + \frac{x-\mu}{\sigma})^{-\alpha}, x > \mu$, and $\alpha, \sigma > 0$, then $E(X) = \frac{\sigma}{\alpha-1} + \mu$, for $\alpha > 1$, and

$$T_X(p) = \frac{\sigma}{\alpha - 1}[\mu - (1-p)^{\frac{\alpha-1}{\alpha}}],$$

and TTTAI function for X is

$$L_X^T(p) = \frac{\alpha - (1-p)^{\frac{\alpha-1}{\alpha}}}{\alpha p + \frac{\alpha}{2\alpha-1}[(1-p)^{\frac{2\alpha-1}{\alpha}} - 1]}, \quad p \in (0,1).$$

In studying the properties of the TTTAI function, as the first result, its relation with total time on test transform can be obtained.

Proposition 3.3. *For non-negative random variable X, $L_X^T(p) = 1$ for $p \in (0,1)$ if and only if $T_X(p)$ is a constant.*

Proof. For proof similar to the method used in [11] can be written $L_X^T(p) = 1$ hence $\frac{T_X(p)}{\frac{1}{p}\int_0^p T_X(u)du} = 1$. therefore

$$\begin{aligned} L_X^T(p) = 1 &\Longleftrightarrow pT_X(p) = \int_0^p T_X(u)du, \forall p \\ &\Longleftrightarrow \frac{\partial}{\partial p}[pT_X(p)] = \frac{\partial}{\partial p}\int_0^p T_X(u)du \\ &\Longleftrightarrow T_X(p) + pT_X^{'}(p) = T_X(p) \\ &\Longleftrightarrow pT_X^{'}(p) = 0 \end{aligned}$$

From the last relation, we can conclude that $T_X(p)$ is a constant value and the proof is complete. □

The following results can be immediately and simply concluded from the proposition 3.4 .

Proposition 3.4. *Let X be a non-negative random variable. Since the $T_X(p)$ is increasing in $p \in (0,1)$, we can conclude that the $L_X^T(p)$ is increasing always.*

Note that the revese of proposition 3.5 is not true necessarily. The following examples illustrate this.

4 Two classes of TTTAI

In this section, two different classes of distributions based are introduced based on the TTTAI concept. Then, the characteristics of these classes are examined.

Definition 5. *A random variable X with distribution function $F(\cdot)$ and mean μ, is said to be increasing(decreasing) in total time on test ageing intensity and its shown by ITTTAI(DTTTAI) if the corresponding TTTAI function $L_X^T(p)$ is increasing(decreasing) in $p \in (0,1)$.*

Proposition 4.1. *Let $F(\cdot)$ be a lifetime distribution with linear total time on test transform $T_X(p) = a+bp$. For $0 < a, b \leq \mu$, the $T_X(p)$ is increasing and by mathematical calculations we can conclude that $L_X^T(p) = \frac{2(a+bp)}{2a+bp}$ is an increasing function. Therefore $F \in ITTTAI$. It can also be easily concluded that $\bar{F}(x) = bf(x)$.*

Proposition 4.2. *Let $T_X(p) = \frac{1}{a+bp}$, for $a, b \geq \frac{1}{\mu}$, then*

$$L_X^T(p) = \frac{bp}{(a+bp)[\ln(a+bp) - \ln a]}$$

which is a decreasing function in p. Therefore $F \in DTTTAI$.

Proposition 4.3. *If $T_X(p) = \exp\{a+bp\}$, for $a, b \leq \ln \mu$, then $L_X^T(p) = \frac{bp \exp\{a+bp\}}{\exp\{a+bp\} - \exp\{a\}}$*

5 TTTAI order

We define a new stochastic order based on the TTTAI function as fallows.

Definition 6. *Let X and Y are lifetime random variables. We say X is said to be smaller than Y in the total time on yest ageing intensity(TTTAI) order and denoted by $X \leq_{TAI} Y$ if $L_X^T(p) \leq L_Y^T(p)$, for all $p \in (0,1)$.*

Proposition 5.1. *Let X and Y are lifetime random variables with TTT transform $T_X(p)$ and $T_Y(p)$ respectively. In this case,*

$$X \leq_{TAI} Y \Longleftrightarrow \frac{\int_0^p T_X(u)du}{\int_0^p T_Y(u)du} \searrow p,$$

wich that $\searrow p$ means decreasing than $p \in (0,1)$.

Proof. For proof suppose $M_X(p) = \int_0^p T_X(u)du$ and $M_Y(p) = \int_0^p T_Y(u)du$. It is clear that $M_X^{'}(p) = T_X(p)$ and also $M_Y^{'}(p) = T_Y(p)$. Now, using the definition of order

$$X \leq_{TAI} Y \Longleftrightarrow \frac{T_X(p)}{\frac{1}{p}\int_0^p T_X(u)du} \leq \frac{T_Y(p)}{\frac{1}{p}\int_0^p T_Y(u)du},$$

according to the above assumptions, we can write:

$$\begin{aligned} X \leq_{TAI} Y &\Longleftrightarrow M_X(p)M_Y^{'}(p) - M_Y(p)M_X^{'}(p) \geq 0 \\ &\Longleftrightarrow \frac{M_X(p)M_Y^{'}(p) - M_Y(p)M_X^{'}(p)}{(M_X(p))^2} \geq 0 \\ &\Longleftrightarrow \frac{\partial}{\partial p}(\frac{M_Y(p)}{M_X(p)}) \geq 0 \end{aligned}$$

the last relation means that $\frac{M_Y(p)}{M_X(p)}$ is increasing in $p \in (0,1)$. Therefore, it can be concluded that $\frac{M_X(p)}{M_Y(p)}$ is decreasing in $p \in (0,1)$, and as a result $\frac{\int_0^p T_X(u)du}{\int_0^p T_Y(u)du}$ is decreasing in $p \in (0,1)$. □

The total time on test ageing intensity order have 3 properties of a stochastic order, namely reflexive, commutative, and antisymmetric properties that will examine them below.

i) Reflexive property: Its obvious for $p \in (0,1)$, $L_X^T(p) \leq L_X^T(p)$, i.e. $X \leq_{TAI} X$.

ii) Commutative property: If $X \leq_{TAI} Y$ then $L_X^T(p) \leq L_Y^T(p)$, and if $Y \leq_{TAI} Z$ then $L_Y^T(p) \leq L_Z^T(p)$. Therefore, it can be concluded that $L_X^T(p) \leq L_Z^T(p)$, and this also means $X \leq_{TAI} Z$. Hence

$$X \leq_{TAI} Y \quad and \quad Y \leq_{TAI} Z \Longrightarrow X \leq_{TAI} Z.$$

iii) Antisymmetric property: If $X \leq_{TAI} Y$ then $L_X^T(p) \leq L_Y^T(p)$, and if $Y \leq_{TAI} X$ then $L_Y^T(p) \leq L_X^T(p)$. So we have $L_X^T(p) = L_Y^T(p)$, therefore $\frac{T_X(p)}{\frac{1}{p}\int_0^p T_X(u)du} = \frac{T_Y(p)}{\frac{1}{p}\int_0^p T_Y(u)du}$. Now, using the method of proving Proposition 5.3, if assume $M_X(p) = \int_0^p T_X(u)du$ and $M_Y(p) = \int_0^p T_Y(u)du$ and

$$L_X^T(p) = L_Y^T(p) \Longleftrightarrow M_X(p)T_Y(p) = M_Y(p)T_X(p).$$

As respects $M_X^{'}(p) = T_X(p)$ and also $M_Y^{'}(p) = T_Y(p)$, it can be concluded that $\frac{\partial}{\partial p}(\frac{M_Y(p)}{M_X(p)}) = 0$, therefore $\frac{M_Y(p)}{M_X(p)} = c$ hence

$$\begin{aligned} L_X^T(p) = L_Y^T(p) &\Longleftrightarrow M_X(p) = cM_Y(p) \\ &\Longleftrightarrow \int_0^p T_X(u)du = c\int_0^p T_Y(u)du \\ &\Longleftrightarrow T_X(p) = T_Y(p), \qquad p \in (0,1). \end{aligned}$$

That is, X and Y have proportional total time on test transform functions.

The next proposition provides an interesting result about the TTTAI of sequences of lifetime random variables.

Proposition 5.2. *Let X_i and Y_j, that $i,j = 1,2,...,n$ be two random sample of X and Y respectively with $T_{X_i}(p) = a + bp$ and $T_{Y_j}(p) = c + dp$, that $a,b,c,d > 0$. If $X_i \leq_{TAI} Y_j$ for all i,j, then $ad - bc \geq 0$*

Proof. Since $X_i \leq_{TAI} Y_j$ it can be concluded that $L_{X_i}^T(p) \leq L_{Y_j}^T(p)$. So

$$\begin{aligned} &\frac{T_{X_i}(p)}{\frac{1}{p}\int_0^p T_{X_i}(u)du} \leq \frac{T_{Y_j}(p)}{\frac{1}{p}\int_0^p T_{Y_j}(u)du} \\ &\Longrightarrow \frac{a+bp}{\frac{1}{p}\int_0^p a + budu} \leq \frac{c+dp}{\frac{1}{p}\int_0^p c + dudu} \\ &\Longrightarrow \frac{a+bp}{ap + b\frac{p^2}{2}} \leq \frac{c+dp}{cp + d\frac{p^2}{2}} \\ &\Longrightarrow (a+bp)(cp + d\frac{p^2}{2}) \leq (c+dp)(ap + b\frac{p^2}{2}), \end{aligned}$$

by doing simple algebraic calculations we have $(ad - bc)p^2 \geq 0$ and as a result $ad - bc \geq 0$. □

Remark 5.3. According to Theorem 4.6 of [11], we can conclude that the total time on test ageing intensity order is preserved by the linear strictly increasing transform $\phi : R^+ \to R^+$ with $\phi(0) = 0$ and $\phi(x) = ax$, namely, if $X \leq_{TAI} Y$ then $\phi(X) \leq_{TAI} \phi(Y)$.

6 conclosions

In this paper, we have introduced the total ime on test intensity function and studied its properties based behaviour of the total ime on test transform and AI function. Also we have defined two new classes, ITTTAI and DTTTAI, and studied closure properties under different reliability concepts.

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Fractional Geometric Programming Problems with Interval Coefficients

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Abstract

Geometric programming (GP) is a powerful technique for solving a special class of nonlinear optimization problems. Originally developed for engineering design, GP has been widely applied to various optimization domains, particularly when decision variables appear in exponential form. This paper investigates a geometric programming model with a fractional objective function and uncertain coefficients. To achieve the optimal solution, the objective function and all constraints—formulated as posynomials—are first transformed into a convex geometric programming problem through convexification techniques. Subsequently, a variable transformation is employed to address the fractional objective, and robust optimization methods are applied to handle uncertainty in the coefficients. The proposed approach provides an effective framework for solving complex optimization problems with both.
Keywords: Global optimization, Geometric programming, Fractional programming, Interval programming.

1 Introduction

Geometric programming (GP) is a well-established method for solving nonlinear programming problems. It was first introduced in 1961 by Clarence Zener in his pioneering paper "Mathematics of Engineering Design Optimization." Later, in 1967, Zener together with Duffin and Peterson presented a comprehensive exposition of GP in their influential book "Geometric Programming: Theory and Applications" [5]. In the optimization literature, fractional geometric programming (FGP) problems play a crucial role due to their wide-ranging applications in engineering design, management, and decision-making processes. Such problems frequently arise in areas such as industrial economics, information systems, and resource allocation [2, 4]. Seraj et al. (2018) investigated posynomial-type fractional GP problems and further extended the study to multi-objective linear GP using the reference point approach [33, 2] . Shirinnejad et al. proposed a technique for solving signomial fractional GP problems, offering a globally optimal solution within a small tolerance from the original problem [9]. In another significant contribution, Westerlund [1] introduced several transformation techniques that are particularly useful for determining global optimal solutions. To address uncertainty in real-world systems, interval programming has been widely applied. This method transforms the original interval-based optimization model into two deterministic sub-models, enabling the

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treatment of interval uncertainty in a systematic manner [7]. In this paper, we focus on a fractional geometric programming problem with uncertain coefficients. Specifically, we consider an FGP model in which both the objective function and the constraints are of the signomial type. By applying a variable transformation, the fractional objective function is converted into a single-objective problem. Convexification techniques are then employed to reformulate the model as a convex GP problem, thereby ensuring the possibility of reaching a global optimum. Subsequently, interval-based optimization methods are applied to handle the uncertainty of coefficients. Finally, a numerical example is presented to illustrate and validate the proposed methodology.

2 geometric programming

Geometric programming was first introduced in 1964 by Peterson, Zener, and Duffin. The objective of geometric programming is to minimize design costs. The general form of a constrained geometric programming problem is as follow:

$$f(x) = \sum_{i=1}^{m} f_i(x) = \sum_{i=1}^{m} \left(c_i \prod_{j=1}^{n} x_j^{a_{ij}} \right) \tag{2.1}$$

$$\text{s.t.} \quad g_k(x) = \sum_{i=1}^{s} \left(c_{ki} \prod_{j=1}^{n} x_j^{c_{kij}} \right) \leq 1, \quad k = 1, \ldots, l \tag{2.2}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \quad a_{ij} \in \mathbb{R} \tag{2.3}$$

In the above functions, if all coefficients are positive, the function is called a posynomial. If for some terms the coefficients are positive and for others negative, the function is referred to as a signomial. There are various methods for solving geometric programming problems, among which the most important are deriving the dual model and solving the dual problem [6].

2.1 The optimality conditions of the problem

In general, problem (1) is non-convex and may therefore lack a global optimal solution.To address this issue, a convexification algorithm is employed to reformulate the problem into a convex geometric programming form.

A convex problem, as opposed to a non-convex geometric problem such as inverse and exponential transformations, has been described and discussed in detail by [8, 10, 6]. A twice differentiable function $f(X) = c\prod_{i=1}^{n} x_i^{a_i}$ is convex if $\forall c > 0,\ x_i > 0, a_i \leq 0$,$(i = 1, ..., n)$.

Definition: We define a real interval as $[x^-, x^+]$ such that $x^- \leq x^+$, and we have,$x = \{x \in R, x^- \leq x \leq x^+\}$,the values $x^- and \quad x^+$ a referred to as the lower and upper bounds of the interval, respectively.

A fractional geometric programming problem with uncertain coefficients can be formulated as follow:

$$\min \ F(X) = \frac{\sum_{i=1}^{m} f_i(x)}{\sum_{i=1}^{n} q_i(x)} = \frac{\sum_{i=1}^{m} [\alpha_i^-, \alpha_i^+] x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}}{\sum_{i=1}^{n} [\beta_i^-, \beta_i^+] x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}} \tag{2.4}$$

$$\text{s.t.} \quad g_k(X) = \sum_{i=1}^{s} [\gamma_{ik}^-, \gamma_{ik}^+] x_1^{c_{ik1}}, x_2^{c_{ik2}} \cdots x_n^{c_{ikn}} \leq 1 \quad k = 1, \ldots, l \tag{2.5}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \tag{2.6}$$

The above problem is decomposed into two subproblems, which establish a lower bound and an upper bound for the main problem.

$$\min\ F(X) = [\frac{\sum_{i=1}^{m} \alpha_i^- x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}}{\sum_{i=1}^{n} \beta_i^+ x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}}, \frac{\sum_{i=1}^{m} \alpha_i^+ x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}}{\sum_{i=1}^{n} \beta_i^- x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}}] \tag{2.7}$$

$$\text{s.t.} \quad g_k(X) = \sum_{i=1}^{s} [\gamma_{ik}^-, \gamma_{ik}^+] x_1^{c_{ik1}}, x_2^{c_{ik2}} \cdots x_n^{c_{ikn}} \leq 1 \quad k = 1, \ldots, l \tag{2.8}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \tag{2.9}$$

In order to transform the fractional geometric model (4), a denominator variable substitution is employed. By this transformation, the objective functions are expressed in terms of the new variables $\cdot$ and $\cdot$. Accordingly, the original problem is reformulated as two single-objective geometric programs with deterministic coefficient.

$$\min\ F^-(X) = \frac{\sum_{i=1}^{m} \alpha_i^- x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}}{\sum_{i=1}^{n} \beta_i^+ x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}} \tag{2.10}$$

$$\text{s.t.} \quad g_k^+(X) = \sum_{i=1}^{s} \gamma_{ik}^+ x_1^{c_{ik1}}, x_2^{c_{ik2}} \cdots x_n^{c_{ikn}} \leq 1 \quad k = 1, \ldots, l \tag{2.11}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \tag{2.12}$$

$$\min\ F^+(X) = \frac{\sum_{i=1}^{m} \alpha_i^+ x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}}{\sum_{i=1}^{n} \beta_i^- x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}} \tag{2.13}$$

$$\text{s.t.} \quad g_k^-(X) = \sum_{i=1}^{s} \gamma_{ik}^- x_1^{c_{ik1}}, x_2^{c_{ik2}} \cdots x_n^{c_{ikn}} \leq 1 \quad k = 1, \ldots, l \tag{2.14}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \tag{2.15}$$

By solving the above model, both an upper bound and a lower bound solution for the problem are obtained. To achieve the global optimal solution, the problem is first convexified and then solved. In the following, we address concepts related to convexification.

$$\min\ F^-(X) = \sum_{i=1}^{m} \alpha_i^- x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}} t^{-1} \tag{2.16}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \beta_i^+ x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}} t^{-1} = 1 \tag{2.17}$$

$$g_k^+(X) = \sum_{i=1}^{s} \gamma_{ik}^+ x_1^{c_{ik1}}, x_2^{c_{ik2}} \cdots x_n^{c_{ikn}} \leq 1 \quad k = 1, \ldots, l \tag{2.18}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \tag{2.19}$$

$$\min\ F^+(X) = \sum_{i=1}^{m} \alpha_i^+ x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}} y^{-1} \tag{2.20}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \beta_i^- x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}} y^{-1} = 1 \tag{2.21}$$

$$g_k^-(X) = \sum_{i=1}^{s} \gamma_{ik}^- x_1^{c_{ik1}}, x_2^{c_{ik2}} \cdots x_n^{c_{ikn}} \leq 1 \quad k = 1, \ldots, l \tag{2.22}$$

$$X = (x_1, x_2, \ldots, x_n) > 0, \tag{2.23}$$

3 Example

Consider the geometric problem in the (p_1) model below.

$$(P_1)\min F = \frac{[4,5]x_1^{1.9}x_2^{-2.5}x_3^{-3} + [2,3]x_4^{-1.3}x_5^{2.6} + [1.7,2.5]x_1^{-1.8}x_5^{-2.5}}{[1.2,2.4]x_1^{0.9}x_2^{-1.9}x_3^{-3} + [1.5,2.5]x_4^{-0.3}x_5^{2.5}}$$

$$\begin{aligned}
\text{s.t.} \quad & [5,6]x_1^{-3.8}x_2^{2.2}x_3^{4.3} + [0.2,0.5]x_4^{-0.7}x_5^{-1.6} + [0.1,0.2]x_1^{2.3}x_3^{-3.9}x_5^{8.5} \leq 1 \\
& [3.1,5]x_1^{2.3}x_2^{1.7}x_3^{4.5} \leq 1 \\
& [1.4,2.6]x_3^{-2.1}x_5^{0.4} \leq 1 \\
& [2.1,3]x_1^{1.6}x_2^{2.4}x_3^{-3.8} \leq 1 \\
& [3.2,4.5]x_4^{3.4}x_5^{2.3} \leq 1 \\
& [4.3,5]x_2^{-1.1}x_4^{7.3}x_5^{5.6} \leq 1 \\
& 0 \leq x_i \leq 10
\end{aligned}$$

Using the basic concepts of interval programming, the example is expressed in the form of (p_2) model .

$$(P_2)\ \min F = [\frac{4x_1^{1.9}x_2^{-2.5}x_3^{-3} + 2x_4^{-1.3}x_5^{2.6} + 1.7x_1^{-1.8}x_5^{-2.5}}{2.4x_1^{0.9}x_2^{1.9}x_3^{-3} + 2.5x_4^{-0.3}x_5^{2.5}}, \frac{5x_1^{0.5}x_2^{-1}x_3^{0.3} + 3x_4^{-1.3}x_5^{2.6} + 2.5x_1^{1.8}x_5^{2.5}}{1.2x_1^{0.9}x_2^{1.9}x_3^{-3} + 1.5x_4^{-0.3}x_5^{2.5}}]$$

$$\begin{aligned}
\text{s.t.} \quad & [5,6]x_1^{-3.8}x_2^{2.2}x_3^{4.3} + [0.2,0.5]x_4^{-0.7}x_5^{-1.6} + [0.1,0.2]x_1^{2.3}x_3^{-3.9}x_5^{8.5} \leq 1 \\
& [3.1,5]x_1^{2.3}x_2^{1.7}x_3^{4.5} \leq 1 \\
& [1.4,2.6]x_3^{-2.1}x_5^{0.4} \leq 1 \\
& [2.1,3]x_1^{1.6}x_2^{2.4}x_3^{-3.8} \leq 1 \\
& [3.2,4.5]x_4^{3.4}x_5^{2.3} \leq 1 \\
& [4.3,5]x_2^{-1.1}x_4^{7.3}x_5^{5.6} \leq 1 \\
& 0 \leq x_i \leq 10
\end{aligned}$$

Modeling (p_2) is expressed as two submodels, representing the lower bound and upper bound, as follows.

$$\min F^- = \frac{4x_1^{1.9}x_2^{-2.5}x_3^{-3} + 2x_4^{-1.3}x_5^{2.6} + 1.7x_1^{-1.8}x_5^{-2.5}}{2.4x_1^{0.9}x_2^{1.9}x_3^{-3} + 2.5x_4^{-0.3}x_5^{2.5}}$$

$$\begin{aligned}
\text{s.t.} \quad & 6x_1^{-3.8}x_2^{2.2}x_3^{4.3} + 0.5x_4^{-0.7}x_5^{-1.6} + 0.2x_1^{2.3}x_3^{-3.9}x_5^{8.5} \leq 1 \\
& 5x_1^{2.3}x_2^{1.7}x_3^{4.5} \leq 1 \\
& 2.6x_3^{-2.1}x_5^{0.4} \leq 1 \\
& 3x_1^{1.6}x_2^{2.4}x_3^{-3.8} \leq 1 \\
& 4.5x_4^{3.4}x_5^{2.3} \leq 1 \\
& 5x_2^{-1.1}x_4^{7.3}x_5^{5.6} \leq 1 \\
& 0 \leq x_i \leq 10
\end{aligned}$$

$$\min F^+ = \frac{5x_1^{0.5}x_2^{-1}x_3^{0.3} + 3x_4^{-1.3}x_5^{2.6} + 2.5x_1^{1.8}x_5^{2.5}}{1.2x_1^{0.9}x_2^{1.9}x_3^{-3} + 1.5x_4^{-0.3}x_5^{2.5}}$$

$$\begin{aligned}
\text{s.t.} \quad & 5x_1^{-3.8}x_2^{2.2}x_3^{4.3} + 0.2x_4^{-0.7}x_5^{-1.6} + 0.1x_1^{2.3}x_3^{-3.9}x_5^{8.5} \leq 1 \\
& 3.1x_1^{2.3}x_2^{1.7}x_3^{4.5} \leq 1 \\
& 1.4x_3^{-2.1}x_5^{0.4} \leq 1 \\
& 2.1x_1^{1.6}x_2^{2.4}x_3^{-3.8} \leq 1 \\
& 3.2x_4^{3.4}x_5^{2.3} \leq 1 \\
& 4.3x_2^{-1.1}x_4^{7.3}x_5^{5.6} \leq 1 \\
& 0 \leq x_i \leq 10
\end{aligned}$$

First, to eliminate the fractional form of the problem, the denominator of each problem is set equal to a new variable. Consequently, the problem is transformed into two geometric problems with deterministic coefficients, as follows.

$$\min F^- = 4x_1^{1.9}x_2^{-2.5}x_3^{-3}t^{-1} + 2x_4^{-1.3}x_5^{2.6}t^{-1} + 1.7x_1^{-1.8}x_5^{-2.5}t^{-1}$$

$$\begin{aligned}
\text{s.t.}\quad & 2.4x_1^{0.9}x_2^{1.9}x_3^{-3}t^{-1} + 2.5x_4^{-0.3}x_5^{2.5}t^{-1} = 1 \\
& 6x_1^{-3.8}x_2^{2.2}x_3^{4.3} + 0.5x_4^{-0.7}x_5^{-1.6} + 0.2x_1^{2.3}x_3^{-3.9}x_5^{8.5} \leq 1 \\
& 5x_1^{2.3}x_2^{1.7}x_3^{4.5} \leq 1 \\
& 2.6x_3^{-2.1}x_5^{0.4} \leq 1 \\
& 3x_1^{1.6}x_2^{2.4}x_3^{-3.8} \leq 1 \\
& 4.5x_4^{3.4}x_5^{2.3} \leq 1 \\
& 5x_2^{-1.1}x_4^{7.3}x_5^{5.6} \leq 1 \\
& 0 \leq x_i \leq 10
\end{aligned}$$

$$\min F^+ = 5x_1^{0.5}x_2^{-1}x_3^{0.3}y^{-1} + 3x_4^{-1.3}x_5^{2.6}y^{-1} + 2.5x_1^{1.8}x_5^{2.5}y^{-1}$$

$$\begin{aligned}
& 1.2x_1^{0.9}x_2^{1.9}x_3^{-3}y^{-1} + 1.5x_4^{-0.3}x_5^{2.5}y^{-1} = 1 \\
& 5x_1^{-3.8}x_2^{2.2}x_3^{4.3} + 0.2x_4^{-0.7}x_5^{-1.6} + 0.1x_1^{2.3}x_3^{-3.9}x_5^{8.5} \leq 1 \\
& 3.1x_1^{2.3}x_2^{1.7}x_3^{4.5} \leq 1 \\
& 1.4x_3^{-2.1}x_5^{0.4} \leq 1 \\
& 2.1x_1^{1.6}x_2^{2.4}x_3^{-3.8} \leq 1 \\
& 3.2x_4^{3.4}x_5^{2.3} \leq 1 \\
& 4.3x_2^{-1.1}x_4^{7.3}x_5^{5.6} \leq 1 \\
& 0 \leq x_i \leq 10
\end{aligned}$$

By employing convexification techniques, the problem is transformed into a convex geometric model. Through the substitution of variable uuu, the problems can be reformulated in convex form as follows:

Accordingly, we obtain the following two problems:

$$\min F^- = 4z_1^{-1.9}x_2^{-2.5}x_3^{-3}t^{-1} + 2x_4^{-1.3}z_5^{-2.6}t^{-1} + 1.7x_1^{-1.8}x_5^{-2.5}t^{-1}$$

$$\begin{aligned}
\text{s.t.}\quad & 2.4z_1^{-0.9}x_2^{-1.9}x_3^{-3}t^{-1} + 2.5x_4^{-0.3}z_5^{-2.5}t^{-1} = 1\\
& 6x_1^{-3.8}z_2^{-2.2}z_3^{-4.3} + 0.5x_4^{-0.7}x_5^{-1.6} + 0.2z_1^{-2.3}x_3^{-3.9}z_5^{-8.5} \le 1\\
& 5z_1^{-2.3}z_2^{-1.7}z_3^{-4.5} \le 1\\
& 6x_3^{-2.1}z_5^{-0.4} \le 1\\
& 3z_1^{-1.6}z_2^{-2.4}x_3^{-3.8} \le 1\\
& 4.5z_4^{-3.4}z_5^{-2.3} \le 1\\
& 5x_2^{-1.1}z_4^{-7.3}z_5^{-5.6} \le 1\\
& x_i^{-1}z_i^{-1} = 1\\
& 0 \le x_i \le 10\\
& 0 \le z_i \le 10
\end{aligned}$$

$$\min F^+ = 5z_1^{-0.5}x_2^{-1}z_3^{-0.3}y^{-1} + 3x_4^{-1.3}z_5^{-2.6}y^{-1} + 2.5z_1^{-1.8}z_5^{-2.5}y^{-1}$$

$$\begin{aligned}
& 1.2z_1^{-0.9}z_2^{-1.9}x_3^{-3}y^{-1} + 1.5x_4^{-0.3}z_5^{-2.5}y^{-1} = 1\\
& 6x_1^{-3.8}z_2^{-2.2}z_3^{-4.3} + 0.5x_4^{-0.7}x_5^{-1.6} + 0.2z_1^{-2.3}x_3^{-3.9}z_5^{-8.5} \le 1\\
& 5z_1^{-2.3}z_2^{-1.7}z_3^{-4.5} \le 1\\
& 6x_3^{-2.1}z_5^{-0.4} \le 1\\
& 3z_1^{-1.6}z_2^{-2.4}x_3^{-3.8} \le 1\\
& 4.5z_4^{-3.4}z_5^{-2.3} \le 1\\
& 5x_2^{-1.1}z_4^{-7.3}z_5^{-5.6} \le 1\\
& x_i^{-1}z_i^{-1} = 1\\
& 0 \le x_i \le 10\\
& 0 \le z_i \le 10
\end{aligned}$$

By solving the two aforementioned problems, the optimal lower and upper bounds of the objective function are obtained $F^- = 6.509692$ $F^+ = 11.01330$ and the solution to the problem is presented as an interval with upper and lower bounds in Table 1

	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*
x_-^*	1.043886	0.1027814	1.183635	0.5287299	
x_+^*	1.05729	0.3103955	1.605401	0.6637904	1.101235
	z_1^*	z_2^*	z_3^*	z_4^*	z_5^*
z_-^*	0.945115	3.221697	0.6228973	1.506500	0.9080712
z_+^*	0.9579593	9.729387	0.8448549	1.891325	0.9570317

4 conclution

In this study, we propose a model for addressing fractional geometric problems with uncertain data. Our approach incorporates the denominator of the objective function as an additional variable within the problem formulation. To ensure the attainment of a global optimal solution, the model is transformed into a convex form. Nevertheless, the algorithm remains applicable to non-convex cases, where independent verification of global optimality is not guaranteed. Furthermore, the proposed technique can be extended to multi-objective geometric problems. In addition, other contemporary approaches—such as treating the denominator as a separate objective function or employing multi-objective optimization methods—can be adopted to derive alternative solutions.

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SEIQR model for analyzing Covid-19 with vaccination via Caputo fractional derivative

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Abstract

In this paper, an SEIQR model for the spread of Covid-19 with vaccination is analyzed via the Caputo fractional derivative. using the fractional Euler approach, an approximate solution to the model is obtained. Our goal of this research is to check how the number of infected people would decrease if vaccination was done in Iran on July 10, 2021.

Keywords: Caputo Fractional derivative, Fractional Euler approach, Epidemic model.

1 Introduction

The Covid-19 pandemic, declared a global crisis, reached Iran on February 19, 2020. The international scientific community swiftly mobilized to develop effective vaccines, resulting in the production of various inoculations, including Pfizer, AstraZeneca, Moderna, and Sinopharm, among others. Unfortunately, vaccination in Iran was delayed and this caused a large number of people to be infected with the Coronavirus. The main goal of this research is to show how much the number of infected cases would be reduced if public vaccination in Iran was done on time. New mathematical models of fractional order have been developed to help predict and control disease outbreaks ([2, 3, 4]).

2 Preliminary Definitions

Here, we start by introducing the fundamental definitions of fractional calculus that are relevant to the present manuscript.

Definition 1. *([5]) For a function $\mathcal{B}$, the Caputo fractional derivative of order $\delta \in (0,1)$ is given by*

$$ {}^{C}\mathcal{D}_{t}^{\delta}[\mathcal{B}(t)] = \frac{1}{\Gamma(j-\delta)} \int_{0}^{t} \frac{\mathcal{B}^{(j)}(\zeta)}{(t-\zeta)^{(\delta-j+1)}}\, d\zeta, \qquad j = [\delta] + 1. $$

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Also, fractional integral of order δ *with* $\delta \in \mathbb{R}^+$ *is defined as*

$$ {}^{C}\mathrm{I}_{\mathsf{t}}^{\delta}[\mathcal{B}(\mathsf{t})] = \frac{1}{\Gamma(\delta)} \int_0^{\mathsf{t}} (\mathsf{t}-\zeta)^{\delta-1} \mathcal{B}(\zeta)\, \mathrm{d}\zeta. $$

3 Model Formulation

In this Section, a model is developed where the population is divided into five distinct classes; namely: Susceptible individuals (S), Exposed individuals (E), Infected individuals (I), Quarantined individuals (Q) and Recovered individuals (R). The proportion of individuals who vaccinated in Iran is denoted by $0 < \varpi \leq 1$ while $0 < \varepsilon \leq 1$ shows the efficacy of the vaccine. The system is as

$$ \begin{cases} \theta^{\delta-1}\mathcal{D}^{\delta}\mathrm{S}(\mathsf{t}) = \Pi - \vartheta(1-\varepsilon\varpi)\mathrm{S}(\mathsf{t})\mathrm{I}(\mathsf{t}) - \zeta\,\mathrm{S}(\mathsf{t}), \\ \theta^{\delta-1}\mathcal{D}^{\delta}\mathrm{E}(\mathsf{t}) = \vartheta(1-\varepsilon\varpi)\mathrm{S}(\mathsf{t})\mathrm{I}(\mathsf{t}) - (\hbar+\zeta)\mathrm{E}(\mathsf{t}), \\ \theta^{\delta-1}\mathcal{D}^{\delta}\mathrm{I}(\mathsf{t}) = \hbar\mathrm{E}(\mathsf{t}) - (\grave{\imath}+\varrho+\phi+\zeta)\mathrm{I}(\mathsf{t}), \\ \theta^{\delta-1}\mathcal{D}^{\delta}\mathrm{Q}(\mathsf{t}) = \varrho\mathrm{I}(\mathsf{t}) - (\grave{\jmath}+\varkappa+\zeta)\mathrm{Q}(\mathsf{t}), \\ \theta^{\delta-1}\mathcal{D}^{\delta}\mathrm{R}(\mathsf{t}) = \grave{\imath}\mathrm{I}(\mathsf{t}) + \grave{\jmath}\mathrm{Q}(\mathsf{t}) - \zeta\,\mathrm{R}(\mathsf{t}), \end{cases} \tag{3.1} $$

for $\mathsf{t} > 0$ where the initial conditions (ICs) are

$$ \mathrm{S}(0) = \mathrm{S}_0 > 0, \quad \mathrm{E}(0) = \mathrm{E}_0 > 0, \quad \mathrm{I}(0) = \mathrm{I}_0 > 0, \quad \mathrm{Q}(0) = \mathrm{Q}_0 > 0, \quad \mathrm{R}(0) = \mathrm{R}_0 > 0. $$

The disease-free equilibrium point is $\mathbb{E}_0 = \left(\frac{\Pi}{\zeta}, 0, 0, 0, 0\right)$.

The basic reproduction number is as

$$ \mathcal{R}_0 = \frac{\hbar\vartheta(1-\varepsilon\varpi)\Pi}{\zeta(\hbar+\zeta)(\grave{\imath}+\varrho+\phi+\zeta)}. $$

Theorem 3.1. *The disease-free equilibrium point* $\mathbb{E}_0$ *of* (3.1) *is locally asymptotically stable if* $\mathcal{R}_0 < 1$.

Theorem 3.2. *The disease-free equilibrium point* $\mathbb{E}_0$ *of* (3.1) *is globally asymptotically stable if* $\mathcal{R}_0 \leq 1$.

4 Numerical Method and Simulation

Consider system (3.1) in a compact form as follows:

$$ \theta^{\delta-1}{}^{C}\mathcal{D}_{\mathsf{t}}^{\delta}\omega(\mathsf{t}) = \mathcal{K}(\mathsf{t}, \omega(\mathsf{t})), \qquad \omega(0) = \omega_0,\ 0 \leq \mathsf{t} \leq T < \infty, \tag{4.1} $$

where $\omega = (\mathrm{S}, \mathrm{E}, \mathrm{I}, \mathrm{Q}, \mathrm{R}) \in \mathbb{R}_+^5$, $\omega_0 = (\mathrm{S}_0, \mathrm{E}_0, \mathrm{I}_0, \mathrm{Q}_0, \mathrm{R}_0)$ is the initial vector, and $\mathcal{K}(\mathsf{t}) \in \mathbb{R}$ is a continuous vector function satisfying Lipschitz condition

$$ ||\mathcal{K}(\omega_1(\mathsf{t})) - \mathcal{K}(\omega_2(\mathsf{t}))|| \leq c||\omega_1(\mathsf{t}) - \omega_2(\mathsf{t})||, \qquad c > 0. $$

Applying a fractional integral operator corresponding to the Caputo derivative to equation (4.1), we obtain

$$ \omega(\mathsf{t}) = \theta^{1-\delta}[\omega_0 + \mathrm{I}^{\delta}\mathcal{K}(\omega(\mathsf{t}))], \qquad 0 \leq \mathsf{t} \leq T < \infty. $$

Set $\mathsf{h} = \frac{T-0}{N}$ and $\mathsf{t_n} = \mathsf{nh}$, where $\mathsf{t} \in [0, T]$ and N is a natural number and $\mathsf{n} = 0, 1, 2, ..., N$.
Let ω_{t} be the approximation of $\omega(\mathsf{t})$ at $\mathsf{t} = \mathsf{t_n}$. Using the fractional Euler approach in [1], we get

$$\omega_{\mathsf{n}+1} = \theta^{1-\delta}[\omega_0 + \frac{\mathsf{h}^{\delta}}{\Gamma(\delta+1)} \sum_{i=0}^{\mathsf{n}} \mathsf{u}_{\mathsf{n}+1,i} \mathcal{K}(\mathsf{t}_i, \omega_i)], \qquad i = 0, 1, 2, ..., N-1,$$

where

$$\mathsf{u}_{\mathsf{n}+1,i} = (\mathsf{n}+1-i)^{\delta} - (\mathsf{n}-i)^{\delta}, \qquad i = 0, 1, 2, ..., \mathsf{n}.$$

Using MATLAB, model (3.1) will be simulated for Iran's deta. The parameters values of model (3.1) are available in Table 1.

Table 1: Details of the model parameters and their numerical value.

Parameter explanation	Parameter	Value	Ref
Susceptible individuals	S(0)	83969998	Estimated
Exposed individuals	E(0)	300000	Estimated
Infected individuals	I(0)	260002	[2]
Quarantined individuals	Q(0)	30000	Estimated
Recovered individuals	R(0)	0	Assumed
Birth rate of population	Π	4041.968	Estimated
Natural death rate	ζ	1.3257×10^{-5}	Estimated
Infection transmission rate	ϑ	5.14×10^{-9}	Fitted
Eefficacy of vaccine	ε	0.92	Assumed
Proportion of individuals who inject vaccines	ϖ	0.4	Assumed
Transmission rate from E class to I class	$\hbar$	0.411	Fitted
Recovery rate in I class	$\grave{\imath}$	0.3	Fitted
Quarantined rate in I class	ϱ	0.06	Fitted
Disease mortality rate in I class	ϕ	0.015	Fitted
Recovery rate in Q class	$\grave{\jmath}$	0.2	Fitted
Disease mortality rate in Q class	$\varkappa$	0.015	Fitted

In the following, we have graphs of the active cases of Covid-19 for fifth peak in Iran with impact of vaccine and without impact of vaccine:

In the paper, what happened in reality was first simulated, on July 10, 2021 the fifth Covid-19 peak began in Iran. In this peak, the most infected cases and the most deaths were recorded, and $\mathcal{R}_0 = 4.17$. At this time, the population of Iran had not been vaccinated yet. It has been shown that if 40% of people had been vaccinated on July 10 then the fifth Covid-19 peak would not have happened in Iran, and the number of infected cases would have decreased significantly and in this case, $\mathcal{R}_0 = 2.64$. Also, according to the calculations, we came to the conclusion that if 85% of people had been vaccinated on July 10, then $\mathcal{R}_0$ would have been equal to 0.91. Hence, the disease would have disappeared.

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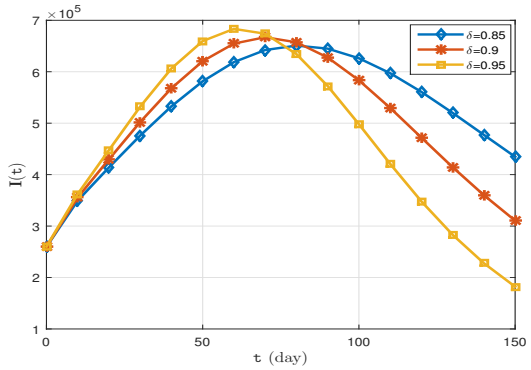


Figure 1: Dynamics of I(t) from July 10 to December 06, 2021 without vaccination and with different fractional order $\delta = 0.85, 0.90, 0.95$.

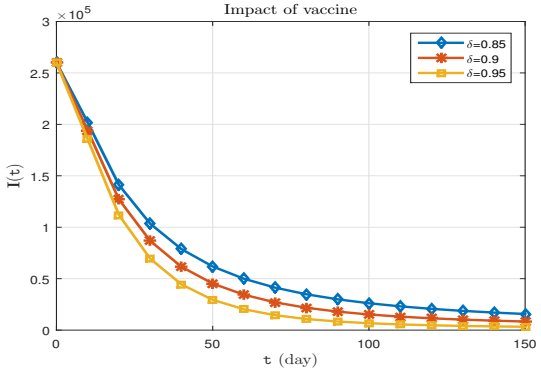


Figure 2: Dynamics of I(t) from July 10 to December 06, 2021 with vaccination and with different fractional order $\delta = 0.85, 0.90, 0.95$.

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Some Laws of Large Numbers for Dependent Banach-Valued Random Elements

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Abstract

In this paper, we first extend certain laws of large numbers for asymptotically pairwise negatively dependent (APND) random variables. We then attempt to establish analogous results for coordinatewise asymptotically pairwise negatively dependent (CAPND) random elements that take values in a compact subset of a separable Banach space.
Keywords: Laws of Large Numbers, Banach-Valued Random Elements

1 Introduction

The limiting behavior of the partial weighted sums of a sequence or an array of independent Separable Banach space valued random elements is a very attractive and important subject in theory of probability and have been discussed by many authors, among them [5],[10], [1], [2], [3].
Although independence assumption is usually made for mathematical convenience, it does not reflect reality in a great number of situations. Different forms of non-independence have been of interest for many years. [4] investigated various concepts of positive and negative dependence in the bivariate case. Strong concepts of bivariate positive and negative dependence were introduced by [15]. The study of the extension to some types of negative dependences have been done by many authors (e.g., [16], [17], [18]). [19], [20] and [21] investigated some various concepts of negative dependence for Banach-valued random elements.
Although the literature on concepts of negative dependence is still very interesting, it is more limited and the positive dependence concept is more practicable. [3] and [15] introduced a new class of dependent random variables (asymptotically pairwise negatively dependent (APND in brief)) that was larger than the negatively dependent random variables class.[10] Extended this class of dependencies to Banach valued random elements. In this paper, we first assume that the random elements take values in a compact set of separable Banach space and prove some laws of large numbers for CAPND random elements, and show that every one of them, are true for WAPND Banach-valued random elements. At the end, we replace the condition that "the random elements take values in a compact set of separable Banach space" with assuming that the random elements are tight and prove some results for them.

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2 Preliminaries

Throughout this paper, $(B, \|.\|)$ and (Ω, F, P) denote, respectively, the separable Banach space and probability space and B^* is the dual of B (set of every linear functional in B). Banach-valued random element V is a measurable function from the probability space, to separable Banach space. Let $\{X_n, n \geq 1\}$ be a sequence of integrable random variables defined on the same probability space.

Definition 1. *[3] The random variables X_i and X_j are said to be asymptotically negatively dependent (AND) if the following inequalities hold,*

$$P(X_i > x_i, X_j > x_j) \leq (1 + q(|i-j|))\, P(X_i > x_i)P(X_j > x_j), \tag{2.1}$$

and

$$P(X_i \leq x_i, X_j \leq x_j) \leq (1 + q(|i-j|))\, P(X_i \leq x_i)P(X_j \leq x_j) \tag{2.2}$$

for all $x_i, x_j \in R$, $i \neq j$, where $q(n) = o(n^{-w})$, $w > 0$.

The sequence $\{X_n, n \geq 1\}$ of random variables is said to be *asymptotically pairwise negatively dependent* (APND) if (X_i, X_j) is AND for every $i \neq j$, $i, j \geq 1$.

Lemma 2.1. *[15] Let $\{X_n, n \geq 1\}$ be a sequence of APND random variables and $\{g_n, n \geq 1\}$ be a sequence of concordinate functions, then*

1- $\{g_n(X_n), n \geq 1\}$ is a sequence of APND random variables with the initial coefficients $q(n)$.

2- The set of inequalities 1.1 and 1.4 are equivalent to those obtained by replacing one or more than one of inequalities $X_i \leq x_i$, $X_j \leq x_j$, $X_i > x_i$ or $X_j > x_j$ by the corresponding inequalities $X_i < x_i$, $X_j < x_j$, $X_i \geq x_i$ or $X_j \geq x_j$.

3- $P(X_i \geq x_i, X_j < x_j) \geq P(X_i \geq x_i)\left[1 - (1 + q(|i-j|))\, P(X_j \geq x_j)\right]$.

4- $P(X_i \leq x_i, X_j > x_j) \geq P(X_i \leq x_i)\left[1 - (1 + q(|i-j|))\, P(X_j \leq x_j)\right]$.

Definition 2. *Let $a > 0$. A positive measurable function u on $[a, +\infty)$, varies regularly at infinity with exponent ρ, $-\infty < \rho < \infty$, denote by $u \in RV(\rho)$, if and only if, for all $x > 0$,*

$$\frac{u(tx)}{u(t)} \overset{t\to\infty}{\to} x^{\rho}.$$

If $\rho = 0$, the function is slowly varying at infinity; $u \in SV$.
Typical examples of regularly varying functions are x^{ρ}, $x^{\rho}\log x$, $x^{\rho}\log\log x$, $x^{\rho}\frac{\log x}{\log\log x}$ and so on.

Lemma 2.2. *[10] Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers and $\{X_n, n \geq 1\}$ a sequence of APND random variables such that there exists a real number $c > 0$ such that, for all $k \geq 1$, $q(k) \leq c$. If $\frac{S_n}{a_n} \overset{a.s.}{\to} 0$ and $\sup\limits_n \frac{a_{n-1}}{a_n} \leq M < \infty$ for some $M \in R$, then $\sum\limits_{n=1}^{\infty} P(|X_n| > a_n) < \infty$.*

Lemma 2.3. *[10] Let $\{X_n, n \geq 1\}$ be a sequence of APND random variables with the same distribution function $F(x)$, such that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} q(k) < \infty$. $\frac{S_n}{n} \overset{a.s.}{\to} a$, for some constant $a \in R$, if and only if $E\,|X_1| < \infty$. If $E\,|X_1| < \infty$, then $a = EX_1$.*

Lemma 2.4. *[16] Let $\{X_n, n \geq 1\}$ be a sequence of APND random variables with parameter $\{q(n), n \geq 1\}$ which are stochastically dominated by random variable X. Let for $x > 0$, $n \geq 1$ and some $\rho \in (0, 1]$, $b(x) = x^{\frac{1}{\rho}} l(x)$ where $l \in SV$, $b_n = b(n)$ and $m_n = EX_i I\{|X_i| \leq b_n\}$. If $\sum_{k=1}^{n} (n-k)(1+q(k)) = O(n^2)$, and $nP(|X| > b_n) \to 0$, $b_n = b(n)$ then*

I- $\frac{1}{b_n} \sum_{i=1}^{n} (X_i - m_{n_i}) \xrightarrow{P} 0,$

II- $\max_{1 \leq k \leq n} \left| \frac{1}{b_n \log_2 n} \sum_{i=1}^{k} (X_i - m_{n_i}) \right| \xrightarrow{P} 0.$

Definition 3. *Let X be a linear topological space. A sequence $\{\beta_n\} \subset X$ is a Schauder basis for X if for each $x \in X$, there exists a unique sequence of scalars $\{t_n\}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} t_k \beta_k$.*

When a linear topological space X has a basis $\{\beta_n\}$, a sequence of linear functionals $\{f_k, k \geq 1\}$ can be defined by letting $f_k(x) = t_k$, $k \geq 1$, where $x \in X$ and $x = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)\beta_k$. The linear functionals $\{f_k, k \geq 1\}$ are called the *coordinate functionals* (for the basis $\{\beta_n\}$). The coordinate functionals depend on the basis and need not be continuous (see Example 1.3.1 of [18]). However, (by Lemma 1.3.1 of [18]), the coordinate functionals of a Banach space are continuous.

Definition 4. *for each $x \in X$ a sequence of linear functions $\{U_n\}$ on X can be defined by letting $U_n(x) = \sum_{k=1}^{n} f_k(x)\beta_k$. The sequence of functions $\{U_n\}$ is called the sequence of partial sum operators (for the basis $\{\beta_n\}$).*

Lemma 2.5. *[22] If X is a Banach space which has a Schauder basis, then there exists a positive constant M such that $\|U_n\| \leq M$ for each n, that is, $\|U_n(x)\| \leq M \|x\|$ for each $x \in X$ and each n. The constant M is referred to as the basis constant.*

Now, if we set $Q_n(x) = x - U_n(x)$, for each $x \in X$ and each n, it is clear that $\|Q_n\| \leq M + 1$.

Lemma 2.6. *[20] Let k be a compact set in a Banach space which has a Schauder basis. For each $\eta > 0$, there exists a positive integer N such that $\|Q_n(x)\| = \|x - U_n(x)\| \leq \eta$ for all $x \in k$ and $n \geq N$.*

Definition 5. *Let B be a Banach space which has a Schauder basis whit coordinate functionals $\{f_j, j \geq 1\}$. A sequence of Banach-valued random elements, $\{V_n, n \geq 1\}$, is said to be coordinatewise asymptotically pairwise negatively dependent (CAPND) with parameter $\{q(n), n \geq 1\}$ if the sequence of random variables $\{f_j(V_n), n \geq 1\}$ is APND with parameter $\{q(n), n \geq 1\}$ for each coordinate functional f_j, $j \geq 0$.*

Definition 6. *A sequence of Banach-valued random elements $\{V_n, n \geq 1\}$ is said to be weakly asymptotically pairwise negatively dependent (WAPND) if the sequence of random variables $\{f(V_n), n \geq 1\}$ is APND for each $f \in B^*$.*

Definition 7. *A sequence of Banach valued random elements $\{V_n, n \geq 1\}$ is said to be stochastically dominated by random variable X if for each $i \geq 1$ and each $x \geq 0$ we have $P(\|V_i\| \geq x) \leq P(|X| \geq x)$.*

Lemma 2.7. *Let $\{V_n, n \geq 1\}$ be a sequence of WAPND (CAPND) random elements with parameter $\{q(n), n \geq 1\}$ such that they are stochastically dominated by random variable X. Then for each $f \in B^*$ (each coordinatewise functional f), $\{f(V_n), n \geq 1\}$ is a sequence of APND random variables with parameter $\{q(n), n \geq 1\}$ which are stochastically dominated by random variable X.*

Proof. Let $f \in B^*$ (coordinatewise functional f). By definition of WAPND (CAPND) random element we know that $\{f(V_n), n \geq 1\}$ is a sequence of APND random variables with parameter $\{q(n), n \geq 1\}$. Also for each $x \geq 0$ and each $1 \leq i \leq n$, Since $|f(V_i)| \leq \|V_i\|$, $P(|f(V_i)| > x) \leq P(\|V_i\| > x) \leq P(|X| > x)$,and the proof is completed. □

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Definition 8. *A random element V in a topological space X is said to be tight if for each $\varepsilon > 0$, there exists a compact subset K_ε of X such that $P(V \in K_\varepsilon) > 1 - \varepsilon$. A sequence $\{V_n, n \geq 1\}$ of random elements is (uniformly) tight if for each $\varepsilon > 0$, there exists a compact subset K_ε of X such that $P(V_n \in K_\varepsilon) > 1 - \varepsilon$ for all n.*

If X is a complete, separable metric space, then each random element in X is tight [[1], page 10]. Hence, any sequence of identically distributed random elements is tight in a separable, complete metric space.

Definition 9. *Let X and Y be linear topological spaces. A one-to-one bicontinuous linear function from X onto Y is called an isomorphism. The two spaces X and Y are said to be isomorphic if there is an isomorphism from each space onto the other.*

Lemma 2.8. *[13] A separable Banach space X is isometrically isomorphic to a closed subspace of the Banach space $C[0, 1]$.*

3 Main Results

In the remainder of this paper, B is a separable banachspace, B^* is its dual and $\{V_n; n \geq 1\}$ is a sequence of Banach valued random elements. we may show some laws of large numbers for CAPND and WAPND Banach valued random elements.

In this subsection, we assume that B has a Schauder basis with coordinate functionals $\{f_k, k \geq 1\}$, $\{V_n; n \geq 1\}$ is a sequence of CAPND random elements in B, k is a compact set such that $V_n \in K$ $(a.s.)$ for all $n \geq 1$, and $S_n = \sum_{k=1}^{n} V_k$.

Theorem 3.1. *Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers and $\{V_n, n \geq 1\}$ be a sequence of CAPND random elements with parameter $\{q(n), n \geq 1\}$, such that there exists a real value $c < \infty$ that $q(n) \leq c$ for all $n \geq 1$. If $\frac{S_n}{a_n} \xrightarrow{a.s.} 0$ and $\sup_n \frac{a_{n-1}}{a_n} \leq M < \infty$ for some $M \in R$, then $\sum_{n=1}^{\infty} P(\|V_n\| > a_n) < \infty$.*

Proof. Due to the existence of the Schauder basis, for each $i, t \geq 1$, we have $V_i = U_t(V_i) + Q_t(V_i)$, where $U_t(V_i) = \sum_{k=1}^{t} f_k(V_i)\beta_k$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} P(\|V_n\| > a_n) &= \sum_{n=1}^{\infty} P(\|U_t(V_n) + Q_t(V_n)\| > a_n) \\ &\leq \sum_{n=1}^{\infty} P(\left\|\sum_{k=1}^{t} f_k(V_n)\beta_k\right\| > \tfrac{a_n}{2}) + \sum_{n=1}^{\infty} P(\|Q_t(V_n)\| > \tfrac{a_n}{2}) \\ &\leq \sum_{k=1}^{t}\sum_{n=1}^{\infty} P(|f_k(V_n)| > \tfrac{a_n}{2\|\beta_k\|}) + \sum_{n=1}^{\infty} \tfrac{2E\|Q_t(V_n)\|}{a_n}.\end{aligned}$$

By Lemma2.6, since $V_n \in K\, a.s.$ and k is a compact set, we can choice t such that $\sum_{n=1}^{\infty} \frac{2E\|Q_t(V_n)\|}{a_n} < \infty$. Since $t < \infty$, it is enough to show that for each $k = 1, ..., t$, $\sum_{n=1}^{\infty} P(|f_k(V_n)| > \frac{a_n}{2\|\beta_k\|}) < \infty$. We know that for

each $k = 1, 2, ..., t$, $\{f_k(V_n), n \geq 1\}$ is a sequence of APND random variables such that, for all $n \geq 1$, there exists a real number $c > 0$ that $q(n) \leq c$. Hence by Lemma2.2, it is enough to show that $\frac{\sum_{i=1}^{n} f_k(V_i)}{a_n/2\|\beta_k\|} \xrightarrow{a.s.} 0$. Since $\frac{\left\|\sum_{i=1}^{n} V_i\right\|}{a_n} \xrightarrow{a.s.} 0$, we have

$$0 \leq \left| \frac{\sum_{i=1}^{n} f_k(V_k)}{\frac{a_n}{2\|\beta_k\|}} \right| = 2\|\beta_k\| \left| \frac{f_k\left(\sum_{i=1}^{n} V_i\right)}{a_n} \right| \leq \frac{2\|\beta_k\| \|f_k\| \left\|\sum_{i=1}^{n} V_i\right\|}{a_n} \xrightarrow{a.s.} 0,$$

and the proof is completed. □

Remark 3.2. In Theorem3.1, if $a_n = n$, assumption $\frac{S_n}{a_n} \xrightarrow{a.s} 0$ can be replaced by $\frac{S_n}{n} \xrightarrow{a.s} a$, for some element $a \in B$.

Theorem 3.3. *Let $\{V_n, n \geq 1\}$ be a sequence of CAPND identically distributed random elements in B with parameter $\{q(n)\}$ such that $\lim_{n\to\infty} \sum_{k=1}^{n} q(k) < \infty$. $\frac{S_n}{n} \xrightarrow{a.s.} a$, for some $a \in B$ if and only if $E\|V_1\| < \infty$.*

Proof. **(Necessity)** We know that if $\lim_{n\to\infty} \sum_{k=1}^{n} q(k) < \infty$ then there exist a real value $c < \infty$ such that, $q(n) \leq c$, for all $n \geq 1$. Hence if $\frac{S_n}{n} \xrightarrow{a.s.} a$, by Remark3.2, we have $E(\|V_1\|) = \sum_{n=1}^{\infty} P(\|V_1\| > n) = \sum_{n=1}^{\infty} P(\|V_n\| > n) < \infty$, and this complete the proof.
(Sufficiency) Now we assume that $E\|V_1\| < \infty$. Due to the existence of the Schauder basis, for each $i \geq 1$ and each $t \geq 1$, we have $V_i = U_t(V_i) + Q_t(V_i)$, where $U_t(V_i) = \sum_{k=1}^{t} f_k(V_i)\beta_k$. Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} P(\left\|\tfrac{S_n}{n}\right\| > \varepsilon) &= \sum_{n=1}^{\infty} P(\left\|\frac{\sum_{i=1}^{n} V_i}{n}\right\| > \varepsilon) = \sum_{n=1}^{\infty} P(\left\|\frac{\sum_{i=1}^{n} (U_t(V_i)+Q_t(V_i))}{n}\right\| > \varepsilon) \\
&\leq \sum_{k=1}^{t} \sum_{n=1}^{\infty} P(\frac{\left|\sum_{i=1}^{n} f_k(V_i)\right|}{n} > \tfrac{\varepsilon}{2\|\beta_k\|}) + \sum_{n=1}^{\infty} P(\frac{\left\|\sum_{i=1}^{n} Q_t(V_i)\right\|}{n} > \tfrac{\varepsilon}{2}) \\
&\leq \sum_{k=1}^{t} \sum_{n=1}^{\infty} P(\frac{\left|\sum_{i=1}^{n} f_k(V_i)\right|}{n} > \tfrac{\varepsilon}{2\|\beta_k\|}) + \sum_{n=1}^{\infty} \frac{2\sum_{i=1}^{n} E\|Q_t(V_i)\|}{n\varepsilon}.
\end{aligned}$$

By Lemma2.6, since $V_n \in K$ *a.s.* and k is a compact set, we can choice t such that $\sum_{n=1}^{\infty} \frac{2\sum_{i=1}^{n} E\|Q_t(V_i)\|}{n\varepsilon} < \infty$. Since $t < \infty$, it is enough to show that for each $k = 1, ..., t$, $\sum_{n=1}^{\infty} P(\frac{\left|\sum_{i=1}^{n} f_k(V_i)\right|}{n} > \frac{\varepsilon}{2\|\beta_k\|}) < \infty$. We know that, for each $k = 1, 2, ..., t$, $\{f_k(V_n), n \geq 1\}$, is a sequence of APND random variables with parameters $\{q(n), n \geq 1\}$, such that $\lim_{n\to\infty} \sum_{k=1}^{n} q(k) < \infty$ and $E|f_k(V_1)| \leq E(\|f_k\| \|V_1\|) \leq E\|V_1\| < \infty$. Hence by Lemma2.3, for each $k = 1, 2, ..., t$, $\frac{\left|\sum_{i=1}^{n} f_k(V_i)\right|}{n} \xrightarrow{a.s.} a_k$ and by Remark3.2 and Lemma2.2, $\sum_{n=1}^{\infty} P(\frac{\left|\sum_{i=1}^{n} f_k(V_i)\right|}{n} > \frac{\varepsilon}{2\|\beta_k\|}) < \infty$, and proof is completed. □

Theorem 3.4. *Let $\{V_n, n \geq 1\}$ be a sequence of CAPND random elements with parameter $\{q(n), n \geq 1\}$ such that they are stochastically dominated by random variable X. Let b_n as in Lemma2.4 and $m_{n_i} = EV_i I\{\|V_i\| \leq b_n\}$, for all $n \geq 1$. If $\sum_{k=1}^{n}(n-k)(1+q(k)) = O(n^2)$ and $nP(|X| > b_n) \to 0$, then*

I- $\frac{1}{b_n}\sum_{i=1}^{n}(V_i - m_{n_i}) \xrightarrow{p} 0.$

II- *For each coordinate functional $\{f_j, j \geq 1\}$,*

$$\max_{1\leq k\leq n}\left|\frac{1}{b_n \log_2 n}\sum_{i=1}^{k} f(V_i - m_{n_i})\right| \xrightarrow{P} 0.$$

III- $\max_{1\leq k\leq n}\left\|\frac{1}{b_n \log_2 n}\sum_{i=1}^{k}(V_i - m_{n_i})\right\| \xrightarrow{P} 0.$

Proof. We know that, for each $n \geq 1$ and each $i = 1, 2, ..., n$, $m_{n_i} \in K$, $V_i - m_{n_i} \in 2K \subseteq B$ and $2K$ is compact. Due to the existence of the Schauder basis, for each $n \geq 1$ and each $t \geq 1$, we have $V_i - m_{n_i} = U_t(V_i - m_{n_i}) + Q_t(V_i - m_{n_i})$, where $U_t(V_i - m_{n_i}) = \sum_{k=1}^{t} f_k(V_i - m_{n_i})\beta_k$.
Since $2K$ is compact, by Lemma2.6 for each $\varepsilon > 0$, we can choice $t \geq 1$ such that $\|Q_t(V_i - m_{n_i})\| \leq \frac{\varepsilon}{2}$, for each $V_i - m_{n_i} \in 2k$. We know that for each $1 \leq i \leq n$ and each $j = 1, 2, ..., t$, $f_j(EV_i) = Ef_j(V_i)$, hence $f_j(m_{n_i}) = f_j(EV_i I\{\|V_i\| \leq b_n\}) = Ef_j(V_i)I\{\|V_i\| \leq b_n\}) \geq Ef_j(V_i)I\{f_j(V_i) \leq b_n\})$. Now we have

$$\begin{aligned}
P\left(\left\|\tfrac{1}{b_n}\sum_{i=1}^{n}(V_i - m_{n_i})\right\| > \varepsilon\right) &= P\left(\left\|\tfrac{1}{b_n}\sum_{i=1}^{n}\left[U_t(V_i - m_{n_i}) + Q_t(V_i - m_{n_i})\right]\right\| > \varepsilon\right) \\
&\leq \sum_{j=1}^{t} P\left(\tfrac{1}{b_n}\left|\sum_{i=1}^{n}(f_j(V_i) - f_j(m_{n_i}))\right| > \tfrac{\varepsilon}{2t\|\beta_j\|}\right) \\
&+P\left(\tfrac{1}{b_n}\sum_{i=1}^{n}\|Q_t(V_i - m_{n_i})\| > \tfrac{\varepsilon}{2}\right) \\
&\leq \sum_{j=1}^{t} P\left(\tfrac{1}{b_n}\left|\sum_{i=1}^{n}(f_j(V_i) - Ef_j(V_i)I\{f_j(V_i) \leq b_n\})\right| > \tfrac{\varepsilon}{2t\|\beta_j\|}\right) \\
&+\frac{2\sum_{i=1}^{n} E\|Q_t(V_i - m_{n_i})\|}{\varepsilon b_n}
\end{aligned}$$

By lemma2.7, for each $j = 1, 2, ..., t$, $\{f_j(V_n), n \geq 1\}$ is a sequence of APND random variables with parameter $\{q(n), n \geq 1\}$ which are stochastically dominated by random variable X, such that $\sum_{k=1}^{n}(n-k)(1+q(k)) = O(n^2)$. Hence by part(I) Lemma2.4

$$P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n}(f_j(V_i) - Ef_j(V_i)I\{f_j(V_i) \leq b_n\})\right| > \frac{\varepsilon}{2t\|\beta_j\|}\right) \to 0,$$

and

$$\sum_{j=1}^{t} P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n}(f_j(V_i) - Ef_j(V_i)I\{f_j(V_i) \leq b_n\})\right| > \frac{\varepsilon}{2t\|\beta_j\|}\right) \to 0.$$

Also $\frac{2\sum_{i=1}^{n} E\|Q_t(V_i - m_{n_i})\|}{\varepsilon b_n} \leq \frac{n}{b_n} \to 0$. Hence proof of (I) is completed.
By Lemma2.7, for each coordinate functional f, $\{f(V_n), n \geq 1\}$ is a sequence of APND random variables

with parameter $\{q(n), n \geq 1\}$ which are stochastically dominated by random variable X, such that $\sum_{k=1}^{n}(n-k)(1+q(k)) = O(n^2)$, and Hence by part(II) of Lemma2.4, proof of (II) is completed.
To prove (III), we have

$$\begin{aligned}
&P\left(\max_{1\leq k\leq n}\left\|\tfrac{1}{b_n\log_2 n}\sum_{i=1}^{k}(V_i-m_{n_i})\right\|>\varepsilon\right)\\
&\qquad=P\left(\left\|\tfrac{1}{b_n\log_2 n}\sum_{i=1}^{n}\left[U_t\left(V_i-m_{n_i}\right)+Q_t\left(V_i-m_{n_i}\right)\right]\right\|>\varepsilon\right)\\
&\qquad\leq\sum_{j=1}^{t}P\left(\max_{1\leq k\leq n}\left|\tfrac{1}{b_n\log_2 n}\sum_{i=1}^{k}f_j(V_i-m_{n_i})\right|>\tfrac{\varepsilon}{2t\|\beta_j\|}\right)\\
&\qquad+P\left(\tfrac{1}{b_n\log_2 n}\sum_{i=1}^{n}\left\|Q_t\left(V_i-m_{n_i}\right)\right\|>\tfrac{\varepsilon}{2}\right)\\
&\qquad\leq\sum_{j=1}^{t}P\left(\max_{1\leq k\leq n}\left|\tfrac{1}{b_n\log_2 n}\sum_{i=1}^{k}f_j(V_i-m_{n_i})\right|>\tfrac{\varepsilon}{2t\|\beta_j\|}\right)\\
&\qquad+\tfrac{2\sum_{i=1}^{n}\left\|Q_t\left(V_i-m_{n_i}\right)\right\|}{\varepsilon b_n\log_2 n}
\end{aligned}$$

Since $t<\infty$, by part (II)

$$\sum_{j=1}^{t}P\left(\max_{1\leq k\leq n}\left|\frac{1}{b_n\log_2 n}\sum_{i=1}^{k}f_j(V_i-m_{n_i})\right|>\frac{\varepsilon}{2t\left\|\beta_j\right\|}\right)\to 0,$$

also $\frac{2\sum_{i=1}^{n}\left\|Q_t\left(V_i-m_{n_i}\right)\right\|}{\varepsilon b_n\log_2 n}\leq\frac{n}{b_n\log_2 n}\to 0$. Hence the proof is completed. □

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Iterative methods for solving a special case of multilinear systems

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Abstract

In this work, based on tensor splittings, we propose some iterative methods and associated theorems for solving the tensor equation $\mathcal{I}_m\mathbf{x}^{m-1} = \mathcal{B}\mathbf{x}^{m-1} + \mathbf{b}$. Also, the concept of the convergent upper bound of tensor splittings is introduced, which makes it possible to investigate the convergence behavior of splittings of tensors. As an application, some theorems about the convergence of the methods are proposed. These theorems contain several known theorems about the convergence of the iteration methods, especially for nonnegative and diagonally dominant real tensors.

Keywords: Multilinear system, Iterative method, Tensor splitting.

1 Introduction

Throughout this paper, scalars are denoted by lower and capital letters $a, A, b, B, \ldots$, vectors are written as bold lower letters $\mathbf{a}, \mathbf{b}, \ldots$, matrices correspond to bold capitals letters $\mathbf{A}, \mathbf{B}, \ldots$, and tensors are written as calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$. The notations $\mathbf{0}$, $\mathbf{O}$, and $\mathcal{O}$ denote null vector, null matrix, and null tensor, respectively. By $\mathbb{R}^{n_1\times\ldots\times n_m}$ we denote all m order real tensors consist of $n_1 \times \ldots \times n_m$ entries and by $\mathbb{R}^{[m,n]}$ we denote the set of all mth order n-dimensional real tensors. When $m = 1$, $\mathbb{R}^{[1,n]}$ is simplified as $\mathbb{R}^n$, which is the set of all n-dimension real vectors. For a tensor $\mathcal{A}$, let $\mathcal{A}$ denotes the tensor whose $(i_1, \ldots, i_m)$th entry is defined as $a_{i_1\ldots i_m}$. For $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$, $\mathcal{A} \leq \mathcal{B}$ means that $a_{i_1 i_2\ldots i_m} \leq b_{i_1 i_2\ldots i_m}$ for every $i_1, i_2, \ldots, i_m$.

Definition 1. *If each entry of $\mathcal{A}$ is nonnegative, then $\mathcal{A}$ is called a nonnegative tensor. The mth order n-dimensional identity tensor is denoted by ${}_m$, where its nonzero entries are ${}_m(i, i, \ldots, i) = 1$, $i = 1, 2, \ldots$. $\mathbf{I}$ is denoted for the identity matrix of size $n \times n$.*

Definition 2. *The majorization matrix of tensor $\mathcal{A}$, denoted by $M(\mathcal{A})$, is defined as a square matrix of size $n \times n$ with its entries*

$$(M(\mathcal{A}))_{ij} = a_{ij\ldots j}, \quad i, j = 1, 2, \ldots, n.$$

Definition 3. *If $M(\mathcal{A})$ is a nonsingular matrix and $\mathcal{A} = M(\mathcal{A})\mathcal{I}_m$, then $(M(\mathcal{A}))^{-1}$ is the order 2 left-inverse of $\mathcal{A}$, i.e. $(M(\mathcal{A}))^{-1}\mathcal{A} = \mathcal{I}_m$, and then we say that $\mathcal{A}$ is a left-invertible tensor or left-nonsingular tensor.*

Definition 4. *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is said a splitting of $\mathcal{A}$ if $\mathcal{E}$ is a left-nonsingular; a regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $(M(\mathcal{E}))^{-1} \geq \mathbf{O}$ and $\mathcal{F} \geq \mathcal{O}$; a weak regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $(M(\mathcal{E}))^{-1} \geq \mathbf{O}$ and $(M(\mathcal{E}))^{-1}\mathcal{F} \geq \mathcal{O}$; a convergent splitting if $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) < 1$.*

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Definition 5. *For a real tensor $\mathcal{A}$, if*

$$a_{ii\ldots i} \geq (>) \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}^{n} a_{ii_2\ldots i_m},$$

then $\mathcal{A}$ is called (strictly) diagonally dominant.

Definition 6. *$\mathcal{A}$ is called a reducible tensor if there exists a nonempty proper index subset $\mathbb{I} \subseteq \{1, 2, \ldots, n\}$ such that*

$$a_{i_1 i_2\ldots i_m} = 0, \ \forall\ i_1 \in \mathbb{I}, \ \forall\ i_2 \ldots i_m \notin \mathbb{I},$$

else, we say that $\mathcal{A}$ is irreducible.

Definition 7. *A nonnegative tensor $\mathcal{P}$ is called an upper bound of the tensor splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$, if*

$$(M(\mathcal{E}))^{-1}\mathcal{F} \leq \mathcal{P}.$$

Theorem 1.1. *If an upper bound of a tensor splitting is convergent then the tensor splitting is convergent.*

Proof. From $(M(\mathcal{E}))^{-1}\mathcal{F} \leq \mathcal{P}$, it follows immediately that $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) \leq \rho(\mathcal{P}) < 1$. □

Lemma 1.2. *Let $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ be a weak regular splitting and a regular splitting, respectively, and $\mathcal{F}_2 \leq \mathcal{F}_1, \mathcal{F}_2 \neq \mathcal{O}$. One of the following statements holds.*

1. $\rho((M(\mathcal{E}_2))^{-1}\mathcal{F}_2) \leq \rho((M(\mathcal{E}_1))^{-1}\mathcal{F}_1) < 1$;
2. $\rho((M(\mathcal{E}_2))^{-1}\mathcal{F}_2) \geq \rho((M(\mathcal{E}_1))^{-1}\mathcal{F}_1) \geq 1$.

If $\mathcal{F}_2 < \mathcal{F}_1, \mathcal{F}_2 \neq \mathcal{O}$ and $\rho((M(\mathcal{E}_1))^{-1}\mathcal{F}_1) > 1$, then the first inequality in part 2 is strict.

2 Main results

Consider the multilinear system

$$\mathcal{I}_m\mathbf{x}^{m-1} = \mathcal{B}\mathbf{x}^{m-1} + \mathbf{b}, \tag{2.1}$$

where $\mathcal{B} \in \mathbb{R}^{[m,n]}$, $\mathbf{b} \in \mathbb{R}^n$ and $\mathcal{I}_m - \mathcal{B}$ is the left-nonsingular tensor. We want to propose some iterative methods and associated theorems for solving this type of tensor equations. Consider the decompose of $\mathcal{B}$ into the sum $\mathcal{B} = \mathcal{D} + \mathcal{L} + \mathcal{U}$. Herein $\mathcal{D} = \mathbf{D}\mathcal{I}_m$ and $\mathcal{L} = \mathbf{L}\mathcal{I}_m$, where $\mathbf{D}$ and $\mathbf{L}$ are diagonal and strictly lower triangle parts of $M(\mathcal{B})$, respectively. Consider three following iteration methods for solving (3.3)

$$\mathbf{x}_{j+1} = [(\mathcal{D} + \mathcal{L} + \mathcal{U})\mathbf{x}_j^{m-1} + \mathbf{b}]^{[\frac{1}{m-1}]}, \qquad general - step; \tag{2.2}$$

$$\mathbf{x}_{j+1} = [(M(\mathcal{I}_m - \mathcal{L}))^{-1}(\mathcal{D} + \mathcal{U})\mathbf{x}_j^{m-1} + (M(\mathcal{I}_m - \mathcal{L}))^{-1}\mathbf{b}]^{[\frac{1}{m-1}]}, \qquad single - step; \tag{2.3}$$

$$\mathbf{x}_{j+1} = [(M(\mathcal{I}_m - \omega\mathcal{L}))^{-1}((1-\omega)\mathcal{I}_m + \omega(\mathcal{D} + \mathcal{U}))\mathbf{x}_j^{m-1} + \omega(M(\mathcal{I}_m - \mathcal{L}))^{-1}\mathbf{b}]^{[\frac{1}{m-1}]} \ relaxation\ method. \tag{2.4}$$

Consider the tensor splitting

$$\mathcal{A} = \mathcal{E}_\omega - \mathcal{F}_\omega,$$

with

$$\mathcal{E}_\omega = \frac{1}{\omega}(\mathcal{I}_m - \omega\mathcal{L}) \ and \ \mathcal{F}_\omega = \frac{1}{\omega}((1-\omega)\mathcal{I}_m + \omega(\mathcal{D} + \mathcal{U})), \ \omega > 0. \tag{2.5}$$

The corresponding iteration tensor is

$$\mathcal{L}_\omega = (M(\mathcal{E}_\omega))^{-1}\mathcal{F}_\omega = (M(\mathcal{I}_m - \omega\mathcal{L}))^{-1}((1-\omega)\mathcal{I}_m + \omega(\mathcal{D}+\mathcal{U})).$$

Let

$$\tilde{\mathcal{A}}_\omega = \alpha\mathcal{I}_m - \mathcal{B},\ with\ \alpha = \frac{1-1-\omega}{\omega},\ \omega > 0, \tag{2.6}$$

and consider the splitting

$$\tilde{\mathcal{A}} = \tilde{\mathcal{E}}_\omega - \tilde{\mathcal{F}}_\omega \tag{2.7}$$

with

$$\tilde{\mathcal{E}}_\omega = \frac{1}{\omega}(\mathcal{I}_m - \omega\mathcal{L})\ and\ \tilde{\mathcal{F}}_\omega = \frac{1}{\omega}((1-\omega)\mathcal{I}_m + \omega(\mathcal{D}+\mathcal{U})),\ \omega > 0. \tag{2.8}$$

The iteration tensor of (1.6) is

$$\tilde{\mathcal{L}}_\omega = (M(\tilde{\mathcal{E}}_\omega))^{-1}\tilde{\mathcal{F}}_\omega = (M(\mathcal{I}_m - \omega\mathcal{L}))^{-1}((1-\omega)\mathcal{I}_m + \omega(\mathcal{D}+\mathcal{U})).$$

Theorem 2.1. *Tensor splitting $\tilde{\mathcal{A}} = \tilde{\mathcal{E}}_\omega - \tilde{\mathcal{F}}_\omega$ is regular and its iteration tensor is the upper bound tensor of $\mathcal{L}_\omega$, that means*

$$\mathcal{L}_\omega \leq \tilde{\mathcal{L}}_\omega.$$

Proof. Because $\tilde{\mathcal{E}}_\omega = \frac{1}{\omega}(\mathcal{I}_m - \omega\mathcal{L})$, $\mathcal{L} \geq \mathcal{O}$ and $\rho(\mathcal{L}) = 0 < \frac{1}{\omega}$, we have $(M(\tilde{\mathcal{E}}_\omega))^{-1} = (M(\frac{1}{\omega}(\mathcal{I}_m - \omega\mathcal{L})))^{-1} \geq 0$. It is obvious that $\tilde{\mathcal{F}}_\omega \geq \mathcal{O}$. Thus $\tilde{\mathcal{A}} = \tilde{\mathcal{E}}_\omega - \tilde{\mathcal{F}}_\omega$ is regular splitting. Furthermore

$$\begin{aligned}
(M(\mathcal{E}_\omega))^{-1} &= (M(\mathcal{I}_m - \omega\mathcal{L}))^{-1} = (M(\mathcal{I}_m) - \omega M(\mathcal{L}))^{-1} \\
&= (\mathbf{I} - \omega M(\mathcal{L}))^{-1} = \mathbf{I} + \omega M(\mathcal{L}) + (\omega M(\mathcal{L}))^2 + \ldots + (\omega M(\mathcal{L}))^{n-1} + \ldots \\
&\leq \mathbf{I} + \omega M(\mathcal{L}) + \omega^2 M(\mathcal{L})^2 + \ldots + \omega^{n-1} M(\mathcal{L})^{n-1} + \ldots \\
&= (\mathbf{I} - \omega M(\mathcal{L}))^{-1} = (M(\mathcal{I}_m - \omega\mathcal{L}))^{-1} \\
&= (M(\tilde{\mathcal{E}}_\omega))^{-1}.
\end{aligned}$$

Thus for $\omega > 0$, $(M(\mathcal{E}_\omega))^{-1} \leq (M(\tilde{\mathcal{E}}_\omega))^{-1}$ and $\mathcal{F}_\omega \leq \tilde{\mathcal{F}}_\omega$. It follows immediately that

$$\mathcal{L}_\omega = (M(\mathcal{E}_\omega))^{-1}\mathcal{F}_\omega \leq (M(\tilde{\mathcal{E}}_\omega))^{-1}\tilde{\mathcal{F}}_\omega = \tilde{\mathcal{L}}_\omega.$$

□

Theorem 2.2. *If $\rho(\mathcal{B}) < 1$, then $\rho(\tilde{\mathcal{L}}_\omega) < 1$ for*

$$0 < \omega < \frac{2}{1+\rho(\mathcal{B})}.$$

Proof. For $0 < \omega \leq 1$ and $\alpha = 1$, we have $\rho(\mathcal{B}) < \alpha = 1$. For $\omega > 1$, $\alpha = \frac{2-\omega}{\omega}$ and therefore $\rho(\mathcal{B}) < \alpha = \frac{2-\omega}{\omega}$ for $1 < \omega < \frac{2}{1+\rho(\mathcal{B})}$. It is also true $\mathcal{B} \geq \mathcal{O}$ and $\rho(\mathcal{B}) < \alpha = \frac{1-1-\omega}{\omega}$ for $0 < \omega < \frac{2}{1+\rho(\mathcal{B})}$. Since the tensor splitting (1.6) is regular by Theorem 2.1, thus

$$\rho(\tilde{\mathcal{L}}_\omega) = \rho((M(\tilde{\mathcal{E}}_\omega))^{-1}\tilde{\mathcal{F}}_\omega) < 1\ for\ 0 < \omega < \frac{2}{1+\rho(\mathcal{B})}.$$

□

Theorem 2.3. *Let $\mathcal{B} \in \mathbb{R}^{[m,n]}$ and $\rho(\mathcal{B}) < 1$. The iteration method (1.4) is convergent for*

$$0 < \omega < \frac{2}{1+\rho(\mathcal{B})}.$$

Corollary 2.4. *For nonnegative tensor $\mathcal{B} \in \mathbb{R}^{[m,n]}$, the iteration method (1.4) is convergent for*

$$0 < \omega < \frac{2}{1+\rho(\mathcal{B})}.$$

Corollary 2.5. *Let $\mathcal{B} \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor. If the iteration method (2.2) is convergent, then (1.3) is convergent.*

Example 2.6. We consider an example of a convergent and nonnegative tensor that we can use when moving from the general step method to the single-step method or the relaxation method within the abovementioned convergence range. We can achieve significant improvements in convergence. Consider $\mathcal{B} = 0.03 \times tenones(5,5,5) \in \mathbb{R}^{[3,5]}$, where "tenones" is a MATLAB function. The spectral radii of nonnegative tensors are computed by the power method given in [1]. Thus $\rho(\mathcal{B}) = 0.7500 < 1$. For the single-step method ($\omega = 1$), one can obtains $\rho(\mathcal{L}_\omega) = 0.6496 < \rho(\mathcal{B})$. According to Corollary 2.3, the relaxation method converges for $0 < \omega < \frac{2}{1.6496} \approx 1.2124$. For $\omega = 0.7$ one gets $\rho(\mathcal{L}_{0.7}) = 0.2704$, so a significant one greater asymptotic convergence speed than in the case of the single-step method.

Let $\mathcal{S} \in \mathbb{R}^{[m,n]}$ be a strictly or irreducible diagonally dominant tensor and $\mathcal{D}$ is the diagonal tensor $\mathcal{D} = \mathcal{S}(i,\ldots,i)$. Set $\mathcal{B} = \mathcal{I}_m - (M(\mathcal{D}))^{-1}\mathcal{S}$. If $\mathcal{S}$ is strictly diagonally dominant, then $\sum_{i_2,\ldots,i_m \neq i} b_{ii_2\ldots i_m} < 1$. If $\mathcal{S}$ is diagonally dominant, then $\sum_{i_2,\ldots,i_m \neq i} b_{ii_2\ldots i_m} \leq 1$, where for at least one i the inequality is strict. In both cases $\rho(\mathcal{B}) < 1$. It is therefore true:

Corollary 2.7. *Let $\mathcal{S} \in \mathbb{R}^{[m,n]}$ be a strictly or irreducible diagonally dominant tensor. Then if the Jacobi method converges for iteration tensor $\mathcal{B} = \mathcal{I}_m - (M(\mathcal{D}))^{-1}\mathcal{S}$, then the relaxation method (1.4) is convergent for*

$$0 < \omega < \frac{2}{1+\rho(\mathcal{B})}.$$

Corollary 2.8. *In the case of a strictly or irreducible diagonally dominant tensor, the Gauss-Seidel method is convergent.*

Theorem 2.9. *Let $\mathcal{B} = \mathcal{D} + \mathcal{L} + \mathcal{U}$ and $\rho(\mathcal{B}) < 1$. Then the single-step method (1.3) is convergent and $\rho(\mathcal{L}_1) = \rho((M(\mathcal{I}_m - \mathcal{L}))^{-1}(\mathcal{D}+\mathcal{U})) \leq \rho(\mathcal{B}) < 1$. If $\mathcal{B}$ is irreducible, then the single-step method converges asymptotically better than the general-step method.*

Proof. We have $\rho(\mathcal{B}) \leq \rho(\mathcal{B}) < 1$. Using Theorem 2.1 and Theorem 2.2 for $\omega = 1$, we have

$$\rho(\mathcal{L}_1) = \rho((M(\mathcal{I}_m - \mathcal{L}))^{-1}(\mathcal{D}+\mathcal{U})) \leq \rho(\tilde{\mathcal{L}}_1) = \rho((M(\mathcal{I}_m - \mathcal{L}))^{-1}(\mathcal{D}+\mathcal{U})) < 1. \tag{2.9}$$

For $\tilde{\mathcal{A}}_1 = \mathcal{I}_m - \mathcal{B}$, we consider two following splittings

$$\tilde{\mathcal{A}}_1 = \tilde{\mathcal{E}}_1 - \tilde{\mathcal{F}}_1 \quad with \quad \tilde{\mathcal{E}}_1 = \mathcal{I}_m - \mathcal{L} \quad and \quad \tilde{\mathcal{F}}_1 = \mathcal{D}+\mathcal{U},$$

$$\tilde{\mathcal{A}}_1 = \mathcal{E} - \mathcal{F} \quad with \quad \mathcal{E} = \mathcal{I}_m \quad and \quad \mathcal{F} = \mathcal{B}.$$

These splittings are regular and $\mathcal{F} \geq \tilde{\mathcal{F}}_1$. It follows from Lemma 1.2 and Eq. (2.9) that

$$\rho(\mathcal{L}_1) \leq \rho(\tilde{\mathcal{L}}_1) = \rho((M(\tilde{\mathcal{E}}_1))^{-1}\tilde{\mathcal{F}}_1) \leq \rho((M(\mathcal{E}))^{-1}\mathcal{F}) = \rho(\mathcal{B}) < 1.$$

□

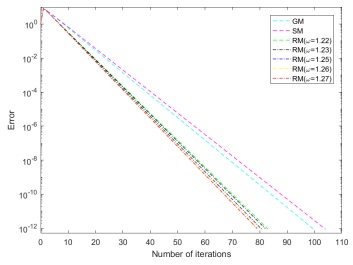


Figure 1: Numerical curves.

Corollary 2.10. *If the general step method (2.2) converges for a nonnegative tensor $\mathcal{B}$, the corresponding single step method (1.3) converges asymptotically at least as well. If $\mathcal{B}$ is irreducible, the single-step method converges asymptotically better than the general-step method.*

Theorem 2.11. *Let $\mathcal{B} = \mathcal{D}+\mathcal{L}+\mathcal{U}$ be a tensor with $\rho(\mathcal{B}) < 1$ and $\mathcal{L}_\omega = (M(\mathcal{I}_m - \omega\mathcal{L}))^{-1}((1-\omega)\mathcal{I}_m + \mathcal{D}+\mathcal{U})$. Then $\rho(\mathcal{L}_\omega) \leq 1 - \omega + \omega\rho(\mathcal{B}) < 1$ for $0 < \omega < \dfrac{2}{1+\rho(\mathcal{B})}$. The strict inequality holds if $\mathcal{B}$ is irreducible.*

Proof. By Theorem 2.1 if $\tilde{\mathcal{L}}_\omega$ is upper bound of $\mathcal{L}_\omega$, then $\rho(\mathcal{L}_\omega) \leq \rho(\tilde{\mathcal{L}}_\omega)$. Herein $\tilde{\mathcal{L}}_\omega$ is the iteration tensor of the single-step method applying for positive tensor $1 - \omega\mathcal{I}_m + \omega\mathcal{B}$. According to Corollary 2.7, we have

$$\rho(\tilde{\mathcal{L}}_\omega) \leq \rho(1 - \omega\mathcal{I}_m + \omega\mathcal{B}) = 1 - \omega + \omega\rho(\mathcal{B}) < 1,$$

for $0 < \omega < \dfrac{2}{1+\rho(\mathcal{B})}$. Herein the strict inequality holds if $\mathcal{B}$ is irreducible. □

Corollary 2.12. *Let $\mathcal{B} = \mathcal{D}+\mathcal{L}+\mathcal{U}$ be a nonnegative tensor with $\rho(\mathcal{B}) < 1$. Then $\rho(\mathcal{L}_\omega) \leq 1 - \omega + \omega\rho(\mathcal{B}) < 1$ for $0 < \omega < \dfrac{2}{1+\rho(\mathcal{B})}$. The inequality is strict here if $\mathcal{B}$ is irreducible.*

Example 2.13. Suppose that $\mathcal{A} \in \mathbb{R}^{[3,10]}$ be a nonnegative tensor with elements $a_{i_1 i_2 i_3} = \tan(i_1 + i_2 + i_3)$. The spectral radius of $\mathcal{A}$ is $\rho(\mathcal{A}) \approx 1450$, which is computed by the power method given in [1]. Consider the multilinear system

$$\mathcal{I}_m\mathbf{x}^2 = \mathcal{B}\mathbf{x}^2 + \mathbf{b},$$

where $\mathcal{B} = \dfrac{1}{2000}\mathcal{A}$. Using the mentioned methods, we want to solve this multilinear system by choosing $\mathbf{b} = (1, 2, ..., 10)^\mathsf{T}$, initial vector as $\mathbf{x}_0 = (1, 1, \ldots, 1)^\mathsf{T}$ and the stopping criterion is

$$Error = \mathcal{I}_m\mathbf{x}^{m-1} - \mathcal{B}\mathbf{x}^{m-1} - \mathbf{b}_2 < 10^{-12}.$$

The numerical curves are depicted in Figure 1, where GM, SM, and RM are denoted for the general-step, single-step, and relaxation methods, respectively. Figure 1 illustrates that the proposed methods are efficient for solving this type of tensor equations.

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Taylor-Telephone spectral method for solving KdV-mKdV equation

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Abstract

A hybrid approximate approach for solving the nonlinear KdV-mKdV equation is proposed. While the time discretization is done via Taylor series procedure, the resulting family of subequations is solved by spectral (novel) Telephone collocation technique. A numerical test cas is provided to show the applicability of the method.

Keywords: Telephone polynomials, Collocation points, Taylor series method.

1 Introduction

The main objective of this work is to apply a combined Taylor series method in time and spectral collocation approach based on novel set of polynomial functions called Telephone polynomials [1] to solve the nonlinear KdV-mKdV equation given in the form [2]:

$$\mu_t(x,t) + q\,\mu(x,t)\,\mu_x(x,t) + p\,\mu^2(x,t)\,\mu_x(x,t) + r\,\mu_{xxx}(x,t) = 0, \quad (x,t) \in \Omega, \tag{1.1}$$

where $\Omega =: \Omega_{LR} \times \Omega_T := (x_L, x_R) \times (0,T)$ supplemented with the following initial and boundary conditions:

$$\mu(x,0) = \mu_0(x),\ x \in \Omega_{LR}, \quad \mu(x_L,t) = \mu_L(t), \quad \mu(x_R,t) = \mu_R(t), \quad t \in \Omega_T. \tag{1.2}$$

1.1 Telephone polynomials

Telephone numbers (TelNs), counting connection configurations for m individuals, are defined recursively by:

$$\mathbb{T}_m = \mathbb{T}_{m-1} + (m-1)\mathbb{T}_{m-2}, \quad \mathbb{T}_0 = \mathbb{T}_1 = 1,$$

and explicitly by:

$$\mathbb{T}_m = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{2^\ell \ell!(m-2\ell)!}.$$

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Their polynomial generalization, Telephone polynomials (TelPs), is given by:

$$\mathbb{T}_m(z) = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{2^\ell \ell! (m-2\ell)!} z^{m-2\ell} = \sum_{\substack{k=0 \\ m-k \text{ even}}}^{m} \frac{m!}{2^{\frac{m-k}{2}} \left(\frac{m-k}{2}\right)! \, k!} z^k,$$

satisfying the recurrence:

$$\mathbb{T}_m(z) = z\mathbb{T}_{m-1}(z) + (m-1)\mathbb{T}_{m-2}(z), \quad \mathbb{T}_0(z) = 1, \quad \mathbb{T}_1(z) = z.$$

2 Main results

2.1 Time advancement

To discretize the KdV-mKdV equation temporally over the interval $[0, T]$, partition it into N uniform subintervals with grid points $t_0 = 0 < t_1 = \Delta t < \cdots < t_N = N\Delta t = T$, where $\Delta t = t_s - t_{s-1}$ is the time step. Employing a Taylor expansion for $\mu_t^s = \mu(x, t_s)$, the time derivative is approximated with second-order accuracy as:

$$\mu_t^s = \frac{\mu^{s+1} - \mu^s}{\Delta t} - \frac{\Delta t}{2} \mu_{tt}^s + \mathcal{O}(\Delta t^2). \tag{2.1}$$

At discrete time t_s, we may write the KdV-mKdV equation (1.1) as

$$\mu_t^s = -q\,\mu^s\,\mu_x^s - p\,(\mu^2)^s\,\mu_x^s - r\,\mu_{xxx}^s. \tag{2.2}$$

By differentiating both sides of Eq. (2.2) with respect to time and replacing the first-order derivatives μ_t^s with the finite difference approximation $\frac{\mu^{s+1}-\mu^s}{\Delta t}$, we derive an expression for the second-order time derivative μ_{tt}^s:

$$\begin{aligned} \Delta t\,\mu_{tt}^s = &-q\left(\mu^{s+1} - \mu^s\right)\mu_x^s - q\,\mu^s\left(\mu_x^{s+1} - \mu_x^s\right) - 2p\,\mu^s\,\mu_x^s\left(\mu^{s+1} - \mu^s\right) \\ &- p(\mu^s)^2\left(\mu_x^{s+1} - \mu_x^s\right) - r\left(\mu_{xxx}^{s+1} - \mu_{xxx}^s\right). \end{aligned} \tag{2.3}$$

Now, by inserting Eqs. (2.2) and (2.3) into Eq. (2.1) and simplifying, we derive the discretized formula:

$$\begin{aligned} \Big(2 + q\Delta t\mu_x^s + 2p\Delta t\,\mu^s\,\mu_x^s\Big)\mu^{s+1} + \Delta t\,\mu^s\Big(q + p\mu^s\Big)\mu_x^{s+1} + r\Delta t\,\mu_{xxx}^{s+1} = -r\Delta t\,\mu_{xxx}^s \\ + \mu^s\Big(2 + p\Delta t\,\mu^s\,\mu_x^s\Big)\mu^s, \quad s = 0, 1, \ldots, N-1, \end{aligned} \tag{2.4}$$

where $\mu^0(x) = \mu_0(x)$ is the given initial condition and the associated B.Cs. are converted as

$$\mu^{s+1}(x_L) = \mu_L(t_{s+1}) =: \mu_L^{s+1}, \quad \mu^{s+1}(x_R) = \mu_R(t_{s+1}) =: \mu_R^{s+1}. \tag{2.5}$$

The aim is to tackle the initial-boundary-value problems (IBVPs) (2.4)- (2.5) with Telephone spectral collocation scheme as describe below.

2.2 The combined Taylor-Telephone scheme

Given the approximate solution $\mathcal{U}_{s,N}(x)$ of μ^s at the current time level t_s, we aim to compute the solution $\mathcal{U}_{s+1,N}(x)$ at the subsequent time level t_{s+1} (for $s = 0, 1, \ldots, N$) using the expansion:

$$\mathcal{U}_{s+1,N}(x) = \sum_{k=0}^{N} a_{k,s}\,\mathbb{T}_k(x) = \boldsymbol{T}_N(x)\,\boldsymbol{A}_{s,N}, \quad x \in \Omega_{LR} = [x_L, x_R], \tag{2.6}$$

where $a_{k,s}$ are the unknown coefficients and $\mathbb{T}_s(x)$ denote Telephone basis functions defined over $[x_L, x_R]$. In addition, we have

$$\boldsymbol{A}_{s,N} = [a_{0,s} \quad a_{1,s} \quad \dots \quad a_{N,s}]^T, \quad \boldsymbol{T}_N(x) = [\mathbb{T}_0(x) \quad \mathbb{T}_1(x) \quad \dots \quad \mathbb{T}_N(x)].$$

In matrix form, we may also represent $\boldsymbol{T}_N(x)$ as

$$\boldsymbol{T}_N(x) = \boldsymbol{Y}_N(x)\,\boldsymbol{E}_N, \quad \boldsymbol{Y}_N(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^N \end{bmatrix}. \tag{2.7}$$

Here, the matrix $\boldsymbol{E}_N$ with dimension $(N+1)\times(N+1)$ has the lower-triangular structure. For instance, for $N=5$, we get

$$\boldsymbol{E}_5 = \begin{pmatrix} 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 & 0 & 15 \\ 0 & 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(\boldsymbol{E}_5) = 1.$$

One can easily verify that [3]

$$\frac{d}{dx}\boldsymbol{Y}_N(x) = \boldsymbol{Y}_N(x)\boldsymbol{M}_N, \quad m_{i,j} := \begin{cases} i+1 & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases} \quad 0 \le i,j \le N-1. \tag{2.8}$$

The immediate conclusion is that

$$\frac{d^2}{dx^2}\boldsymbol{Y}_N(x) = \boldsymbol{Y}_N(x)(\boldsymbol{M}_N)^2, \quad \frac{d^3}{dx^3}\boldsymbol{Y}_N(x) = \boldsymbol{Y}_N(x)(\boldsymbol{M}_N)^3. \tag{2.9}$$

2.3 Spatial Discretization via Telephone Polynomials

In view of relations (2.7)-(2.9), we are able to represent $\mathcal{U}_{s+1,N}(x)$ and its derivatives in (2.6) as

$$\begin{aligned} &\mathcal{U}_{s+1,N}(x) = \boldsymbol{Y}_N(x)\,\boldsymbol{E}_N\,\boldsymbol{A}_{s,N}, \quad \mathcal{U}^{(1)}_{s+1,N}(x) = \boldsymbol{Y}_N(x)\,\boldsymbol{M}_N\,\boldsymbol{E}_N\,\boldsymbol{A}_{s,N}, \\ &\mathcal{U}^{(3)}_{s+1,N} := \frac{d^3}{dx^3}\mathcal{U}_{s+1,N}(x) = \boldsymbol{Y}_N(x)\,(\boldsymbol{M}_N)^3\,\boldsymbol{E}_N\,\boldsymbol{A}_{s,N}. \end{aligned} \tag{2.10}$$

By introducing the functions

$$\begin{aligned} &a_0^s(x) := 2 + q\Delta t\mu_x^s + 2p\Delta t\,\mu^s\,\mu_x^s, \quad a_1^s(x) := \Delta t\,\mu^s\Big(q + p\mu^s\Big), \quad a_2^s(x) := r\Delta t, \\ &b^s(x) := -r\Delta t\,\mu_{xxx}^s + \mu^s\Big(2 + p\Delta t\,\mu^s\,\mu_x^s\Big)\mu^s, \end{aligned}$$

followed by inserting the collocation points $x_i = x_L + \frac{i}{N}(x_R - x_L)$ for $i = 0,1,\dots,N$, into (2.4), we arrive at

$$a_0^s(x_i)\mathcal{U}_{s+1,N}(x_i) + a_1^s(x_i)\mathcal{U}^{(1)}_{s+1,N}(x_i) + a_2^s(x_i)\mathcal{U}^{(3)}_{s+1,N}(x_i) = b^s(x_i), \quad i = 0,1,\dots,N,$$

for $s = 0,1,\dots,N-1$. In matrix form, we are able to represent the set of former equations as

$$\boldsymbol{A}_0^s\,\boldsymbol{U}_{s+1} + \boldsymbol{A}_1^s\,\dot{\boldsymbol{U}}_{s+1} + \boldsymbol{A}_2^s\,\ddot{\boldsymbol{U}}_{s+1} = \boldsymbol{B}_s, \tag{2.11}$$

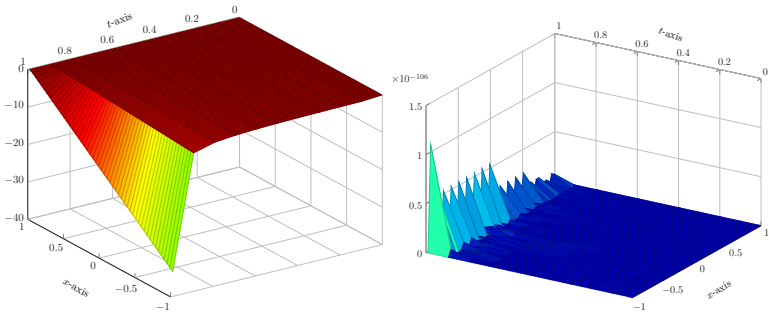


Figure 1: A visual representation of numerical solution (left) and its absolute error (right) in Example 2.1 for $(x,t) \in (-1,1) \times (0, 0.99)$.

where

$$\boldsymbol{A}_i^s = \begin{bmatrix} a_i^s(x_0) & 1 & 0 & \dots & 0 \\ 0 & a_i^s(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_i^s(x_N) \end{bmatrix}, \quad \boldsymbol{U}_{s+1} = \begin{bmatrix} \mathcal{U}_{s+1,N}(x_0) \\ \mathcal{U}_{s+1,N}(x_1) \\ \vdots \\ \mathcal{U}_{s+1,N}(x_N) \end{bmatrix}, \quad \boldsymbol{B}_s = \begin{bmatrix} b^s(x_0) \\ b^s(x_1) \\ \vdots \\ b^s(x_N) \end{bmatrix}$$

$$\dot{\boldsymbol{U}}_{s+1} = \begin{bmatrix} \mathcal{U}^{(1)}_{s+1,N}(x_0) \\ \mathcal{U}^{(1)}_{s+1,N}(x_1) \\ \vdots \\ \mathcal{U}^{(1)}_{s+1,N}(x_N) \end{bmatrix}, \quad \dddot{\boldsymbol{U}}_{s+1} = \begin{bmatrix} \mathcal{U}^{(3)}_{s+1,N}(x_0) \\ \mathcal{U}^{(3)}_{s+1,N}(x_1) \\ \vdots \\ \mathcal{U}^{(3)}_{s+1,N}(x_N) \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_N(x_0) \\ \boldsymbol{Y}_N(x_1) \\ \vdots \\ \boldsymbol{Y}_N(x_N) \end{bmatrix}$$

Using the matrix representations (2.3), one can express the system (2.11) in matrix format:

$$\underbrace{\left[\boldsymbol{A}_0^s \boldsymbol{Y} + \boldsymbol{A}_1^s \boldsymbol{Y}\, \boldsymbol{M}_N + \boldsymbol{A}_2^s \boldsymbol{Y}\, (\boldsymbol{M}_N)^3\right] \boldsymbol{E}_N}_{\boldsymbol{C}_s} \boldsymbol{A}_{s,N} = \boldsymbol{B}_s, \quad \text{or} \quad [\boldsymbol{C}_s;\ \boldsymbol{B}_s], \tag{2.12}$$

for $s = 0, 1, \dots, N-1$. By utilizing the relation (2.3), the boundary conditions (2.5) can be enforced by replacing the first and last rows of the system (2.12) with:

$$\boldsymbol{C}_L\, \boldsymbol{A}_{s,N} = \mu_L^{s+1}, \quad \boldsymbol{C}_L := \boldsymbol{Y}_N(x_L)\, \boldsymbol{E}_N,$$
$$\boldsymbol{C}_R\, \boldsymbol{A}_{s,N} = \mu_R^{s+1}, \quad \boldsymbol{C}_R := \boldsymbol{Y}_N(x_R)\, \boldsymbol{E}_N.$$

This results in a modified linear system:

$$\bar{\boldsymbol{C}}_s\, \boldsymbol{A}_{s,N} = \bar{\boldsymbol{B}}_s, \quad \text{or} \quad [\bar{\boldsymbol{C}}_s;\ \bar{\boldsymbol{B}}_s]. \tag{2.13}$$

The solution of this system at each time step provides the coefficients $\boldsymbol{A}_{s,N}$ needed to construct the approximate solution $\mathcal{U}_{s+1,N}(x)$ via equation (2.6).

Example 2.1. Let us take $p = -6$, $q = 0$, $r = 1$ to arrive at the following equation

$$\mu_t - 6\mu\, \mu_x + u_{xxx} = 0, \quad |t| < 1,\ x \in \mathbb{R}.$$

The initial condition is $\mu_0(x) = (x-1)/6$ and the boundary conditions are exacted from the given exact true solution $\mu(x,t) = \frac{1}{6}\frac{x-1}{1-t}$, which is valid for $t \in (-1,1)$ and $x \in \mathbb{R}$. Using $N = 1$ and $\Delta t = 0.1$, the results of approximations are plotted in Fig. 1 over the space-time domain $(-1,1) \times (0, 0.99)$.

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Finiteness of Gorenstein Projective Dimension over endomorphism algebra

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Abstract

Gorenstein homological algebra is a branch of relative homological algebra that has been developed to a high level. Its main idea is to replace projective modules with Gorenstein projective modules. In this article, we focus on the endomorphism algebra $\Gamma = \mathrm{End}_\Lambda(M)^{op}$ and using the concept of $\mathcal{X}$- totally acyclic complex, we show that if M is a $\mathcal{G}$-representation generator then every finitely presented Γ-module has finite Gorenstein projective dimension.

Keywords: Artin algebra, Endomorphism algebra, Gorenstein projective module, Gorenstein Projective Dimension.

1 Introduction and preliminaries

Relative homological algebra is initiated by Eilenberg and Moore [4]. The idea of relative homological algebra is that one might replace projective modules by certain classes of modules and then "pretends" that these modules are projective. Gorenstein homological algebra is a kind of relative homological algebra, where the relatively projective modules are chosen to be Gorenstein-projective modules. Finitely generated Gorenstein-projective modules over a noetherian ring are introduced by Auslander and Bridger under the name "modules of G-dimension zero" [1]. Over a commutative Gorenstein ring these modules are equal to the maximal Cohen-Macaulay modules. Auslander and Bridger introduce the notion of G-dimension for a finitely generated module and then they generalize the famous Auslander-Buchbaum formula with projective dimension replaced by the G-dimension. The notion of arbitrary Gorenstein projective modules over an arbitrary ring is invented by Enochs and Jenda [3]. Later the theory of Gorenstein-projective modules is studied intensively by Enochs's school and others.

In general, we assume that Λ is an arbitrary ring, associative with the identity element. We let ModΛ, resp. modΛ, denotes the category of all, resp. finitely presented, right Λ-modules. Let $\mathcal{M}$ be a class of Λ-modules. We denote by Add-$\mathcal{M}$ the class of all modules that are isomorphic to a direct summand of a direct sum of modules in $\mathcal{M}$. Moreover, add$\mathcal{M}$ denotes the class of all direct summands of finite direct sums of objects in $\mathcal{M}$. If $\mathcal{M} = \{M\}$ contains a single object, we write Add-M, resp. add$M$, instead of Add-$\{M\}$, resp. add$\{M\}$. We also denote by $\mathcal{P}(\Lambda)$ the full subcategory of all projective Λ-modules.

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Definition 1. *A module $G \in \mathrm{Mod}\Lambda$ is Gorenstein-projective if there exists an exact sequence*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective Λ-module, which stays exact after applying $\mathrm{Hom}_\Lambda(-, P)$ for each projective Λ-module P, such that $G \cong \ker(P_0 \to P^0)$. We denote by $\mathcal{GP}(\Lambda)$ the full subcategory of Gorenstein-projective modules.

Definition 2. *For any Λ-module M, the Gorenstein projective dimension of M, denoted $\mathrm{Gpd}_\Lambda M$, is defined to be the minimum number n such that there exists an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where the G_i lie in $\mathcal{GP}(\Lambda)$, for any $i \geq 0$. If no such number n exists, we define $\mathrm{Gpd}_\Lambda M := \infty$. For a ring Λ, the Gorenstein global dimension of Λ, denoted $\mathcal{G}$-gldim(Λ), is defined as*

$$\mathcal{G}\text{-gldim}(\Lambda) = \sup\{\mathrm{Gpd}_\Lambda M \ : \ M \in \mathrm{mod}\Lambda\}.$$

In the rest of the paper we assume that Λ is an Artin algebra. For $M \in \mathrm{mod}\ \Lambda$ we denote the functor $\mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}\ \Lambda \longrightarrow \mathrm{mod}\ \Gamma$ by e_M and call it the *evaluation functor* at M, where $\Gamma = \mathrm{End}_\Lambda(M)^{op}$. The following lemma shows the evaluation functor can represent Projective modules over endomorphism algebra $\Gamma = \mathrm{End}_\Lambda(M)^{op}$ when we pass from an Artin algebra Λ to endomorphism algebra $\Gamma = \mathrm{End}_\Lambda(M)^{op}$

Lemma 1.1. *Let Λ be an artin algebra and $M \in \mathrm{mod}\ \Lambda$. Then the evaluation functor $e_M : \mathrm{mod}\ \Lambda \longrightarrow \mathrm{mod}\ \Gamma$ has the following properties:*
(a) If $X \in \mathrm{add}M$, then $e_M(X) \in \mathcal{P}(\Gamma)$
(b) $e_M|_{\mathrm{add}M} : \mathrm{add}M \longrightarrow \mathcal{P}(\Gamma)$ is an equivalence.

Proof. We refer to [1, Proposition 2.1]. □

2 Main Result

In this section we generalized Lemma 1.1 to the Gorenstein case and then show that every finitely presented Γ-module has finite Gorenstein projective dimension and as a result the Gorenstein global dimension of Γ is finite. We start with the following definition:

Definition 3. *Let $\mathcal{X}$ be a class of objects of* mod Λ *closed under isomorphism. An $\mathcal{X}$-totally acyclic complex is a complex*

$$X_\bullet = \cdots \to X_2 \to X_1 \to X_0 \to X^0 \to X^1 \to X^2 \to \cdots$$

of modules in $\mathcal{X}$ such that the induced sequence $\mathrm{Hom}_\Lambda(X, X_\bullet)$ and $\mathrm{Hom}_\Lambda(X_\bullet, X)$ are acyclic for all X in $\mathcal{X}$.

Now let $\mathcal{X} = \mathrm{add}M$. Consider

$$\mathcal{C}_{tac}(\mathrm{add}M) = \{\text{All } \mathcal{X} - \text{totally acyclic complex in mod } \Lambda\}$$

and let $\mathcal{G}\mathrm{add}M$ be all syzygies of all elements of $\mathcal{C}_{tac}(\mathrm{add}M)$. It is clear that $\mathrm{add}M \subseteq \mathcal{G}\mathrm{add}M$.

The following result gives the basis for our reduction:

Lemma 2.1. *Let $M \in \mathrm{mod}\ \Lambda$ where Λ is an artin algebra.Then the evaluation functor $e_M : \mathrm{mod}\ \Lambda \longrightarrow \mathrm{mod}\ \Gamma$ has the following properties:*
(a): If $X \in \mathcal{G}\mathrm{add}M$ then $e_M(X) \in \mathcal{GP}(\Gamma)$

(b): $e_M|_{\mathcal{G}\mathrm{add}M} : \mathcal{G}\mathrm{add}M \longrightarrow \mathcal{GP}(\Gamma)$ is an equivalence.

Proof. (a)Let $X \in \mathcal{G}\mathrm{add}M$, so there exists complex $L_\bullet$ as follows:

$$L_\bullet = \cdots \longrightarrow L_2 \xrightarrow{\rho_2} L_1 \xrightarrow{\rho_1} L_0 \xrightarrow{\rho_0} L^0 \xrightarrow{\sigma^0} L^1 \xrightarrow{\sigma^1} \cdots$$

where $L_i, L^i \in \mathrm{add}M$, $\mathrm{Hom}_\Lambda(X', L_\bullet)$ and $\mathrm{Hom}_\Lambda(L_\bullet, X')$ are exact for all $X' \in \mathrm{add}M$ and $X \cong \ker \rho_0$. Since $\mathrm{Hom}_\Lambda(M, L_\bullet)$ is exact then we have complete projective resolution as follows:

$$\mathcal{P}^\Gamma_\bullet = \cdots \longrightarrow \mathrm{Hom}_\Lambda(M, L_1) \xrightarrow{\overline{\rho_1}} \mathrm{Hom}_\Lambda(M, L_0) \xrightarrow{\overline{\rho_0}} \mathrm{Hom}_\Lambda(M, L^0) \longrightarrow \cdots$$

Note that $L_i, L^i \in \mathrm{add}M$, hence $\mathrm{Hom}_\Lambda(M, L^i)$ and $\mathrm{Hom}_\Lambda(M, L_i)$ are projective as Γ-module where $\Gamma = \mathrm{End}(\Lambda)^{op}$. By Yoneda philosophy for every $Q \in \mathrm{add}M$ we have

$$\mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(M, L_\bullet), \mathrm{Hom}_\Lambda(-, Q)) \cong \mathrm{Hom}_\Lambda(L_\bullet, Q))$$

Hence $\mathrm{Hom}(-, \mathrm{Hom}(-, Q))$ is exact for all $Q \in \mathrm{add}M$, furthermore $\mathrm{Hom}(M, X)$ is kernel of $\overline{\rho_0}$, hence $\mathrm{Hom}(M, X)$ is Gorenstein projective Γ-module.

(b) : Let $Y \in \mathcal{GP}(\Gamma)$, then we have the following complete projective resolution

$$\mathcal{P}^\Gamma_\bullet = \cdots \longrightarrow \mathrm{Hom}_\Lambda(M, L_1) \xrightarrow{\overline{\rho_1}} \mathrm{Hom}_\Lambda(M, L_0) \xrightarrow{\overline{\rho_0}} \mathrm{Hom}_\Lambda(M, L^0) \longrightarrow \cdots$$

such that $Y \cong \ker \overline{\rho_0}$ and $L_i, L^i \in \mathrm{add}M$. By Yoneda Lemma we have a complex as follows:

$$L_\bullet = \cdots \longrightarrow L_2 \xrightarrow{\rho_2} L_1 \xrightarrow{\rho_1} L_0 \xrightarrow{\rho_0} L^0 \xrightarrow{\sigma^0} L^1 \xrightarrow{\sigma^1} \cdots$$

$L_\bullet$ is $\mathrm{add}M$-totally acyclic, because

$$\forall X' \in \mathrm{add}M \qquad \mathrm{Hom}_\Lambda(L_\bullet, X') \cong \mathrm{Hom}(\mathrm{Hom}_\Lambda(M, L_\bullet), \mathrm{Hom}_\Lambda(-, X'))$$

is exact, since $\mathcal{P}^\Gamma_\bullet$ is complete projective resolution. Furthermore $\mathrm{Hom}_\Lambda(X, L_\bullet)$ is exact, since $\mathrm{Hom}_\Lambda(M, L_\bullet)$ is exact and X' is direct summand of direct sum of M. Now let $X = \ker(L_0 \longrightarrow L^0)$ then $X \in \mathcal{G}\mathrm{add}M$ and $e_M(X) = Y$, hence $e_M|_{\mathcal{G}\mathrm{add}M}$ is dense.
Since $e_M|_{\mathcal{G}\mathrm{add}M}$ is dense and by Yoneda Lemma $e_M|_{\mathcal{G}\mathrm{add}M}$ is full and faithful we can say that $e_M|_{\mathcal{G}\mathrm{add}M} : \mathcal{G}\mathrm{add}M \longrightarrow \mathcal{GP}(\Gamma)$ is an equivalence. □

Definition 4. *Consider a fix object $M \in \mathrm{mod}\,\Lambda$. An $\mathcal{G}_M$-resolution of an object $X \in \mathrm{mod}\,\Lambda$ is an exact sequence*

$$\cdots \to M_2 \to M_1 \longrightarrow M_0 \to X \to 0$$

in $\mathrm{mod}\,\Lambda$ *such that $M_i \in \mathcal{G}\mathrm{add}M$ for all i and the induced sequence*

$$\cdots \to \mathrm{Hom}_\Lambda(M, M_2) \to \mathrm{Hom}_\Lambda(M, M_1) \to \mathrm{Hom}_\Lambda(M, M_0) \to \mathrm{Hom}_\Lambda(M, X) \to 0$$

of abelian groups is exact. The resolution has finite length if $M_i = 0$ for $i \gg 0$. We call M a $\mathcal{G}$-representation generator if every object $X \in \mathrm{mod}\,\Lambda$ admits an $\mathcal{G}_M$-resolution of finite length.

Theorem 2.2. *Consider* $\mathrm{mod}\,\Lambda$ *with a $\mathcal{G}$-representation generator M. Then every finitely presented Γ-module has finite Gorenstein projective dimension.*

Proof. Take a Γ-module Y and take a projective presentation of Y, say

$$\mathrm{Hom}_\Lambda(M, M_1) \xrightarrow{h} \mathrm{Hom}_\Lambda(M, M_0) \longrightarrow Y \longrightarrow 0$$

with $M_1, M_0 \in \mathrm{add}M$. Its kernel is $\mathrm{Hom}_\Lambda(M, X)$, where X is the kernel of $M_1 \to M_0$ (since applying $\mathrm{Hom}_\Lambda(M, -)$ to the exact sequence $0 \to X \to M_1 \to M_0$ yields an exact sequence). But by assumption X admits an $\mathcal{G}_M$-resolution

$$0 \to M'_n \to \cdots \to M'_1 \to M'_0 \to X \to 0$$

applying $\mathrm{Hom}_\Lambda(M, -)$, we obtain a Gorenstein projective resolution of length n of $\mathrm{Hom}_\Lambda(M, X)$. Thus $Gpd_\Gamma Y \leq n + 2$ □

Corollary 2.3. *Let $M \in \mathrm{mod}\ \Lambda$ be a $\mathcal{G}$-representation generator. Then $\mathcal{G}$-gldim(Γ) is finite.*

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A Nonlinear mathematical model and its application to solving a real industrial problem

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Abstract

This article addresses mixed integer fractional signomial geometric programming (MIFSGP) problems, which have been widely used in industrial design. In this paper, first, we convert fractional signomial programming into a non fractional problem so that it maintains its geometric structure. Then, convex relaxation is used to reach a mixed integer global solution. Although, in many cases, we obtain a better objective function value with this process, designers may still be dissatisfied with the rupture between the original objective function value and the relaxed value. Therefore, we apply a spatial branch and bound algorithm to decrease that distance to an acceptable extent and maintain the global solution. Finally, a real design problem is considered to evaluate the efficiency and accuracy of the proposed technique.
Keywords: geometric programming, fractional programming, mixed integer programming, nonconvex functions, spatial branch and bound algorithm.

1 Introduction

Fractional geometric programming (FGP) is applied to solve a class of geometric programming problems to minimize the fractional objective function under definite constraints. A few methods have been utilized in recent decades to convert a fractional signomial objective function into a non fractional signomial objective function to reach the optimal solution via common mathematical programming techniques. In a mixed integer fractional signomial geometric programming problem (MIFSGP), the objective function is a quotient of two signomial functions subject to certain constraints with integer and continuous decision variables; this approach is an essential part of geometric programming problems in the wide scope of engineering design, management and finance. For instance, Ray and Saini (2001) [2], Arora (1989) [3], Tsai (2005) [3], Bazikar et al(2018)[4] and Ahmad, S(2022)[2] addressed a few methods to solve real (FGP) problems in design engineering.

In a real non fractional signomial case, the number of iterations of the solver is also significantly reduced, and an integer solution is obtained by reformulating the problem, changing negative power variables to positive power variables and applying our proposed approach. In this work, first, we define a new variable for the fractional objective function. This technique formulates an MIFGP problem by adding new constraints to

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a non fractional mixed integer geometric programming (MIGP) problem. Since the formulated problem still contains some concave terms in the objective functions or constraints and is sometimes more than before, this feature still results in local solutions to the problem. In addition, the existence of integer variables generally makes the feasible region nonconvex. Therefore, applying convex relaxation in most cases will yield the lowest possible value for the objective function of the original (MIFGP) problem. Therefore, by using a spatial branch and bound algorithm (SBB), we find a feasible solution for the MIGP problems and the tightest global lower bound to the local original lower bound for the objective function simultaneously.

2 Mixed integer signomial geometric fractional programming (MISGFP) problems

A signomial function consists of a sum of positive or negative terms that are products of power functions, i.e.,

$$p(x) = f(x_1, x_2, \cdots, x_n) = \sum_{t=1}^{T} \quad \sigma_t c_t \Pi_{j=1}^{n} x_j^{\alpha_{tj}} \tag{2.1}$$

where x is a vector that contains positive variables of real or integer types. In each signomial function, T and n represent the number of terms and variables, respectively. Ct is the absolute value of the coefficients, and σ_t is the sign of the coefficient ($+1$ or -1). If $\sigma_t = +1$ in all terms of a signomial function, the function is called a posynomial. The A (MISGFP) programming problem is defined in its typical form as follows: Obtain $x = (x_1, x_2, x_3, \cdots, x_n)^T$ to

$$Minimize f(x) = \frac{p(x)}{q(x)}, \quad q(x) > 0 \tag{2.2}$$

Subject to$g_i(x) \leqslant \xi_i, \xi_i = \pm 1, i = 1, 2, \cdots, m.$ where $p(x), q(x)$ and $g_i(x)$ are signomial functions and $x = (x_1, x_2, x_3, \ldots, x_n)^T in X, X$ is a vector of real or integer positive variables

$$x, 0 \leqslant \underline{x_j} \leqslant x_j \leqslant \bar{x}_j, j = 1, 2, \cdots, n.$$

It is assumed that the mentioned problem is feasible and has an optimal solution.

3 Strategy of reformulation

This article presents a convenient technique for converting a nonconvex problem (MISGFP) into a convex non fractional mixed integer signomial geometric programming problem (MISGP). Consider the following signomial geometric fractional programming problem:

$$Minimize f(x) = \frac{p(x)}{q(x)} = \frac{\sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \Pi_{j=1}^{n} x_j^{\alpha_{0tj}}}{\sum_{t=1}^{T_0'} \sigma'_{0t} c'_{0t} \Pi_{j=1}^{n} x_j^{\alpha'_{0tj}}} \tag{3.1}$$

S.t

$$g_i(x) \leqslant \xi_i, \xi_i = \mp 1, \quad i = 1, 2, \cdots, m.$$

where

$$
\begin{aligned}
&g_i(x) = \sum_{t=1}^{T_i} \sigma_{it} c_{it} \Pi_{j=1}^{n} x_j^{\alpha_{it}} \\
&q(x) > 0, \\
&for \quad x \in X \quad and \quad \forall x = (x_1, x_2, x_3, \cdots, x_n)^T \in X \subseteq R, \\
&0 \leqslant \underline{x_j} \leqslant x_j \leqslant \bar{x}_j, \\
&T_0 = Number \quad of \quad terms \quad in \quad p(x), \\
&T_0' = number \quad of \quad terms \quad in \quad q(x), \\
&c_{0t}, c_{0t}' \in R^+ \\
&0 \leqslant \underline{x_j} \leqslant x_j \leqslant \bar{x}_j, \\
&j = 1, 2, \cdots, n.
\end{aligned}
$$

Suppose x_{n+1}, is a positive variable; let $q(x) = x_{n+1}$. We have:

$$f(x) = x_{n+1}^{-1} * p(x),$$

To find the lower and upper bounds of x_{n+1}, we should solve the following two sub problems:

$$L_{n+1} := \min q(x),$$

Subject to

$$
\begin{aligned}
&g_i(x) \leqslant \xi_i, \\
&\xi = \mp 1, \\
&i = 1, 2, \cdots, m, \\
&for \quad x \in X
\end{aligned}
$$

and

$$
\begin{aligned}
&U_{n+1} := \max q(x), \\
&subject \quad to \quad g_i(x) \leqslant \xi_i, \\
&\xi = \mp 1, \\
&i = 1, 2, \cdots, m, \\
&for \quad x \in X.
\end{aligned}
$$

Therefore, problem 2.7 leads to the following signomial geometric programming problem, which contains a new equality constraint,$x_{n+1}^{-1} * q(x) = 1$.

$$
\begin{aligned}
&Minimize \quad x_{n+1}^{-1} * p(x) && (3.2) \\
&subject \quad to \quad x_{n+1}^{-1} * q(x) = 1 && (3.3) \\
&\xi_i = \mp 1, i = 1, 2, \cdots, m, \\
&g_i(x) \leqslant \xi_i, \\
&L_{n+1} \leqslant x_{n+1} \leqslant U_{n+1}, x > 0
\end{aligned}
$$

We can replace equation 1.2 with two inequalities as follows:

$$x_{n+1}^{-1} * q(x) \leqslant 1 \tag{3.4}$$

$$-x_{n+1}^{-1} * q(x) \leqslant -1 \tag{3.5}$$

Or equivalent of6

$$x_{n+1} * q^{-1}(x) \leqslant 1 \tag{3.6}$$

Therefore, the original MISGFP problem is reformulated to:

$$Minimize \quad (x_{n+1}^{-1}) * p(x),$$

Subject to

$$\begin{aligned}
&x_{n+1}^{-1} * q(x) \leqslant 1\\
&x_{n+1} * q^{-1}(x) \leqslant 1\\
&g_i(x) \leqslant \xi_i,\\
&\xi_i = \mp, i = 1, 2, \cdots, m\\
&L_{n+1} \leqslant x_{n+1} \leqslant U_{n+1}, x > 0\\
&p(x) = \sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \Pi_{j=1}^{n} x_j^{\alpha_{0tj}},\\
&q(x) = \sum_{t=1}^{T_0'} \sigma'_{0t} c'_{0t} \Pi_{j=1}^{n} x_j^{\alpha'_{0tj}},\\
&g_i(x) \sum_{t=1}^{T_i} \sigma_{it} c_{it} \Pi_{j=1}^{n} x_j^{\alpha_{itj}}\\
&q(x) > 0, \quad \forall x = (x_1, x_2, \cdots, x_n)^T \in X \subseteq R.\\
&0 \leqslant \underline{x_j} \leqslant x_j \leqslant \bar{x}_j,\\
&j = 1, 2, \cdots, n
\end{aligned} \tag{3.7}$$

Proposition 3.1. *proposition: If $0 < q(x^*) = x_{n+1}^*$, where x^*, is the optimal solution of 1.2, $x_n^*$is the optimal solution of 2.7.*

$$\forall x \neq x^*, (x_{n+1}^*)^{-1} * p(x^*) \leqslant (x_{n+1})^{-1} * p(x) \tag{3.8}$$

Proof. Suppose $x^* = (x_1^1, x_2^*, \cdots, x_n^*)$ is an optimal solution of 1.2; consequently,

$$\begin{aligned}
&(x_{n+1}^*)^{-1} * q(x^*) = 1\\
&g_i(x^*) \leqslant \xi_i, \xi = \mp 1,\\
&i = 1, 2, \ldots, m
\end{aligned}$$

Thus, x^*, is satisfied in the constraint of 2.7. Additionally, by replacing $q(x) = x_{n+1}$ and $q(x^*) = x_{n+1}^*$ in the objective function 1.2, we have:

$$\begin{aligned}
&x_{n+1}^{*-1} * \sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \Pi_{j=1}^{n} x_j^{*\alpha_{0tj}} \leqslant x_{n+1}^{-1} * \sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \Pi_{j=1}^{n} x_j^{\alpha_{0t}}\\
&\frac{\sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \Pi_{j=1}^{n} x_j^{*\alpha_{0tj}}}{\sum_{t=1}^{T_0'} \sigma'_{0t} c'_{0t} \Pi_{j=1}^{n} x_j^{*\alpha'_{0tj}}} \leqslant \frac{\sum_{t=1}^{T_0} \sigma_{0t} c_{0t} \Pi_{j=1}^{n} x_j^{\alpha_{0tj}}}{\sum_{t=1}^{T_0'} \sigma'_{0t} c'_{0t} \Pi_{j=1}^{n} x_j^{\alpha'_{0tj}}}
\end{aligned} \tag{3.9}$$

Hence, x^* is an optimal solution of 2.7. □

4 Convexification strategies

Converting to convex relaxation is an efficient tool for obtaining a global solution in nonconvex (MIGP) problems. It is applied to expand the feasible set of 1.2 and achieves a lower bound on the optimal solution of the (MIGP) problems. Since the nonlinear functions in the constraints and objective of (4) are signomial functions, each signomial function is convex if all the terms are convex [7]. The requirements of convexification for every signomial term are provided with the following theorem. See [8].
Theorem 4.1. (Maranas and Floudas (1995). A positive signomial term $f(x) = c\Pi_{i=1}^{n} x_i^{\alpha_i}$is convex if one of the following requirements holds:

- i)$\alpha_i \leqslant 0(i = 1 \cdots n)$.
- ii)$\exists k \neq i, \alpha_k > 0, \alpha_i \leqslant 0, \sum_{i=1}^{n} \alpha_i \geqslant 1$.

A negative signomial term $f(x) = c\Pi_{i=1}^{n} x_i^{\alpha_i}(c < 0)$is convex if $\alpha_i > 0(for \quad i = 1 \cdots n)$and $\sum_{i}^{n} \alpha_i \leqslant 1$. Theorem 1 states that it is possible to convexify every positive or negative signomial term by using power transformations; therefore, this fact is applied in this paper to convexify nonconvex signomial functions. After utilizing power transformations to convexify a signomial term, the power transformation functions need to be approximated by piecewise linear functions. This study used a standard piecewise linear approximation, special order set type 2 (SOS2) (Beale and Tomlin) [5].

5 Bound assessment algorithm

The spatial branch-and-bound algorithm is one of the best-known techniques for obtaining a precise or at least an ϵ-approximate solution to a nonconvex mixed integer geometric programming problem. The lower bound of the objective function is computed for the entire feasible region. The SBB algorithm works in a common application of tight lower bounds estimated through convex relaxation. To improve the quality of our method used to determine the lower bounds of the MIFGP 2, we applied (using AMPL [6]), a streamlined "partial SBB" algorithm [33, 9], to find ϵ- approximate solutions for every small positive . This algorithm works recursively by dividing the search area along the coordinate direction that contributes most to the difference between the lower and upper bounds on the optimal objective function value determined in every subproblems. For a nonlinear minimization problem, the lower bound is usually computed by constructing and solving a convex relaxation. The upper bound can simply be a local minimum found by an NLP solver. At each branching step, the most reassuring node is registered, and the others are discarded. Since our purpose is to solve an MIGFP problem with integer variables, branching may be necessary for both continuous and integer decision variables.

6 Proposed Algorithm: The partial depth-first SBB algorithm

Input P as a nonconvex MIGP problem. Devote count while converting the nonconvex problem to a convex one.

- 1) Allocate o→ count and find x^*, through an objective function with the value S^*,, which can solve P in local form.



- 2) A convex relaxation R is used for the primal problem P.
- 3) find the minimum value of $\grave{S}$ to obtain the optimum solution S^* using relaxation R.
- 4)Choose branching point $\grave{x_i}$ for the ith variable.
- 5)We add a new constraint as follows to define P_0 as$P$

$$x_i^L \leqslant x_i \leqslant \grave{x}_i$$

- 6) P_1 is defined as P by adding the constraint $\grave{x}_i \leqslant x_i \leqslant x_i^U$
- 7) P_k is used as a convex relaxation for $R_k, K \in \{0, 1\}$.
- 8)Similarly, we used $\grave{x_k}$as the optimum of P_k by the S_k value,$k \in \{0, 1\}$.
- 9) Let $l = (S_0, S_1)$, (the best lower bound to S^*).
- 10)For $S_l > S^*$, the algorithm is stopped since the node cannot be further improved.
- 11)For $S_l < S^*$, devote $S_l \to S$ and$\grave{x_l} \to \grave{x}$.
- 12)In the case of the feasibility of x^* in P_l, solve local P_1 to find x_i^* with S_i^*.
- 13)For $S_i^* < S^*$, let $S_i^* \to S^*$and $x_i^* \to x^*$, to improve the nonconvex problem solution.
- 14) The global optimum is achieved if $|S^* - S| < \epsilon.\epsilon$ is an acceptable error.
- 15)Dedicates $P_l \to P$ and $\grave{x_l} \to \grave{x}$ to update the ith branching variable and branching point $\grave{x_i}$. Finally, the count is increased, and the algorithm is repeated.

7 Numerical example

Example 7.1.(see[2]) This case pertains to a design problem of a journal bearing. Its structure is an inverse problem, where the eccentricity and attitude angle are achieved for a certain load and speed. The volume of steel, the thickness of the intermediate layer and nickel barrier, and the dimensions of the plated overlay of the journal bearing are assumed to be unknown. Hence, some of these parameters of the model are known and estimated by engineers. Suppose that x_1 is the radial clearance, x_2 is the fluid force, x_3 is the diameter, x_4 is the rotation speed and x_5 is the length-to-diameter ratio. The variables of these parameters are concentrated between some specified lower and upper bounds. The following mathematical programming

formulation depicts the above mentioned problem as a signomial geometric programming problem:

$$\begin{aligned}
&minimize \quad 0.5x_1^2x_2x_4x_5 + 1.1x_1^{-1}x_2^{-1}x_3^{-1}\\
&s.t \quad 8.4x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5 \leqslant 4.2\\
&0.5x_2x_3 + x_1 + x_4^{-1}x_5^{-1} + 1.6x_3x_4 \leqslant 1
\end{aligned}$$

The above problem is a concave signomial programming problem. Upon solving the problem, the local objective value was found to be $F = 3.561$. The objective function of the above problem can be easily converted into a fractional function as follows:

$$\frac{0.5x_1^3x_2^2x_3x_4x_5 + 1.1}{x_1x_2x_3}$$

Therefore, the original problem is modified by defining x_6 as a new variable where $x_6 = x_1x_2x_3$.

$$\begin{aligned}
&minimize \quad 0.5x_1^3x_2^2x_3x_4x_5x_6^{-1} + 1.1x_6^{-1}\\
&subject \quad to \quad x_1x_2x_3x_6^{-1} \leqslant 1\\
&x_1^{-1}x_2^{-1}x_3^{-1}x_6 \leqslant 1\\
&8.4x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5 \leqslant 4.2\\
&0.5x_2x_3 + x_1 + x_4^{-1}x_5^{-1} + 1.6x_3x_4 \leqslant 1.
\end{aligned}$$

The mentioned problem has seven nonconvex terms for obtaining a global solution. Several power transformations and piecewise linear approximations are applied to underestimate convexified terms. Therefore, the nonconvex primary programming problem is converted to:

$$\begin{aligned}
&minimize \quad 0.5Z_1^6y_2^{-2}y_3^{-1}y_4^{-1}y_5^{-1}x_6^{-1} + 1.1x_6^{-1}\\
&y_1^{-1}y_2^{-1}y_3^{-1}x_6^{-1} \leqslant 1\\
&8.4k_1^5x_2^{-1}x_3^{-1}x_4^{-1}y_5^{-1} \leqslant 4.2\\
&0.5y_2^{-1}z_3^2 + k_1^5x_4^{-1}x_5^{-1} + 1.6z_3^2y_4^{-1} \leqslant 1\\
&x_1^{-1}x_2^{-1}x_3^{-1}y_6^{-1} \leqslant 1
\end{aligned}$$

where

$$\begin{aligned}
&z_1 = x_1^{0.5}, Z_1 = L(x_1^{0.5})\\
&k_1 = x_1^{0.2}, K_1 = L(x_1^{0.2})\\
&z_3 = x_3^{0.5}, Z_3 = L(x_3^{0.5})\\
&y_i = x_i^{-1}, Y_i = L(x_i^{-1}), i = 1, 2, 3, 4, 5, 6\\
&x_2 \in Z, x_1, x_2, x_3, x_4, x_5, x_6 \in R\\
&0.1 \leqslant x_1 \leqslant 0.5, 9 \leqslant x_2 \leqslant 17, \quad 0.01 \leqslant x_3 \leqslant 0.21, 0.1 \leqslant x_4 \leqslant 1.7,\\
&0.1 \leqslant x_5 \leqslant 1.\dot{8}, 0.1 \leqslant x_6 \leqslant 0.5
\end{aligned}$$

$L(x_1^{0.5}), L(x_1^{0.2}), L(x_3^{0.5}), L(x_i^{-1})$are the piecewise linearized expressions of the nonconvex terms $x_1^{0.5}, x_1^{0.2}, x_3^{0.5}$and convex terms x_i^{-1} for $i = 1, 2, .., 6$, respectively. Since $x_2 \in Z$, we choose integer break points to have an integer solution directly. By solving this program with LINGO 18.0, where the tolerable error was specified as 0.001, the following solution was obtained. A comparison of the original nonconvex

Table 1: The optimal conciliation obtained solutions.

Problem	F	x_1	x_2	x_3	x_4	x_5	iteration
Nonconvex(P)	$F^* = 3.55257$	0.3317	9.7297	0.1199	0.8687	1.5275	130
Convex(R)	$\hat{F} = 2.82846$	$0\dot{5}$	9	0.1275	0.9	0.9	43

Table 2: Algorithm steps.

steps	B. V	V. v inR_0	O. V$\grave{F}_0$	V. V inR_1	O. V$\hat{F}_1$	$\lvert\grave{F}_0 - F_0^*\rvert$	$\lvert\grave{F}_1 - F_0^*\rvert$	pruned node
1	x_5	0.85	1.97	1.5	2.85b	b1.58257	b0.7	R_0
2	x_5	1.5	2.875	1.52	2.895	0.67757	0.65757	R_0
3	x_2	8	2.895	10	2.82125	0.67757	0.73132	R_1
4	x_2	8	2.895	9	2.921	0.67757	0.63157	R_1
5	x_3	0.120	3.752271	0.2	2.41922	0.1997	1.1335	R_0
6	x_3	0.11	3.41	0.12	3.75227	0.14257	0.1997	R_0
7	x_3	0.105	3.85211	0.11	3.41	0.25985	0.14257	R_0
8	x_3	0.108	3.81242	0.110	3.41	0.25985	0.14257	R_0
9	x_3	0.109	3.43	0.110	3.41	0.12257	0.14257	R_1
10	x_3	0.1085	3.85	0.109	3.43	0.24743	0.12257	R_0
11	x_3	0.1088	3.44	0.109	3.43	0.11257	0.12257	R_1
12	x_4	0.7	3.44372	0.85	3.6887	0.10885	0.13613	R_1
13	x_4	0.76	3.4801	0.85	3.6887	0.07247	0.13613	R_1
14	x_4	0.756	3.54006	0.8	3.8799	0.01251	0.32733	R_1
15	x_1	0.45	3.5422	0.5	3.54006	0.01073	0.0125	R_1
16	x_1	0.43	3.54398	0.45	3.5422	0.0086	0.01037	R_0

problem and the convex problem is shown in Table 1. Although the optimum value of the convex problem is much better than that of the nonconvex problem and the number of iterations is much less than that of the original nonconvex problem, this inequality may not be desirable for decision makers. Therefore, the difference between the objectives of two problems is:

$$|F^* - \hat{F}| = 0.72511$$

Suppose the maximum difference between objectives F^* and $\hat{F}$is accepted for $\epsilon = 10^{-2}$. This means that we should find a tighter relaxation when implementing the proposed Algorithm: Applying the suggested Algorithm, the following steps are implemented to reach the minimum value of $\epsilon \leqslant 10^{-2}$. Therefore, the global solution of the convex problem is obtained with an acceptable difference from that of the primary nonconvex problem. The remaining steps are presented in Table 2 as follows: Hint: The branching variable, objective value and punching node were abbreviated as B.V., O.V., and P.N., respectively. Table 2 shows the global solution obtained after 16 iterations with an acceptable tolerance of $\epsilon = 0.0086$ as follows.

The results of the present algorithm indicate the effectiveness of the spatial branch and bound technique. According to Table 3,we conclude that with the help of the proposed algorithm, we reach a global optimum, which is very close to the solution of the original problem and is sometimes even an integer solution to the problem of interest.

Table 3: Comparison of the results obtained from the initial problem and the relaxed problem.

Problem	O.V	x_1	x_2	x_3	x_4	x_5
Original (P)	$F^* = 3.55257$	0.3317	9.729	0.1199	0.8687	1.5275
Relaxed (R)	$\hat{F} = 3.54398$	0.43	9	0.1088	0.706	1.52

8 Conclusion

In this paper, a mixed-integer fractional geometric programming problem (MIFGP) as an NP-hard problem is discussed. To reach a global solution, we first convert the nonconvex geometric programming problem to a convex problem to underestimate the original nonconvex problem. We used the spatial branch and bound technique to increase the consistency and efficiency of the relaxation strategies. The results showed that implementing this algorithm on the convexified signomial programming problem can make the lower bound of the relaxed problem much closer to the lower bound of its nonconvex programming problem. This approach could be proposed as a practical and essential step for convex relaxation for every mixed integer nonlinear programming problem. In this paper, a mixed-integer fractional geometric programming problem (MIFGP) as an NP-hard problem is discussed. Since nonconvex geometric programming problems must be transformed into convex relaxation problems to reach a global solution, a power transformation and a modified form of piecewise linear approximation are applied to the convex and underestimation problems of the original nonconvex problem. We use the spatial branch and bound technique to increase the consistency and efficiency of the relaxation strategies. The results showed that implementing this algorithm on convexified signomial programming problems can make the lower bound of a relaxed problem closer to the lower bound of its nonconvex programming problem. This work can be proposed as a practical and essential step for convex relaxation for every mixed integer nonlinear programming problem.

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The Mixed integer bilevel linear fractional/linear programming problem

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Abstract

In this paper, we present the Branch and bound algorithm for solving the mixed integer bilevel linear fractional/linear programming (MIBLFPLP) problems. In this paper, we consider MIBLFPLP problem on a compact polyhedron and the variables take non negative integral values. An algorithm to find the optimal integer solution is developed. A numerical example is presented and the results are verified as well.

Keywords: Mixed integer bilevel programming; Linear fractional programming; Branch and bound algorithm.

1 Introduction

Bilevel programming deals with decision making situations in which two decision makers (DMs) are arranged with in a hierarchical structure. Bilevel programming occurs in a variety of applications such as supply chain management [9], and capacity planning [1]. In this paper, we are concerned with gaining a better understanding of the feasible set of MIBLFPLP problems, often denoted as the inducible region (IR). The paper is organized as follows: Section 2 illustrates formulation of the MIBLFPLP problem. Section 3 provides the main theoretical result on optimality. In Subsection 3.1 the algorithm is proposed to solve the MIBLFPLP problem and a formal proof of its correctness is given.

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2 Formulation of the problem

In this paper, we consider the MIBLFPLP problem formulated as follows:

$$\begin{aligned} &\max_{x,y} \quad f_1(x,y) = \frac{c_{11}x + c_{12}y + \alpha_1}{d_{11}x + d_{12}y + \beta_1} \\ &\max_{y} \quad f_2(y) = c_{22}y \\ &s.t. \begin{cases} Ax + By = b \\ 0 \le x \le K, \\ y \ge 0 \\ x_i \in \mathbb{Z}, \ \forall i \in I = \{1, \cdots, n_1\} \\ y_j \in \mathbb{Z}, \ \forall j \in J \subseteq \{1, \cdots, n_2\} \end{cases} \end{aligned} \tag{2.1}$$

Where $c_{11},\ d_{11},\ K \in \mathbb{R}^{n_1 \times 1}$, $c_{12},\ c_{22},\ d_{12} \in \mathbb{R}^{n_2 \times 1}$ and $\alpha_1,\ \beta_1$ are constants. The matrices $A \in \mathbb{Z}^{m \times n_1}, B \in \mathbb{R}^{m \times n_2}$ are of appropriate dimensions and the variables $x \in \mathbb{Z}^{n_1 \times 1}, y \in \mathbb{Z}^{n_2 \times 1}$ is under the control of the 1st level and 2nd level DMs, respectively. The $f_1,\ f_2$ are well-defined and continuous linear fractional and linear functions on the constraint region S, respectively. Now, we state some definitions and conceptions regarding the MIBLFPLP problem:

Definition 1. *The Constraint region of MIBLFPLP (2.1):*

$$S = \{(x,y) : \quad Ax + By = b,\ 0 \le x \le K,\ y \ge 0\}.$$

Definition 2. *The image of S onto the upperel decision space:*

$$S(x) = \{x \ge 0 : \exists y| \quad (x,y) \in S\}.$$

Definition 3. *The point-to-set map is defined by:*

$$P(x) = \{y| \quad (x,y) \in S\}.$$

Definition 4. *The point-to-set map is defined by:*

$$M(x) = \{y|\ y = argmax\{f_2(x,\hat{y}) : \hat{y} \in P(x)\}\}.$$

Actually, $M(x)$ is the rational reaction set of the lower level DM (LLDM), to a given x of upper level DM (ULDM).

Definition 5. *For a given parameter x, we define the lower level linear problem LLP(x) as the following ILP:*

$$\begin{aligned} &\max_{y} \quad f_2 = c_{22}y \\ &s.t. \begin{cases} Ax + By = b, \\ y \ge 0,\ y_j \in \mathbb{Z},\ \forall j \in J. \end{cases} \end{aligned} \tag{2.2}$$

Definition 6. *For a given set of parameters ($l \in^{m \times 1}, u \in^{m \times 1}, r \in$), we define B(l,u,r) as the following*

IBLFPL:

$$\max_{x,y} \quad f_1 = \frac{c_{11}x+c_{12}y+\alpha_1}{d_{11}x+d_{12}y+\beta_1}$$
$$s.t. \begin{cases} l \le Ax \le u \\ c_{22}y \ge r \\ 0 \le x \le K, \ x_i \in \mathbb{Z}, \ \forall i \in I \\ \max\limits_{y} \quad f_2 = c_{22}y \\ s.t. \begin{cases} Ax + By = b \\ y \ge 0, \ y_j \in \mathbb{Z}, \ \forall j \in J. \end{cases} \end{cases} \tag{2.3}$$

Definition 7. *For a given set of parameters (l, u, r), we define U(l,u,r) as the following the integer upper level relaxed linear fractional problem (IULFP):*

$$\max_{x,y} \quad f_1 = \frac{c_{11}x+c_{12}y+\alpha_1}{d_{11}x+d_{12}y+\beta_1}$$
$$s.t. \begin{cases} Ax + By = b \\ l \le Ax \le u \\ c_{22}y \ge r \\ 0 \le x \le K, \ x_i \in \mathbb{Z}, \ \forall i \in I \\ y \ge 0, \ y_j \in \mathbb{Z}, \ \forall j \in J. \end{cases} \tag{2.4}$$

If we apply C & C change of variable, $t = \frac{1}{d_{11}x+d_{12}y+\beta_1}$, $v = tx$, $w = ty$, then we have the upper level linear problem ULP(l,u,r) as follows:

$$\max_{v,w,t} \quad z_1 = \alpha_1 t + c_{11}v + c_{12}w$$
$$s.t. \begin{cases} \beta_1 t + d_{11}v + d_{12}w = 1 \\ Av + Bw - bt = 0 \\ Av - ut \le 0, \qquad Av - lt \ge 0 \\ c_{22}w - rt \ge 0 \\ v - Kt \le 0 \\ v, w, \ t \ge 0. \end{cases} \tag{2.5}$$

Definition 8. The IR of problem (2.1):

$$IR := \{(x, y) \in S | \quad y \in M(x)\}.$$

Indeed, IR represents the feasible solutions set of the problem (2.1). Therefore, for solving the problem (2.1), it suffices to solve the following problem:

$$\max\{f_1(x, y) | \quad (x, y) \in IR\}.$$

In other words, each optimal solution to the above problem is an optimal solution to the problem (2.1). We will also use the following assumptions to come up with the existence of an optimal solution.

Assumption 2.1. The constraint region S is a non-empty and compact polyhedron.

Assumption 2.2. $M(x)$ is non-empty and single-valued for each x forwarding of ULDM.

Assumption 2.3. All variables in vector x be in $I = \{1, \cdots, n_1\}$.

Assumption 2.4. All variables in vector x have $0 \le x \le K$.

Assumption 2.5. Matrix $A \in \mathbb{Z}^{m \times n_1}$.

3 Theorical properties

Before proving the main result on the optimal solution of problem (2.1), we list some preliminary definitions and results. Since f_1, f_2 are quasiconcave and the set of linear constriants S is not necessarily bounded, the MIBLFPLP problem is a particular case of quasiconcave bilevel problem [28].

Lemma 1. *Let (x^*, y^*) be an optimal solution to B(l,u,r) and y^0 be any solution such that $y^0 \geq 0$ and $y^0_j \in \mathbb{Z}, \forall j \in J$. We have either $c_{22}y^* \geq c_{22}y^0$ or $(Ax^* + By^0)_i > b_i, \exists i$.*

Proof. The proof is referred to [5]. □

Lemma 2. *Let (x^U, y^U) be an optimal solution to $U(\hat{l}, \hat{u}, \hat{r})$. Suppose y^L is an optimal to $LLFP(x^U)$ but y^U is not. The following (m+1) new nodes, denoted as $B(l^k, u^k, r^k), \forall k = 1, \cdots, (m+1)$, can be created from the parent node $B(\hat{l}, \hat{u}, \hat{r})$:*

For $k = 1, \cdots, m$, we have:

$$l^k_j = \begin{cases} \hat{l_j} & j = 1, \cdots, k-1, k+1, \cdots, m \\ [(b - By^L)_k] + 1 & j = k \end{cases}$$

$$u^k_j = \begin{cases} \min\{(b - By^L)_j], \hat{u_j}\} & j = 1, \cdots, k-1 \\ \hat{u_j} & j = k, \cdots, m \end{cases}$$

$r^k = r$.
For $k = m + 1$, we have:

$l^{m+1} = \hat{l}, \quad u^{m+1} = \min\{[(b - By^L)], \hat{u}\}, \quad r^{m+1} = c_{22}y^L$.

Proof. The proof is referred to [5]. □

3.1 Algorithm

We are now in a position to present the algorithm for solving the MIBLFPLP problem.

Step 3.1. Create the root node $B(l^1 = -\infty, u^1 = \infty, r^1 = -\infty)$ with $z^1 = \infty$. Initialize $x^* = \emptyset, y^* = \emptyset, \xi^* = -\infty, N = 1$. Go to Step 3.2.

Step 3.2. For all $k \in \{1, \cdots, N\}$ such that $z^k \leq \xi^*$ or $l^k_j > u^k_j, \exists j$; remove node k. Update N as the number of remaining nodes.

If $N = 0$ and $x^* \neq \emptyset$ then (x^*, y^*, ξ^*) is an optimal solution to the problem (2.1). Else, MIBLFPLP (2.1) is infeasible.

Select a node k from $\{1, \cdots, N\}$, set $(\hat{l} = l^k, \hat{u} = u^k, \hat{r} = r^k)$, remove node k. Reorder the remaining nodes from 1 to $N - 1$. Reduce N by 1 and go to Step 3.3.

Step 3.3. Solve $U(\hat{l}, \hat{u}, \hat{r})$. If $U(\hat{l}, \hat{u}, \hat{r})$ is infeasible then go to Step (3.2). Else, let $(x^H, y^H) = (\frac{v^H}{t}, \frac{w^H}{t})$ where $t > 0$ denote an optimal solution to $U(\hat{l}, \hat{u}, \hat{r})$. If $\frac{c_{11}x^H + c_{12}y^H + \alpha_1}{d_{11}x^H + d_{12}y^H + \beta_1} \leq \xi^*$ then go to Step (3.5). Else, go to Step 3.4.

Step 3.4. Solve $L(x^H)$. Let y^L denote an optimal solution to $L(x^H)$.

If $c_{22}y^H = c_{22}y^L$ then set $(x^* = x^H, y^* = y^H, \xi^* = \frac{c_{11}x^H + c_{12}y^H + \alpha_1}{d_{11}x^H + d_{12}y^H + \beta_1})$.

go to Step 3.2.

If $c_{22}y^H \neq c_{22}y^L$ and $\frac{c_{11}x^H+c_{12}y^L+\alpha_1}{d_{11}x^H+d_{12}y^L+\beta_1} > \xi^*$ then set $(x^* = x^H, y^* = y^L, \xi^* = \frac{c_{11}x^H+c_{12}y^L+\alpha_1}{d_{11}x^H+d_{12}y^L+\beta_1})$. Go to Step 3.5.

Step 3.5. Create $m+1$ new nodes, increase N by $m+1$, and go to Step 3.2. For $k = 1, \cdots, m$, node $(N+k)$, $B(l^{N+k}, u^{N+k}, r^{N+k})$ is defined as:

$$l_j^{N+k} = \begin{cases} \hat{l_j} & j = 1, \cdots, k-1, k+1, \cdots, m \\ [(b - By^L)_k] + 1 & j = k \end{cases}$$

$$u_j^{N+k} = \begin{cases} \{\min\{(b - By^L)_j], \hat{u_j}\} & j = 1, \cdots, k-1 \\ \hat{u_j} & j = k, \cdots, m \end{cases}$$

$$r^{N+k} = \hat{r}$$

$$z^{N+k} = \frac{c_{11}x^H+c_{12}y^H+\alpha_1}{d_{11}x^H+d_{12}y^H+\beta_1}$$

The $(N + m + 1)$th node is characterized by:

$$\begin{cases} l^{N+m+1} = \hat{l} \\ u^{N+m+1} = \min\{[b - By^L], \hat{u}\} \\ r^{N+m+1} = c_{22}y^L \\ z^{N+m+1} = z^{N+1} \end{cases}$$

We demonstrate the finite termination and the accuracy of the algorithm.

Theorem 1. *For any input MIBLFPLP (2.1), the algorithm terminates finitely.*

Theorem 2. *The output from the algorithm for the problem (2.1) is correct.*

4 Numerical Examples

In order to illustrate the procedure we consider the following bilevel problem:

Example 4.1. Consider the MIBLFPLP problem as follows:

$$\begin{aligned} &\max_{(x_1,x_2)} \quad f_1 = \tfrac{2x_1+3x_2+x_3}{x_1+4x_2+6} \\ &\max_{(x_3,x_4)} \quad f_2 = x_4 \\ &s.t. \quad x_1 + x_2 + x_3 - x_4 \leq 5 \\ &\qquad 2x_1 + x_2 + x_3 - x_4 \leq 9 \\ &\qquad 0 \leq x_1 \leq 5, \quad 0 \leq x_2 \leq 3 \\ &\qquad x_3,\ x_4 \geq 0. \end{aligned}$$

According to the proposed algorithm, the MIBLFPLP optimal solution is obtained $(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 9, 4)$ with $f_1 = \frac{3}{2}, f_2 = 4$.

5 Conclusion

In this paper, we introduce the branch and bound algorithm for the MIBLFPLP problem based on five simplifying assumptions. Our future work involves relaxing the simplifying assumptions 2, 3, 4 and 5 that the algorithm is dependent upon.

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The Solution of Two-Dimensional Fuzzy Nonlinear Volterra Integral Equations

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Abstract

Fuzzy integral can solve many problems in the sciences and furthermore these equations are applied to find the solutions of fuzzy integro-differential equations. These equations applied mathematics, physics and mechanics. The numerical solution of the two-dimensional nonlinear fuzzy Volterra integral equation of the second-kind (F-VIE2) with a based on a spectral method are discusses in this manuscript. The spectral method is them used to find the approximate solution of the two-dimensional nonlinear fuzzy Volterra integral equation of the second-kind (F2D-VIE2). The parametric form is applied to solve F-VIE2 and F2D-VIE2 by using the spectral scheme. The Gauss Legendre points and Legendre weights are utilized in this procedure. It is prove that the spectral method is a reliable, accurate and useful approach to solving these equations. Eventually, the practical examples are illustrated to demonstrate the improvement of results. A comparison between the solution of spectral method and Adomian method is presented, and the absolute error between the results obtained from the spectral method and the exact solution is calculated.

Keywords: Spectral method, Fuzzy Second-kind two-dimensional Volterra Integral equation (F2D-VIE2), Gauss-Legendre points.

1 Introduction

In engineering, especially in mechanics, applied mathematics and physics, integral equation is important. The solution of these equations is necessary, but obtaining the exact solution is challenging. It is clear that a large category of the boundary and initial value differential equations can be transformed into Volterra integral equations, and partial differential equations, such as heat and wave equations, can be converted to Volterra integral equations of the second kind. As we know, many integral equations have fuzzy elements with fuzzy variable parameters.

For the first time, fuzzy numbers were introduced by Zadeh [10]. Then the numerical solution of fuzzy integral equations was introduced by Friedman et al. in [3]. They used the embedding method, and sufficient conditions for the convergence of solutions were provided. The spectral schemes used in mathematics help in the scientific calculations for the numerical solution of equations, integro-differential equations, differential and integral equations. In this scheme the solution of the equation is expressed as a sum of certain basis functions. Spectral methods use basis functions on a nonzero domain. Essentially, the spectral method provides a method with convergence at an exponential rate [33].

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Jafarian et al. found the solution of fuzzy integral equations using the Taylor expansion method in [9]. Narayanamoorthy and Sathiyapriya studied the Homotopy perturbation method to solve the fuzzy Volterra integral equations in [1]. Salehi and Nejatiyan used Nystorm techniques in [12] to solve them, and Ebadian et al. studied the Picard method to find the solution of the fuzzy Volterra-Fredholm two-dimensional integral equations in [4]. The approximate solution of the fuzzy Fredholm two-dimensional integral equation of the second kind was found by Ezzati and Ziari, they used fuzzy polynomials of the kind of Bernstein to solve the equations in [5]. A modified homotopy perturbation method is employed for solving the two-dimensional fuzzy Fredholm integral equation by Rivaz and Yousefi in [11]. Hassasi and Ezzati used a fuzzy wavelet collocation procedure to find the numerical solutions of fuzzy Fredholm integral two-dimensional equations in [8]. The numerical solution of two-dimensional fuzzy Fredholm integral equations by using triangular functions is studied by Mirzaee et al. [10]. Alidema and Geogieva, solved nonlinear Volterra fuzzy two-dimensional integral equations by applying the Adomian decomposition method in [1]. Geogieva studied the nonlinear Volterra fuzzy two-dimensional integral equations using the homotopy analysis method, [7]. Tang et al. used a spectral scheme for solving crisp Volterra integral equations and conducted a convergence analysis of them in [33].

In this manuscript, the spectral method is applied to find the solution of the fuzzy two-dimensional nonlinear Volterra integral equations as follows:

$$\tilde{u}(x,y) \bigoplus \int_a^x \int_a^y K(x,y,s,t,\tilde{u}(s,t))dsdt = \tilde{g}(x,y), \qquad x,y \in [a,b], \tag{1.1}$$

where $K(x,y,s,t,\tilde{u}(s)) = K(x,y,s,t)\tilde{u}^n(s,t)$.

Some of the necessary definitions are provided briefly in the section 2. Section 3 expresses the spectral scheme using Legendre points and Lagrange polynomials to find the solution of the fuzzy nonlinear Volterra integral equation of the second kind. The procedure of spectral scheme is applied solving the nonlinear two-dimensional fuzzy Volterra integral equation of the second kind in section 4. Finally, examples are given to illustrate our method further. In one of the examples, a comparison between the spectral scheme and the Adomian decomposition method is displayed. Note that the symbol $\tilde{a}$ is indicated for the fuzzy set of a throughout this paper.

2 Basic definitions

Some basic definitions of fuzzy numbers are reviewed, in this section [2, 3, 10, 1]:

Definition 1. *Fuzzy set is a like* $\tilde{F} : \mathbb{R} \to [0,1]$ *is a fuzzy number which satisfies the below terms:*

a. Function $\tilde{F}$ *is an upper semi continuous function,*

b. Outside interval [a,d] $\tilde{F}(x) = 0$ *,*

c. For arbitrary real numbers b, c *such as* $a \leq b \leq c \leq d$ *and*

c.1 $\tilde{F}(x)$ *is a monotonic increasing function on [a, b],*

c.2 $\tilde{F}(x)$ *is a monotonic decreasing function on [c, d],*

c.3 $\tilde{F}(x) = 1$ *for all* $x \in [b,c]$.

Definition 2. *Pair* $(\underline{F}, \overline{F})$ *is a fuzzy number* $\tilde{F}$ *in the parametric form. Functions* $\underline{F}(r)$, $\overline{F}(r)$, $0 \leq r \leq 1$, *satisfy the requirements as follow:*

a. $\underline{F}(r)$ *is a bounded non-decreasing left continuous function in* $(0,1]$*, and right continuous at 0,*

b. $\overline{F}(r)$ *is a bounded non-increasing left continuous function in* $(0,1]$*, and right continuous at 0,*

c. $\underline{F}(r) \leq \overline{F}(r),\ 0 \leq r \leq 1.$

Definition 3. *Suppose that the fuzzy numbers* $\tilde{F} = (\underline{F}(r), \overline{F}(r))$ *and* $\tilde{G} = (\underline{G}(r), \overline{G}(r))$*. and for crisp number* k*, the addition and scalar multiplication of the fuzzy numbers are defined using in the following:*
a) $\tilde{F} = \tilde{G}$ *if and only if* $\underline{F}(r) = \underline{G}(r)$ *and* $\overline{F}(r) = \overline{G}(r)$*,*
b) $\tilde{F} \bigoplus \tilde{G} = (\underline{F}(r) + \underline{G}(r), \overline{F}(r) + \overline{G}(r))$*,*
c) $k \bigotimes \tilde{F} = \begin{cases} (k\underline{F}(r), k\overline{F}(r)), & k \geq 0, \\ (k\overline{F}(r), k\underline{F}(r)), & k < 0 \end{cases}$

3 Fuzzy Legendre-Spectral Method to solve Fuzzy F2D-VIE2

In this section, the two-dimensional nonlinear fuzzy Volterra integral equation of the second-kind in Eq. (1) is considered. Let this equation inserted at the Legendre point pairs (x_i, y_j), for $i, j = 1, 2, ..., N$. The linear transformation and techniques are applied in the two-dimensional (2D) case yields

$$\tilde{u}(\sigma, \delta) \approx \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{u}_{i,j} F_{i,j}(\sigma, \delta), \tag{3.1}$$

where

$$\tilde{u}_{i,j} \bigoplus \frac{x_i + 1}{2} \frac{x_j + 1}{2} \bigotimes \sum_{p=0}^{N} \sum_{l=0}^{N} K(x_i, y_j, s(x_i, \theta_p), s(x_i, \theta_l) \tilde{u}(s(x_i, \theta_p), s(x_i, \theta_l))) \omega_p \omega_l = \tilde{g}(x_i, y_j). \tag{3.2}$$

By putting the nods as the set of $(N+1)$ Legendre $\{x_i\}_{i=1}^{N}$ and $\{y_j\}_{j=1}^{N}$, Eq. (2) holds that the following term:

$$\tilde{u}(x_i, y_j) \bigoplus \int_{-1}^{x_i} \int_{-1}^{y_j} K(x_i, y_j, s, t, \tilde{u}(s, t)) ds dt = \tilde{g}(x_i, y_j), \qquad 0 \leq i, j \leq N, \tag{3.3}$$

with the following linear convert:

$$s(x, \theta) = \frac{x+1}{2}\theta + \frac{x-1}{2}, \qquad s(y, \beta) = \frac{y+1}{2}\beta + \frac{y-1}{2}, \qquad -1 \leq \theta, \beta \leq 1, \tag{3.4}$$

with placement Eq. (5) in Eq. (4) becomes:

$$\tilde{u}(x_i, y_j) \bigoplus \frac{1 + x_i}{2} \frac{1 + y_j}{2} \bigotimes \int_{-1}^{1} \int_{-1}^{1} K(x_i, y_j, s(x_i, \theta), s(y_j, \beta), \tilde{u}(s(x_i, \theta), s(y_j, \beta))) d\theta d\beta = \tilde{g}(x_i, y_j), \quad 0 \leq i, j \leq N \tag{3.5}$$

Using the $(N+1)-$point Gauss quadrature formula relative to the Legendre weights $\{\omega_k\}$ we have the following equation:

$$\tilde{u}(x_i, y_j) \bigoplus \frac{1 + x_i}{2} \frac{1 + y_j}{2} \bigotimes \sum_{p=0}^{N} \sum_{l=0}^{N} K(x_i, y_j, s(x_i, \theta), s(y_j, \beta)), \tilde{u}(s(x_i, \theta), s(y_j, \beta)) \omega_p \omega_l = \tilde{g}(x_i, y_j), \quad 0 \leq i, j \leq N \tag{3.6}$$

The value of $\tilde{u}(s(x_i,\theta),s(y_j,\beta))$ is approximated by $\tilde{u}_{i,j}$ using the relation between the Lagrange interpolation polynomials associated with Lagrange collocation points, similar to the one-dimensional (1D) case. expanded $\tilde{u}$ is expanded using Lagrange interpolation polynomials as follows:

$$\tilde{u}(\sigma,\delta) \approx \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{u}_{i,j} F_{i,j}(\sigma,\delta), \tag{3.7}$$

and

$$K(x_i,y_j,s(x_i,\sigma),s(y_j,\delta),\tilde{u}(s(x_i,\sigma),s(y_j,\delta))) = K(x_i,y_j,s(x_i,\sigma),s(y_j,\delta)) \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{u}_{i,j}^{n} F_{i,j}(\sigma,\delta), \tag{3.8}$$

where $F_{i,j}$ is the j-th Lagrange basic function. Combining two Equations (7) and (9) then yields

$$\tilde{u}_{i,j} \bigoplus \frac{1+x_i}{2}\frac{1+y_j}{2} \sum_{q=0}^{N}\sum_{r=0}^{N} \tilde{u}_{q,r}^{n} \bigotimes (\sum_{p=0}^{N}\sum_{l=0}^{N} K(x_i,y_j,s(x_i,\theta),s(y_j,\beta)) F_{pl}((s(x_i,\theta)),(s(y_j,\beta)))\omega_p\omega_l) = \tilde{g}(x_i,y_j), \tag{3.9}$$

where $0 \le i,j \le N$. Denoting

$$\tilde{U}_{MN} + B \bigotimes \tilde{U}_{MN}^{n} = \tilde{G}_N, \tag{3.10}$$

where the entries of matrix B is given by

$$B_{i,j} = \frac{1+x_i}{2}\frac{1+y_j}{2} \bigotimes \sum_{p=0}^{N}\sum_{l=0}^{N} K(x_i,y_j,s(x_i,\theta),s(y_j,\beta)), F_{i,j}(s(x_i,\theta),s(y_j,\beta))\omega_p\omega_l, \tag{3.11}$$

of the matrix form:

It is know that

$$\tilde{U}_{M,N} = \begin{bmatrix} \tilde{u}_{0,0} & \tilde{u}_{0,1} & ... & \tilde{u}_{0,N}, \\ \tilde{u}_{1,0} & \tilde{u}_{1,1} & ... & \tilde{u}_{1,N}, \\ \vdots & & & \\ \tilde{u}_{N,0} & \tilde{u}_{N,1} & ... & \tilde{u}_{N,N}, \end{bmatrix}, \tag{3.12}$$

and

$$\tilde{G}_N = \begin{bmatrix} \tilde{g}(x_0,y_0) & \tilde{g}(x_0,y_1) & ... & \tilde{g}(x_0,y_N), \\ \tilde{g}(x_1,y_0) & \tilde{g}(x_1,y_1) & ... & \tilde{g}(x_1,y_N), \\ \vdots & & & \\ \tilde{g}(x_N,y_1) & \tilde{g}(x_N,y_1) & ... & \tilde{g}(x_N,y_N), \end{bmatrix}, \tag{3.13}$$

$$F_{i,j}(s,t) = \sum_{p=0}^{N}\sum_{l=0}^{N} \alpha_{p,j} L_{p,l}(x,y), \tag{3.14}$$

where $L_j(s)$ are Legendre functions and $L_{i,j} = L_i(x)L_j(y)$ for $0 \le i,j \le N$, $L_{i,j}(x_r,y_s) = \begin{cases} 1, & i=r, j=s, \\ 0, & otherwise \end{cases}$
and $\alpha_{p,j}$ is discrete polynomial coefficients.

$$\alpha_{p,j}(s) = \frac{1}{\gamma_p}\sum_{i=0}^{N} F_{i,j}(x_i, y_j)L_p(x_i)L_p(y_j)\omega_i\omega_j = \frac{L_p(x_j)\omega_j}{\gamma_p} \tag{3.15}$$

where

$$\gamma_p = \sum_{i=0}^{N} L_p^2(x_i)\omega_i = (p+\frac{1}{2})^{-1}, p < N \tag{3.16}$$

which $\gamma_N = (N+\frac{1}{2})^{-1}$. By using the Gauss formulas we have a following equation:

$$F_{i,j}(s,t) = \sum_{p=0}^{N}\sum_{l=0}^{N} \frac{L_p(x_j)L_l(y_j)L_p(s)L_l(t)\omega_j}{\gamma_p\gamma_l} \tag{3.17}$$

In above equations we have $\tilde{U}_{MN} = (\underline{U}_{MN}, \overline{U}_{MN})$ and $\tilde{G}_N = (\underline{g}_N, \overline{g}_N)$.

4 Numerical Examples

Two examples are presented in this section. They are solved using the spectral method with Legendre nodes and weights. The exact solution and the approximate solution of the spectral scheme are compared, and comparative tables have been provided. Without loss of generality, the Gauss-Laguerre points are used, meaning that the zeros of $L_{(N+1)}(x)$ are applied for the nods. The weights of Legendre-Gauss-Laguerre points are as follows:

$$w_j = \frac{x_j}{(n+1)^2[L_{n+1}(x_j)]^2}, \quad 0 \le j \le n.$$

In example (4.1), the Adomian decomposition method (ADM) is applied to compare the Legendre-spectral scheme with other methods. This example is illustrated in [1]. The comparison between ADM and the Gauss Legendre spectral method is presented.

Example 4.1. Nonlinear fuzzy two-dimensional Volterra integral equation is considered as follows:

$$u(x,y)\bigoplus[\frac{1}{16}x^5y^5(2+r)^2 - xy(2+r), \frac{1}{16}x^5y^5(4-r)^2 - xy(4-r)] = \int_0^y\int_0^x (xyst)u^2(s,t)dsdt.$$

$u(x,y) = [xy(2+r), xy(4-r)]$ is the exact solution. We used the spectral scheme for it. The solution of the Adomian perturbation method is considered and the absolute errors for $\underline{u}(x,y)$ and $\overline{u}(x,y)$ for $x = 0.2$ and $y = 0.7$ are shown in Table 4.1. The results in Table 4.) confirm that the spectral method is accurate and fast and it can be used to find the solutions.

Example 4.2. In this example the nonlinear fuzzy two-dimensional Volterra integral equation is considered in the following term:

$$u(x,y)\bigoplus[(xsiny - \frac{x^5}{4} + \frac{x^5cosy}{4} - \frac{x^2sin^2y}{4})(r^2+r), (xsiny - \frac{x^5}{4} + \frac{x^5cosy}{4} - \frac{x^2sin^2y}{4})(4-r^3-r)]$$

$$= \int_0^y\int_0^x (xt^2 + cost)u^3(s,t)dsdt.$$

The exact solution is $u(x,y) = [xsiny(r^2+r), xsiny(4-r^3-r)]$. We used our method for this equation and the absolute errors for $\underline{u}(x,y)$ and $\overline{u}(x,y)$ for $x = 0.5$ and $y = 0.7$ are shown in Table 4.2.
The results in this example also demonstrates the high precision of the spectral method.

Table 4.1: Results for Example 4.1//

r	$\lvert e(\underline{u})\rvert, N = 4$	$\lvert e(\underline{u})\rvert, N = 8$	$\lvert e(\underline{u})\rvert, N = 12$	$\lvert e(\underline{u})\rvert, N = 20$	$e(\underline{u})(ADM)$	$\lvert e(\overline{u})\rvert, N = 4$	$\lvert e(\overline{u})\rvert, N = 8$	$\lvert e(\overline{u})\rvert, N = 12$	$\lvert e(\overline{u})\rvert, N = 20$	$e(\overline{u})(ADM)$
0	0	0.003E(-4)	3.903E(-8)	5.344E(-10)	0	2.812E(-4)	2.230E(-4)	3.589E(-6)	3.528E(-12)	0.00043
0.1	4.34E(-2)	2.923E(-5)	4.321E(-8)	3.431E(-10)	0.000015	2.129E(-4)	2.146E(-6)	3.341E(-8)	2.458E(-11)	0.00016
0.2	0.12E(-2)	2.005E(-4)	0.181E(-6)	4.013E(-10)	0.000039	2.025E(-3)	3.763E(-5)	2.227E(-6)	2.399E(-10)	0.00016
0.3	1.32E(-2)	3.293E(-4)	1.750E(-6)	3.451E(-10)	0.000061	3.531E(-3)	4.421E(-5)	3.473E(-7)	3.219E(-10)	0.00015
0.4	1.34E(-3)	5.305E(-5)	2.234E(-7)	4.801E(-11)	0.000039	0.238E(-3)	3.817E(-4)	1.649E(-6)	4.292E(-10)	0.00015
0.5	0.03E(-3)	1.243E(-5)	1.263E(-6)	1.220E(-12)	0.000043	3.050E(-2)	4.137E(-4)	3.471E(-6)	3.318E(-12)	0.00016
0.6	2.25E(-2)	4.011E(-4)	2.053E(-6)	2.231E(-12)	0.000051	4.104E(-3)	3.564E(-4)	4.672E(-6)	3.291E(-12)	0.00013
0.7	4.295E(-2)	0.241E(-4)	4.123E(-6)	1.323E(-12)	0.000016	3.431E(-3)	3.671E(-5)	3.714E(-6)	4.008E(-11)	0.00012
0.8	1.25E(-2)	3.235E(-4)	3.363E(-6)	8.387E(-12)	0.000029	3.463E(-3)	3.034E(-5)	2.313E(-7)	0.330E(-11)	0.00011
0.1	2.73E(-2)	3.202E(-4)	3.334E(-6)	2.732E(-12)	0.000071	2.334E(-3)	3.001E(-6)	2.073E(-6)	2.231E(-10)	0.00025
1	1.84E(-2)	2.9E(-5)	5.641E(-7)	5.861E(-12)	0.000053	3.04E(-2)	3.132E(-4)	1.275E(-6)	0.623E(-12)	0.00009

Table 4.2: Result in Example 4.2//

r	$\lvert e(\underline{u})\rvert, N = 4$	$\lvert e(\underline{u})\rvert, N = 8$	$\lvert e(\underline{u})\rvert, N = 12$	$\lvert e(\underline{u})\rvert, N = 20$	$\lvert e(\overline{u})\rvert, N = 4$	$\lvert e(\overline{u})\rvert, N = 8$	$\lvert e(\overline{u})\rvert, N = 12$	$\lvert e(\overline{u})\rvert, N = 20$
0	1.254E(-2)	3.275E(-4)	3.378E(-7)	1.142E(-12)	3.232E(-2)	3.500E(-5)	0.708E(-6)	1.128E(-12)
0.1	3.208E(-2)	2.103E(-4)	4.334E(-6)	2.214E(-12)	2.239E(-3)	4.345E(-5)	1.102E(-6)	0.048E(-11)
0.2	4.256E(-3)	4.733E(-5)	1.138E(-7)	2.113E(-13)	0.985E(-2)	0.003E(-6)	2.033E(-6)	9.045E(-12)
0.3	1.021E(-2)	5.374E(-5)	0.043E(-6)	2.442E(-13)	3.543E(-3)	4.763E(-5)	3.643E(-7)	7.228E(-12)
0.4	0.187E(-2)	3.03E(-5)	1.213E(-7)	3.304E(-13)	1.229E(-3)	4.233E(-5)	1.449E(-6)	2.870E(-13)
0.5	1.183E(-3)	1.318E(-5)	5.143E(-6)	1.483E(-12)	3.823E(-4)	0.231E(-5)	3.761E(-6)	2.252E(-13)
0.6	2.325E(-3)	1.316E(-6)	3.333E(-7)	2.043E(-12)	1.021E(-3)	5.133E(-6)	4.062E(-8)	4.146E(-13)
0.7	3.125E(-3)	1.043E(-5)	2.229E(-6)	2.123E(-12)	9.232E(-4)	5.234E(-5)	0.281E(-7)	2.008E(-12)
0.8	2.142E(-3)	0.122E(-5)	9.004E(-7)	1.319E(-12)	1.002E(-4)	9.703E(-6)	3.334E(-6)	2.113E(-12)
0.9	0.101E(-2)	1.043E(-6)	0.534E(-7)	1.033E(-12)	0.402E(-3)	5.203E(-6)	9.233E(-8)	3.078E(-12)
1	3.321E(-3)	1.239E(-5)	2.043E(-7)	9.002E(-13)	1.907E(-3)	4.633E(-5)	1.001E(-6)	2.237E(-12)

5 Conclusion

A successful method to solve fuzzy two-dimensional Volterra integral equations of the second kind with fuzzy nonlinear kernel was studied in this manuscript. The The approximate and numerical method named spectral scheme is applied to solve these equations. Legendre-spectral scheme is highly accurate method with the Gauss Legendre nods that use Lagrange interpolation polynomials. The absolute error of the applying procedure was obtained. The obtained data prove that the convergence rate is very fast. A comparison between the Legendre-spectral method and the Adomian decomposition method was shown, and the results proves that the Legendre-spectral scheme is much more accurate. In this article, all calculations were done using Maple software 20.

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Mathematical Modeling of Biochemical Nonlinear Systems: Basic Enzyme Kinetics and Feedback Controls Reactions

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Abstract

Nowadays, any bioscientist knows biochemical reactions play the key role in all processes of living organisms. The biological and biochemical mechanisms have complexity process, and so, development of a simplifying model is essential to understand the phenomenon under some considerations. Thus, we should use reaction mechanisms which are plausible biochemically. The main goal is to present mathematical models for enzymes immobilized within membranes. In this paper, we model the second order biochemical reaction mechanism which is known as basic enzyme reaction. This model mirror, not only a large number of real reactions but also some general types of reaction phenomena. Furthermore, their corresponding mathematical realizations may be understood. Regarding most of the biological models have feedback control built into them; we want to extend the above second order reaction model to the biochemical reaction mechanism having more than one autocatalysis. The process of catalysis and autocatalysis indeed play the important role in these reactions. By help of mathematical modeling for the above reactions, we obtain systems of ordinary differential equations. Having initial condition for the reactants at the start of the process, we find out an initial value problem. The main aim is investigation of solutions behavior.

Keywords: Enzyme Reaction, Laws of Mass Action, Autocatalysis, Logistic growth, Lotka-Volterra Differential Equations.

1 Introduction

We first present short literature of mathematics, bioscience, and needful definitions; and then approach to mathematical modeling in biochemistry, especially enzyme. Mathematics has benefited from its involvement with developing sciences. Both science of biochemistry and biology for their continuing have to not only to involve with mathematics but also use mathematics more very more than last centuries. Mathematicians, chemists should take help from each other to do the above important job. Practical application of mathematical models in helping to unravel the underlying mechanisms involved in the chemical and process is so important. Biochemical reactions play the key roles in all processes of living organisms such that: biology, ecology, environmental sciences. It is obvious that the first model to be studied may itself be a model of a more realistic, but still to more complicated biochemical model. Most of the biochemical reactions

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Table 1: Denoted Symbols to Introduce Meaning

Symbol	Meaning
$\rightleftharpoons$	Invertible reaction
$\rightharpoonup$	Reaction only one way
$\Longrightarrow$	Imply
$\longrightarrow$	Tend
$[]$	Reactantcon centration
$\equiv$	Proportional
$\approx$	Nearlye quality
$\ll$	Less than so much
$\forall$	Universal quantifier
$\exists$	Existential quantifier
$\mathbb{R}$	The set of real numbers
IVP	Initial Value Problem

are catalyzed by enzymes. But in living cell reaction interacts with diffusion because enzymes act within structured systems: either bound to cell organelles such as mitochondria. Though enzyme reactions have been used for thousands of years in fermentations, it was not before the nineteenth century that the idea became clear that the biochemical reactions are not under the influence of some mysterious principle of Life, but are subject to the usual Law of Mass Action like other chemical reactions, and are catalyzed by enzymes. In 1835, Berzelius suggested that most of the biochemical reactions could occur under the influence of a new force which he calls "catalytic". In 1878, Kiihne coined the term "enzyme" for this catalytic force, from a Greek word. For more details on enzyme reactions, [1] and [3]. In 1902, Brown suggested that the enzyme E combines with its substrate, S, to form an intermediate complex which then decomposes into the product, P, and E. The most basic enzyme reactions in which introduced by Michaelis Menten in 1913, [4]. Before modeling, we introduce the Table of symbols and their meaning (Table 3).

Definition 1. *Molecules which catalyze the biochemical reactions in the metabolic pathways of living organisms, whence their importance are called enzymes.*

Definition 2. *The process of making a chemical reaction happen quickly by using a catalyst is called catalysis. Indeed a substance catalyst modifie the rate of chemical reaction without undergoing any change itself.*

Definition 3. *The process that a chemical is involved in its own production is called autocatalysis.*

Definition 4. *A point that all of its derivatives of system are zero is called as equilibrium, critical or stationary point.*

2 Main results

Now, we are going to form and model the second order invertible biochemical reaction mechanism. This is known as basic enzyme reaction. In this reaction, an enzyme plays as catalysis role. This model to be studied itself may be more realistic, but still too complicated biochemical model. Because of it is qualitative understanding that we want initially, models of models are first steps. The first model which will be analyzed is the second order invertible having the enzyme as catalysis. This reaction is starting point to our work. The second one is case of having autocatalysis process. The models related to the last reaction employ the logistic growth rate, Lotka reaction mechanism and the system of Lotka-Volterra equations. The case of having at least two reactions led to a system of nonlinear ordinary differential equations. These are known as nonlinear systems of the Lotka-Volterra equations. After it, we model some reactions including at least

three autocatalysis. By finding out the solutions related models, we interpret the concentration function on the basis of time an existing parameter in reaction. Reading any differential equation is led to an integral, we see that for most cases, we cannot find integral solution. That is the solution of differential equation may not be obtained. And so, we have to interpret the reaction on the basis of the behavior of solution by using theory of differential equations, bifurcation, chaos, etc. In this regard, it is obvious without having the solutions of system of differential equation, we find their behavior. For more details on differential equations, [6]. In this section, we study some of the second order reactions. This section indeed consists two subsections entitled: I The series of invertible reactions having an enzyme as catalysis process, II Biochemical reactions having feedback controls, multi autocatalysis process. The Law of Mass Action help us so much to model the above biochemical reactions.

2.1 The series of invertible reactions having an enzyme as catalysis process

Firstly, we are going to formulate the simplest form an enzyme reaction. Assume that firstly there are two reactants enzyme and substrate which are denoted by E and S respectively. Moreover, one molecule of S combines with a molecule of E to form one molecule of SE. Finally produce molecule of P and one molecule of E. In fact, there is a series of reactions where the first reaction is invertible, meanwhile the second one is not. The following reaction, therefore, may be obtained:

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} SE \overset{k_2}{\rightharpoonup} P + E \tag{2.1}$$

If $k_1 + k_2 \geq k_{-1}$ and when time approaches to infinity, the overall reaction is a conversion of and to form product with constant parameter $S \rightharpoonup P$. Now, we complete the above mathematical modeling. We consider initial conditions which one may take as those at the start of the process which converts S to P. Both of substrate S and enzyme E are zero at the final stage state complex concentration. To model of the above reaction which is a phenomenon, some notations, symbolling, assumptions and laws for reactants will be considered as follows:

1. The Law of mass action describes that the rate of reaction is proportional to the product of the concentrations of the reactant;

2. In any reaction kinetics problem, concentration is nonnegative;

3. Considering symbol [ ] to denote concentration. Thus $s = [S]$, $e = [E]$, $c = [SE]$ and $p = [P]$ show the concentration of the existing reactants in the reaction 2.1;

4. The capital letters S, E, SE and P are considered to show reactants matters as: substrate; enzyme, complex obtained matter and product respectively;

5. The constant parameters associated with the rates of reactions are denoted by k_1, k_{-1} and k_2. respectively. These parameters are positive.

Paying attention to reaction 2.1 one is able to see the system of proportions in which statements the rate of concentrations change as follows:

$$\frac{ds}{dt} \equiv s, e \quad \frac{de}{dt} \equiv s, e, c \quad \frac{dc}{dt} \equiv s, e, c \quad \frac{dp}{dt} \equiv c \tag{2.2}$$

Using the Law of Mass Action in reactants and 2.2, one is able to obtain the following nonlinear system of ordinary differential equations:

$$\begin{cases} \frac{ds}{dt} = -k_1 es + k_{-1} c \\ \frac{de}{dt} = -k_1 es + (k_{-1} + k_2) c \\ \frac{dc}{dt} = k_1 es - (k_{-1} + k_2) c \\ \frac{dp}{dt} = k_2 c \end{cases} \tag{2.3}$$

The four equations in above system are exhibit as follows:

1. The first equation statements that the rate of concentration change $[S]$ is made up of decreasing rate of $[S][E]$and again rate proportional to $[SE]$.
2. The second one statements that the rate of concentration change $[E]$ is made up of a loos rate of $[S][E]$ and again rate proportional to$[C]$.
3. The third one statements that the rate of concentration change $[C]$ is made up of increasing rate of $[S][E]$ and decreasing rate proportional to $[C]$.
4. The last equation statements that the rate of concentration change $[P]$ is made up just increasing rate of $[C]$. Moreover we see that this equation is uncoupled from the first three equations.

Theorem 2.1. *Consider the Reaction 2.1 with the following initial values:*

$$s(0) = s_0, \quad e(0) = e_0, \quad c(0) = 0, \quad p(0) = 0.$$

Then the reactants concentration are given by following implicit functions:

$$\begin{cases} s(t) + k_m \ln s(t) = s_0 + k_m \ln s_0 \\ c(t) = \frac{e_0 s(t)}{s(t)+k_m}, \quad k_m = \frac{k_{-1}+k_2}{k_1} \\ e(t) = e_0 - c(t) = e_0 - \frac{e_0 s(t)}{s(t)+k_m} \\ p(t) = k_2 \int_0^t c(x)\, dx = k_2 \int_0^t \frac{e_0 s(x)}{s(x)+k_m} dx. \end{cases}$$

Proof. The system 2.3 with the initial values constructs following IVP:

$$\begin{cases} \frac{ds}{dt} = -k_1 es + k_{-1} c & s(0) = s_0 \\ \frac{de}{dt} = -k_1 es + (k_{-1} + k_2) c & e(0) = e_0 \\ \frac{dc}{dt} = k_1 es - (k_{-1} + k_2) c & c(0) = 0 \\ \frac{dp}{dt} = k_2 c & p(0) = 0 \end{cases} \tag{2.4}$$

Solving the 2.3, one is able to find out the amount of concentration and the rate of reactions for reaction 2.1. If we combine the second and third equations in 2.3, we obtain $\frac{de}{dt} + \frac{dc}{dt} = 0$, which implis that

$$e(t) = e_0 - c(t) \tag{2.5}$$

As the last equation in system 2.3 is uncoupled from another, we obtain $p(t)$ as

$$p(t) = k_2 \int_0^t c(x)\, dx \tag{2.6}$$

And so IVP 2.4 may be reduced to the following IVP:

$$\begin{cases} \frac{ds}{dt} = -k_1 e_0 s + k_1 s + k_{-1} c & s(0) = s_0 \\ \frac{dc}{dt} = k_1 e_0 s - (k_1 s + k_{-1} + k_2) c & c(0) = 0 \end{cases} \tag{2.7}$$

Because of no reactant $c = [SE]$ at initial stage and moreover in this stage, the concentrations of initial reactants $s = [S]$, $e = [E]$ are so much, we can assume formation complex $c = [SE]$ is very fast at equilibrium point.

$$\frac{dc}{dt} \approx 0, \quad c(t) = \frac{e_0 s(t)}{s(t) + k_m}, \quad k_m = \frac{k_{-1} + k_2}{k_1} \tag{2.8}$$

Where k_m is known as Michaelis constant.

$$\frac{ds}{dt} = \frac{k_2 e_0 s}{s + k_m}, \quad s(0) = s_0 \tag{2.9}$$

As regarding an enzyme is concerned to be present in small amount compared with the substrate, the assumption is that the substrate concentration does not change during this initial transit stage. In this case, dynamics is governed by 2.9 approximately. This state is called as quasi-steady state approximation. and so er get the solution as the following implicit function:

$$s(t) + k_m \ln s(t) = s_0 + k_m \ln s_0 \tag{2.10}$$

Paying using 2.1, 2.6, 2.9 and 2.10, we hence obtain the solution of 2.4 in the form of following functions:

$$\begin{cases} s(t) + k_m \ln s(t) = s_0 + k_m \ln s_0 \\ c(t) = \frac{e_0 s(t)}{s(t)+k_m}, \quad k_m = \frac{k_{-1}+k_2}{k_1} \\ e(t) = e_0 - c(t) = e_0 - \frac{e_0 s(t)}{s(t)+k_m} \\ p(t) = k_2 \int_0^t c(x)\, dx = k_2 \int_0^t \frac{e_0 s(x)}{s(x)+k_m}\, dx \end{cases} \tag{2.11}$$

The solution of system 2.4 is the reactants concentrations for reaction 2.1. Therefore, the proof is done. □

We see that all of solutions of 2.3 are obtained as implicit functions. As they are not explicit functions, we are going to study the behavior of solutions. First of all, we should determine the equilibrium points.

2.1.1 Equilibrium point

Theorem 2.2. *The system 2.3 has just trivial equilibrium point.*

Proof. To find out the equilibrium point of system 2.3, we should take

$$\frac{ds}{dt} = \frac{de}{dt} = \frac{dc}{dt} = \frac{dp}{dt} = 0. \tag{2.12}$$

It implies that

$$\begin{cases} -k_1 es + k_{-1} c = 0 \\ -k_1 es + k_{-1} + k_2 c = 0 \\ k_1 es - (k_{-1} + k_2) c = 0 \\ k_2 c = 0 \end{cases} \tag{2.13}$$

First, from the last equation in 2.13, one sees that $c = 0$. Setting this obtained value of c in other equations in (18), we see that

$$es = 0 \quad \Rightarrow \quad (e = 0 \text{ or } s = 0).$$

For the last component, there is no need to set a condition, that is, p is an arbitrary non-negative real number of $\mathbb{R}$. Thus, the equilibrium point $E = (s, e, c, p)$ should be as follows if it exists:

$$\begin{cases} E_1 = (0, e, 0, p) \quad \text{for} \quad \forall e, p \in \mathbb{R} \\ E_2 = (s, 0, 0, p) \quad \text{for} \quad \forall s, p \in \mathbb{R}. \end{cases} \tag{2.14}$$

The first relation in 2.14 states that no substrate reactant S exists in the reaction at the initial stage. And so, in this case, in the absence of substrate reactant S or enzyme reactant E, any reaction may not appear. Thus, the system 2.3 in the absence of substrate reactant S has no equilibrium point. Therefore, the point $E_1 = (0, e, 0, p)$ cannot be an equilibrium point. Also, the second relation in 2.14 states that the amount of enzyme reactant E in the reaction is zero at the initial stage. It means that there is no enzyme which plays a catalytic role in the reactio, and so, there is no any equilibrium point. Therefore, the proof is done. □

2.2 Biochemical Reactions Having Feedback Controls, Multi Autocatalysis Process

Regarding most of the biological models have feedback control built into them; we are going to extend basic enzyme reaction to bio-reaction mechanisms which having enzyme reactants as autocatalysis. Then we consider some cases including at least two enzyme reactants as autocatalysis. In these models, autocatalysis are very important and they have feedback control as activation or inhibition which their effects is nonlinear on reaction. Simply, autocatalysis is the process where a chemical is involved in its own production. In this subsection some models will be described which are models such as invertible enzyme reaction, series models of two parallel reactions, series of three parallel reactions, series of four parallel reactions; and finally model of series parallel multi reactions.

2.2.1 Model of Invertible Enzyme Reaction as Autocatalysis Process

To form the first model, let us consider a molecule of A combines with one of X to form two molecules of X. Moreover, assume that A is maintained at a constant concentration $a = [\mathrm{A}]$. Also assume that the concentration of X is denoted by $x = [\mathrm{X}]$. Thus, the following reaction may be obtained:

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X \tag{2.15}$$

we model enzyme reaction with autocatalysis process. Let us consider a molecule of A combines with one of X to form two molecules of X and $a = [\mathrm{A}]$. After modeling we obtain the logistic growth model [2] [5], [6].

Theorem 2.3. *Consider reaction 2.15 and assume that time variable t tend to infinty. Then*

$$x(t) \to x_s = \frac{k_1 a}{k_{-1}}.$$

Proof. The proof is presented in brief. Using the Law of Mass Action and regarding the above model as second order and invertible, we see that $\frac{dx}{dt} \equiv a, x$ and x^2.

In the above formula, a and x play increasing roles in the product ax, while x^2 acts as a decreasing factor. This leads to the well-known logistic growth model:

$$\frac{dx}{dt} = k_1 ax - k_{-1}x^2 = x(k_1 a - k_{-1}x)$$

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}) \tag{2.16}$$

where the parameters $r = k_1 a$ and $K = \frac{k_1 a}{k_{-1}}$ represent the growth rate and carrying capacity respectively. Equation 2.16 has two equilibrium points:

$$x = 0 \text{ and } x_s = \tfrac{k_1 a}{k_{-1}}.$$

The solution to equation 2.16 is $x(t) = \frac{x_0}{x_0+(1-x_0)e^{-rt}}$ which implies that

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^{-k_1 at}} \tag{2.17}$$

As $t \to \infty$, we get

$$x(t) \to x_s = \frac{r}{k_{-1}} = \frac{k_1 a}{k_{-1}} \tag{2.18}$$

Therefore, the proof is done. □

2.3 Model of Series of Parallel Reactions with Enzyme Autocatalysis

Extending reaction (2.15) by adding reactant B:

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X,\ B + X \xrightarrow{k_2} C \tag{2.19}$$

Theorem 2.4. *Consider 2.19 and let* $r = k_1 a - k_2 b$ *and* $K = \frac{k_{-1}}{k_1 a - k_2 b}$.
The eqution related to reaction this reaction is following logistic equation

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}).$$

Proof. The proof is presented in brief. The rate equation becomes:

$$\frac{dx}{dt} = k_1 ax - k_{-1}x^2 - k_2 bx$$

$$\frac{dx}{dt} = x(k_1 a - k_2 b) - k_{-1}x^2$$

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) \tag{2.20}$$

where $r = k_1 a - k_2 b$ and $K = \frac{k_{-1}}{k_1 a - k_2 b}$. □

2.4 Biochemical Reactions Having Feedback Controls, Multi Autocatalysis Process

$$A + X \xrightarrow{k_1} 2X,\ \rightharpoonup X + Y \xrightarrow{k_2} 2Y\ Y \xrightarrow{k_3} B \tag{2.21}$$

Theorem 2.5. *Consider 2.21. The equation related to reaction this reaction is following nonlinear systems Lotka- Volterra equations:*

$$\frac{dx}{dt} = x(k_1 a - k_2 y), \frac{dy}{dt} = y(k_2 x - k_3).$$

2.5 Model of Series of Parallel Four Reactions Having Three Enzymes as Autocatalysis

We consider an extension reaction 2.21 while by adding another reactant Z such that a molecule of Y combines with one of Z to form two molecules of Z . We denote concentration of the reactant by lowercase letter as $z = [Z]$.

$$A + X \overset{k_1}{\rightharpoonup} 2X,\ X + Y \overset{k_2}{\rightharpoonup} 2Y\ Y + Z \overset{k_3}{\rightharpoonup} 2Z\ \ Z \overset{k_4}{\rightharpoonup} B \tag{2.22}$$

Theorem 2.6. *Consider 2.22. The eqution related to reaction this reaction is following nonlinear systems Lotka- Volterra equations:*

Consider 2.22. The eqution related to reaction this reaction is following nonlinear systems Lotka- Volterra equations:

$$\frac{dx}{dt} = k_1 ax - k_2 xy, \qquad \frac{dy}{dt} = k_2 xy - k_3 yz, \qquad \frac{dz}{dt} = k_3 yz - k_4 z.$$

2.6 Generalized Model of Series of Parallel Reactions

we want to extend the reaction (2.22). Indeed this series is conctructed from n reactions. For simplifying, we change our notation. By adding some another reactants $X_1, X_2, X_3, X_4, \ldots,$ and finally X_n . We form a series of chain reactions. The first reaction is similar the first reaction in reaction (2.22). Indeed, a molecule of A combines with one of X_1 to form two molecules of $X_1 2$. The second reaction describes that a molecule of X_1 combines with one of X_2 to form two molecules of X_2 . In continuity and also same last reaction appears. In the stage of n^{th} reactions, a molecule of X_{n-1} combines with one of X_n to form two molecules of X_n . And finally for last reaction like preciously models assume that a molecule of X_n forms one molecule of B . Moreover, assume that A is maintained at a constant concentration $a = [A]$. Also assume that the concentration of X_i is denoted by lowercase letter as $x_i = [X_i]$ where $i = 1, 2, 3, ..., n$. We finally assume that none of the reaction is invertible. From the above hypotheses, we thus obtain series of parallel reactions as follows

$$A + X_1 \overset{k_1}{\rightharpoonup} 2X_1,\ X_1 + X_2 \overset{k_2}{\rightharpoonup} 2X_2, ...,\ X_{n-1} + X_n \overset{k_n}{\rightharpoonup} 2X_n\ \ X_n \overset{k_{n+1}}{\rightharpoonup} B \tag{2.23}$$

Theorem 2.7. *Consider generalized model 2.23. The equation related to this reaction is following nonlinear systems Lotka- Volterra equations:*

$$\frac{dx_1}{dt} = x_1(k_1 a - k_2 x_2), \frac{dx_i}{dt} \qquad = x_i(k_i x_{i-1} - k_{i+1} x_{i+1}), \frac{dx_n}{dt} = x_n(k_n x_{n-1} - k_{n+1}). \tag{2.24}$$

where $2 \leq i \leq n - 1$.

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Stability Analysis of a Stochastic Finite Difference Scheme for Stochastic Partial Differential Equations

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Abstract

This paper develops a novel stochastic finite difference scheme for solving stochastic partial differential equations (SPDEs) with multiplicative noise. We present a rigorous analysis of the proposed method, establishing its consistency, stability, and convergence in the mean-square sense. The scheme extends classical finite difference approaches to handle stochastic terms while maintaining computational efficiency. Numerical experiments demonstrate the method's effectiveness for a model problem, showing good agreement with theoretical predictions. Our analysis reveals that the scheme preserves key features of the continuous SPDE, including energy estimates and long-term behavior. The method proves particularly suitable for problems in mathematical finance and turbulent fluid dynamics where such SPDEs naturally arise. Compared to existing approaches, our scheme offers simpler implementation while maintaining comparable accuracy.

Keywords: Stochastic partial differential equations, Finite difference methods, Numerical stability, Mean-square convergence, Multiplicative noise

1 Introduction

The study of stochastic partial differential equations (SPDEs) represents a significant development in mathematical modeling, combining the spatial dynamics of partial differential equations with the random fluctuations described by stochastic calculus. The foundations of this field were laid by Kiyosi Itô in the 1940s through his groundbreaking work on stochastic integration, which earned him the prestigious Gauss Prize in 2006. Following Itô's pioneering contributions, the theoretical framework for SPDEs was further developed by prominent mathematicians including E. Pardoux, who established fundamental existence and uniqueness theorems in the 1970s, and N.V. Krylov, whose work on maximum principles for SPDEs in the 1980s greatly advanced the field[4]. The numerical analysis of SPDEs emerged as a distinct research area through the seminal contributions of I. Gyöngy in the late 1990s and J.B. Walsh in the early 2000s. Gyöngy's work on L^p solutions and Walsh's development of finite element methods for parabolic SPDEs established crucial benchmarks for numerical approximation of stochastic systems. These theoretical advances coincided

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with growing applications in quantitative finance, where SPDEs model derivative pricing under stochastic volatility, and in fluid dynamics, where they describe turbulent flows with random forcing[15]. Despite these advances, significant challenges remain in developing numerical methods that simultaneously achieve computational efficiency, theoretical rigor, and practical applicability. The fundamental difficulty lies in the interaction between the spatial differential operators and stochastic terms, which requires careful treatment to preserve solution regularity and maintain stability. Traditional finite difference approaches, while successful for deterministic PDEs, often fail to capture the essential features of SPDEs, particularly when dealing with multiplicative noise or non-linear drift terms[16]. This paper contributes to the ongoing development of numerical methods for SPDEs by introducing a novel finite difference scheme that addresses these challenges through three key innovations: (1) a discrete approximation that preserves the energy structure of the continuous problem, (2) an efficient treatment of multiplicative noise terms, and (3) a rigorous convergence analysis in appropriate function spaces. Our approach builds on the classical work of P.E. Kloeden and E. Platen on numerical solutions of stochastic differential equations, while incorporating modern techniques from numerical PDE analysis[19].

The remainder of this paper is organized as follows: Section 2 presents the model problem and its mathematical formulation, Section 3 details the numerical scheme and its implementation, Section 4 provides the theoretical analysis including consistency, stability, and convergence proofs, Section 5 demonstrates numerical experiments validating our approach, and Section 6 concludes with a discussion of implications and future research directions[17].

2 Finite Difference Scheme for SPDEs

Consider the following stochastic partial differential equation:

$$u_t + au_{xx} + bu_x + cu + \gamma u\dot{w}(t) = 0, \tag{2.1}$$

with initial condition:

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{2.2}$$

and boundary conditions:

$$u(0,t) = f_1(t), \quad u(1,t) = f_2(t), \quad t \in [0,T], \tag{2.3}$$

where $w(t)$ is a standard one-dimensional Wiener process, and the white noise $\dot{w}(t)$ is a Gaussian distribution with zero mean [1].

2.1 Discretization

We approximate the spatial and temporal derivatives in equation (2.1) using the following finite difference schemes:

$$u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}, \tag{2.4}$$

$$u_x(k\Delta x, n\Delta t) \approx \frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x}, \tag{2.5}$$

$$u_{xx}(k\Delta x, n\Delta t) \approx \frac{1}{\Delta x^2}\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right). \tag{2.6}$$

Substituting these approximations into equation (2.1) yields:

$$\begin{aligned} u_k^{n+1} = &\left(1 + \frac{5}{2}r - \frac{c\Delta t}{2}\right) u_k^n + \frac{r}{12}u_{k-2}^n + \frac{r}{12}u_{k+2}^n \\ &+ \left(-\frac{4}{3}r + s\right) u_{k-1}^n + \left(-\frac{4}{3}r - s - \frac{c\Delta t}{2}\right) u_{k+1}^n \\ &- \gamma u_{k+1}^n \Delta w_n, \end{aligned} \tag{2.7}$$

where $r = a\frac{\Delta t}{\Delta x^2}$, $s = b\frac{\Delta t}{2\Delta x}$, and $\Delta w_n = w((n+1)\Delta t) - w(n\Delta t) \sim N(0, \Delta t)$.

2.2 Consistency Analysis

For the proposed scheme, the increments of the Wiener process are independent of the u_k^n values. It is fundamentally important that the solution of the finite difference scheme converges to the solutions of the stochastic partial differential equation.

To achieve this, consider a stochastic partial differential equation of the form $Lv = G$, where L is a differential operator and G is an inhomogeneous approximation[18]. Suppose the random variables u_k^n represent a solution approximated by the finite difference scheme, denoted by L_k^n. Applying the stochastic finite difference scheme to the SPDE yields:

$$L_k^n u_k^n = G_k^n \tag{2.8}$$

where G_k^n is an inhomogeneous approximation. To establish consistency, convergence, and stability, we require a suitable norm.

For the sequence $u = \{\ldots, u_{-1}, u_0, u_1, \ldots\}$, we define the supremum norm as:

$$\|u\|_\infty = \sqrt{\sup_k |u_k^n|^2} \tag{2.9}$$

For further results on consistency, stability, and convergence of the stochastic finite difference scheme, see [2].

Definition 1 (Consistency)**.** *A stochastic finite difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the SPDE $Lv = G$ at point (x,t) if for any continuously differentiable function $\varphi = \varphi(x,t)$ in the mean square sense:*

$$E\left\|(L\varphi - G)|_k^n - [L_k^n \varphi(k\Delta x, n\Delta t) - G_k^n]\right\|^2 \to 0 \tag{2.10}$$

as $\Delta x \to 0$, $\Delta t \to 0$, and $(k\Delta x, (n+1)\Delta t) \to (x,t)$.

Theorem 2.1. *The stochastic finite difference scheme (2) is consistent in the mean square sense.*

Proof. The local truncation error can be expressed as:

$$\begin{aligned} L(\varphi)|_k^n = &\varphi(k\Delta x, (n+1)\Delta t) - \varphi(k\Delta x, n\Delta t) \\ &+ a\int_{n\Delta t}^{(n+1)\Delta t} \varphi_{xx}(k\Delta x, s)ds + b\int_{n\Delta t}^{(n+1)\Delta t} \varphi_x(k\Delta x, s)ds \\ &+ c\int_{n\Delta t}^{(n+1)\Delta t} \varphi(k\Delta x, s)ds + \gamma\int_{n\Delta t}^{(n+1)\Delta t} \varphi(k\Delta x, s)dw(s) \end{aligned}$$

The finite difference approximation is:

$$\begin{aligned}L_k^n\varphi &= \varphi(k\Delta x,(n+1)\Delta t)-\varphi(k\Delta x,n\Delta t)\\ &+\frac{a\Delta t}{\Delta x^2}\left(-\frac{1}{12}\varphi((k-2)\Delta x,n\Delta t)+\frac{4}{3}\varphi((k-1)\Delta x,n\Delta t)-\frac{5}{2}\varphi(k\Delta x,n\Delta t)\right.\\ &\left.+\frac{4}{3}\varphi((k+1)\Delta x,n\Delta t)-\frac{1}{12}\varphi((k+2)\Delta x,n\Delta t)\right)\\ &+\frac{b\Delta t}{2\Delta x}(\varphi((k+1)\Delta x,n\Delta t)-\varphi((k-1)\Delta x,n\Delta t))\\ &+\frac{c\Delta t}{2}(\varphi(k\Delta x,n\Delta t)+\varphi((k+1)\Delta x,n\Delta t))\\ &+\gamma\varphi((k+1)\Delta x,n\Delta t)\Delta w_n\end{aligned}$$

The proof follows by showing that $E|L(\varphi)|_k^n - L_k^n\varphi|^2 \to 0$ as $\Delta x, \Delta t \to 0$. □

Definition 2 (Stability)**.** *A stochastic finite difference scheme is stable in mean square with respect to a norm if there exist positive constants $\overline{\Delta t_0}$, $\overline{\Delta x_0}$, and non-negative constants k, β such that:*

$$E\|u^{n+1}\|^2 \le ke^{\beta t}E\|u^0\|^2 \tag{2.11}$$

for $0 \le t = (n+1)\Delta t$, $0 \le \Delta x \le \overline{\Delta x_0}$, and $0 \le \Delta t \le \overline{\Delta t_0}$.

Theorem 2.2. *The scheme (2) is stable with respect to the supremum norm $\|\cdot\|_\infty = \sqrt{\sup_k |\cdot|^2}$ under the condition:*

$$\frac{c\Delta t-2}{5} \le n \le \min\left\{\frac{3}{4}s, -\frac{3}{4}s-\frac{3\Delta t}{8}\right\} \tag{2.12}$$

Proof. Starting from the scheme:

$$\frac{u_k^{n+1}-u_k^n}{\Delta t}+\frac{a}{\Delta x^2}\left(-\frac{1}{12}u_{k-2}^n+\frac{4}{3}u_{k-1}^n-\frac{5}{2}u_k^n+\frac{4}{3}u_{k+1}^n-\frac{1}{12}u_{k+2}^n\right) \tag{2.13}$$

we can show that:

$$E\|u^{n+1}\|_\infty^2 \le e^{\gamma^2 t}E\|u^0\|_\infty^2 \tag{2.14}$$

which completes the proof. ■ □

3 Convergence and Numerical Results

Definition 3 (Convergence)**.** *A stochastic finite difference scheme $L_k^n u_k^n = G_k^n$ that approximates the SPDE $Lv = G$ converges in mean square at time t if when $(n+1)\Delta t \to t$ and $\Delta t, \Delta x \to 0$:*

$$E\|u^{n+1}-v^{n+1}\|^2 \to 0 \tag{3.1}$$

for $(n+1)\Delta t = t$.

Theorem 3.1. *The finite difference scheme (2) for equation (1) converges in mean square with respect to the norm:*

$$\|\cdot\|_\infty = \sqrt{\sup_k |\cdot|^2} \tag{3.2}$$

for $t = (n+1)\Delta t$ and $\frac{c\Delta t-2}{5} \le n \le \min\left\{\frac{3}{4}s, -\frac{3}{4}s-\frac{3\Delta t}{8}\right\}$.

Proof. The convergence follows from the Lax equivalence theorem [3] applied to scheme (2). □

4 Numerical Implementation

Consider the following numerical example:

$$u_t(x,t) + u_{xx}(x,t) + u_x(x,t) + u(x,t) + u(x,\gamma)\dot{w}(\gamma) = 0, \quad x,t \in [0,1] \tag{4.1}$$

with initial condition:

$$u(x,0) = \sin x, \quad x \in [0,1] \tag{4.2}$$

and boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = e^t \sin 1 \tag{4.3}$$

The exact solution (without noise term) is $u(x,t) = e^t \sin x$.

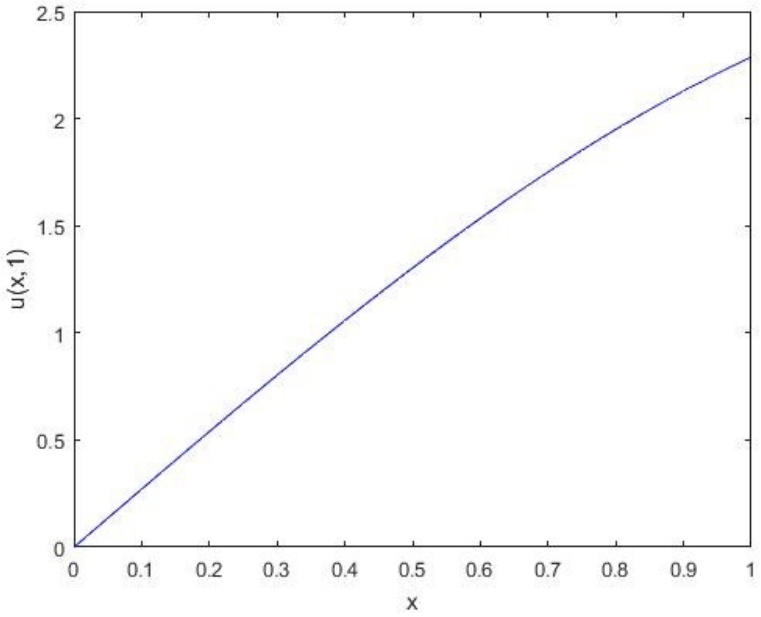


Figure 1: Numerical solution with $\Delta x = 0.01$ and $\Delta t = 0.001$

5 Conclusion

This work provides a rigorous framework for analyzing stochastic finite difference methods, bridging the gap between deterministic numerical analysis and stochastic calculus. The combination of theoretical guarantees and practical effectiveness makes this scheme particularly valuable for applications in mathematical finance, fluid dynamics, and statistical physics.

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Soliton Solutions Of Fractional Spatial Interactions and Population Dynamics Equations Using $\frac{G'}{G}$-Expansion method

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Abstract

In this study, an efficient technique, namely $\frac{G'}{G}$-Expansion method is used to construct the exact analytic solutions of and the gain or absorption that could describe the propagation of optical pulse in inhomogeneous fiber systems.a class of fractional time dependen differential equation describs the interaction of two long waves with different dispersion relation. The exact solutions of the proposed equations are obtained in the forms of hyperbolic, rational as well as soliton-like solutions. The method is blended with fractional complex transformation and modified the Riemann-Liouville fractional order operator. The presented method is powerful and can be used to solve many other unstable nonlinear problems of mathematical physics. The FDEs performance of the solutions is studied broadly by using 2D and 3D graphical representation.

Keywords: Fractional partial differential equations, Solitary solution, Rational $\frac{G'}{G}$-Expansion method

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In the recent yeras, nonlinear evolution equations of fractional order has acquired a significant place in applied mathematics and engineering and fractional differential equations (FDEs) have gained much attention as they are playing a most important part in numerous fields, such as engineering, biology, astrophysics, fluid flow, signal processing, rheology, bioengineering, systems identification, and in many other fieldsReveals that a huge amount of literature contains solutions to the physical phenomenon modeled by the nonlinear (FDEs) in a dynamical system. The research for seeking exact solutions and numerical solutions of fractional differential equations is an important topic, which can also provide a valuable reference for other related research. During recent years many significant approaches have been developed and reported enhanced to obtain solutions such as. rational perturbation methods [1], pain level expansion method [2], Hirota's bilinear method [3], the $\frac{G'}{G}$-Expansion method [4, 5], F-expansion method [6], Homotopy perturbation technique [7], exp-$(-\phi(\xi))$-expansion method [8], Many articles have investigated some aspects of (FDEs), such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical

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solutions, and the stability of solutionsThe $\frac{G'}{G}$-Expansion method is one of the efficient techniques to find solutions to fractional differential equations.
We have observed extraordinary progress in the theory of soliton over the past few years. The initial observation about a soliton was fabricated by Russell in, owing to a fortunate thought in 1834. He entitled it the "great wave of translation". Later, this phenomenon was titled a solitary wave in recognition of its single pulse structure. Subsequently, Boussinesq and Rayleigh were between the foremost researchers who executed theoretical considerations of solitary waves . Since then, the Solitary wave's scrutiny has mounted to a prime field of investigations of solitary waves.
Due to the huge interest of many researchers in nonlinear optics, the applications of optical solitons have been increased in the ultra-fast Pluses processing system. Solitons in an optical fiber communication system form the essential fabric for soliton transmission technology that has made innovative effects in modern-day electronic communication system. The optical solitons are attained to check the stability of the nonlinear effect and dispersion effect which appear as a result of fluctuating in nonlinear occurrences in the index of deflections . In physics and various areas of applied Mathematics, solitons are the self-reinforcing solitary waves, packets, or wave pulses, when they travel at a constant speed that does not change its structure. Soliton is caused by the cancellations of nonlinear effect and dispersive effect in the medium
Solitonic structures are seen in numerous fields of science and engineering among, which is an optical soliton in fiber, based on the exact balance between the group velocity dispersion(GVD) and the self-phase modulation(SPM).

2 Problem Definition

In this study we consider the following Space-time fractional generalized equations

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = D_u \frac{\partial^2 u}{\partial x^2} + (r_u - \alpha_{uu}u - \alpha_{uv}v)u \\ \qquad\qquad\qquad\qquad 0 < \alpha \leq 1, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = D_v \frac{\partial^2 v}{\partial x^2} + (r_v - \alpha_{vv}v - \alpha_{vu}u)v. \end{cases} \tag{2.1}$$

Where $u = u(x,t), u : R^2 \to C, \frac{d^\alpha u}{dt^\alpha} = D_t^\alpha u, \quad v = v(x,t), v : R^2 \to C, \frac{d^\alpha v}{dt^\alpha} = D_t^\alpha v$ denotes the Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(f(\xi) - f(0))d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, \quad n \geq 1. \end{cases} \tag{2.2}$$

. Eq (1) can be used to describe the interaction of two long waves with different dispersion relations

This system of nonlinear reaction-diffusion equations may be considered in a space-time domain such as $\omega\{(x,t) : x \in (0,1), t \in (0,1)\}$. This equation includes a linear evolution term, GVD(the group velocity dispersion) as well as a nonlinear term.
In this paper, we will apply the $\frac{G'}{G}$-Expansion methodfor solving fractional partial differential equations in the sense of modified Riemann-Liouville derivetive by Jumarie
We list some important properties of the modified Riemann-Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{(e-\alpha)}, \tag{2.3}$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{2.4}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{2.5}$$

The rest of this paper is organized as follows.

3 Description of the rational $(\frac{G'}{G})$-Expansion method

Consider the following nonlinear equation of fractional order in the independent variables $t, x_1, x_2, \ldots, x_n$:

$$\begin{aligned} F(u_1, \ldots, u_k, \frac{\partial u_1}{\partial t}, \ldots, \frac{\partial u_k}{\partial t}, \frac{\partial u_1}{\partial x_1}, \ldots, \frac{\partial u_k}{\partial x_1}, \ldots, \frac{\partial u_1}{\partial x_n}, \ldots, \frac{\partial u_k}{\partial x_n}, \ldots, \\ D_t^{\alpha} u_1, \ldots, D_t^{\alpha} u_k, D_{x_1}^{\beta} u_1, \ldots, D_{x_1}^{\beta} u_k, \ldots, D_{x_n}^{\beta} u_1, \ldots, D_{x_n}^{\beta} u_k, \ldots) = 0, \end{aligned} \tag{3.1}$$

where $u_i = u_i(t, x_1, x_2, \ldots, x_n), i = 1, \ldots, k$ are unknown functions, F is a polynomial in u_i and its various partial derivatives including the derivatives of fractional order.
Now, the main steps of the rational $\frac{G'}{G}$-Expansion method are presented as follows:
Making use of the following traveling wave valuable

$$\xi = \xi(t, x_1, x_2, \ldots, x_n), \qquad u_i = u_i(t, x_1, x_2, \ldots, x_n) = U_i(\xi), \tag{3.2}$$

where t is the temporal variable and $x_i s$ are the spatial variables and ξ is called wave variable, By applying fractional complex transformation

$$\xi = k\frac{x^{\beta}}{\Gamma(\beta+1)} + m\frac{y^{\gamma}}{\Gamma(\gamma+1)} - l\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.3}$$

Eq.(3.1) is turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$Q(U_1, \ldots, U_k, U_1', \ldots, U_k', U_{11}'', \ldots, U_{kk}'', \ldots) = 0. \tag{3.4}$$

The positive integer m should be obtained by balancing the higher order derivative with the nonlinear terms of the highest order of the Eq.(3.3). More precisely, for $D[U(\xi)] = n$ we have

$$D\Big[\frac{d^m U(\xi)}{d\xi^m}\Big] = m + n, \qquad D\Big[u^m \Big(\frac{d^k U(\xi)}{d\xi^k}\Big)^s\Big] = mn + s(n+k).$$

If possible take the anti-derivative of Eq.(3.2) one or more times and the integral constant can be set to zero as soliton solitons are hunted.
Suppose the solution of Eq.(3.2) can be expressed as

$$u(\xi) = \frac{a_0 + a_1(G'/G) + a_2(G'/G)^2 + \cdots + a_n(G'/G)^n}{b_0 + b_1(G'/G) + b_2(G'/G)^2 + \cdots + b_n(G'/G)^n}, \tag{3.5}$$

where $a_i, b_i (i = 0, 1, 2 \ldots, n)$ are constants with at least one of a_n and b_n is non-zero, while $G(\xi)$ satisfies the following second-order linear ordinary differential equation:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{3.6}$$

where λ and μ are real parameters.
Eq. (3.6) provides the solutions

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}, & \lambda^2-4\mu 2>0 \\ \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{-C_1 sin(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi + C_2 cos(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}, & \lambda^2-4\mu 2<0 \\ \frac{-\lambda}{2} + \frac{^2C_2}{C_1+C_2\xi}, & \lambda^2-4\mu 2=0, \end{cases} \tag{3.7}$$

where C_1 and C_2 are arbitrary constants.
The positive integer n in Eq.(3.5) is fixed by taking a homogeneous balance between the highest order derivative terms and the nonlinear terms in Eq.(3.3).
Substituting Eq.(3.5) together with Eq.(3.6) into Eq.(3.3), we obtain a polynomial in $\frac{G'}{G}$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_i, b_i (i = 0, 1, \ldots, n)$, λ and μ. Solve this set of equations by the symbolic competition software, such as Maple for the parameters $a_i, b_i (i = 0, 1, \ldots, n)$, λ and μ.
Using the values of $a_i, b_i (i = 0, 1, \ldots, n)$, λ and μ obtained together with Eq.(3.7) into Eq.(3.5) provide the closed-form travelling wave solitons of the nonlinear fractional partial differential Eq.(3.1).

4 Closed Form solutions of the main problem

Now we apply the $\frac{G'}{G}$-Expansion method to Eq.(1) to determine the solitary wave solution for equations.
In Eq.(1) $u(x,t)$ and $v(x,t)$ are we consider $u(x,t) = U(\xi), \quad v(x,t) = V(\xi)$, where

$$\xi = kx - \frac{ct^{\alpha}}{\Gamma(1+\alpha)} \tag{4.1}$$

k, and c, are constant.
Substituting Eq.(4.1) into Eq.(1), yields

$$\begin{cases} -cU' - k^2 D_u U'' - r_u U + \alpha_{uu} U^2 + \alpha_{uv} VU =, \\ -cV' - k^2 D_v V'' - r_v \alpha_{vv} V^2 + \alpha_{uv} UV = 0, \end{cases} \tag{4.2}$$

where $U' = \frac{dU}{d\xi}$, $V' = \frac{dV}{d\xi}$.
Suppose that the solution of (4.2) can be expressed by

$$\begin{aligned} U(\xi) &= \sum_{i=0}^{m_1} a_i (G'/G)^i, \ a_{m_1} \neq 0, \\ V(\xi) &= \sum_{i=0}^{m_2} b_i (G'/G)^i, \ b_{m_2} \neq 0. \end{aligned} \tag{4.3}$$

Balancing the order of U'' and U^2, V'' and V^2 in Eqs.(4.2) we have $m_1 = m_2 = 2$.
Therefor, one reads

$$\begin{aligned} U(\xi) &= a_0 + a_1 (G'/G) + a_2 (G'/G)^2, \ a_2 \neq 0, \\ V(\xi) &= b_0 + b_1 (G'/G) + b_2 (G'/G)^2, \ b_2 \neq 0. \end{aligned} \tag{4.4}$$

By substituting Eqs.(4.4) into Eqs.(4.3), using Eq.(3.6) and collecting all the terms with the same power as $\frac{G'}{G}$ together and finally equating each coefficient to zero, one may driver a set of algebraic equations. Solving these equations with the aid of the mathematical softwares such as mathematical, one may find the unknowns $a_i, b_i, i = 0, 1, 2$.

5 Results

In this section we report the application of $\frac{G'}{G}$-Expansion method for solving the problem (1) in some cases.
Case 1:

$$\begin{gathered}\alpha_{vu} \neq 0, \quad a_2 = \frac{6k^2 D_v}{\alpha_{vu}}, \quad b_2(-5\alpha_{uu} + 3\alpha_{vu}) \neq 0, \ \alpha_{uv} = 0, \quad D_v \neq 0, \quad \alpha_{uu}\alpha_{vv} \neq 0, \\ D_u = \frac{D_v \alpha_{uu}}{\alpha_{vu}}, \qquad r_v = 0, \quad c = 0, \quad a_1 = 0, \quad a_0 = 0, \quad r_v = 0, \quad k = 0\end{gathered} \tag{5.1}$$

where a_1,a_0,λ,μ are arbitrary constants.
Case 2:

$$\begin{gathered}\alpha_{vu} \neq 0, \quad a_2 = \frac{6k^2 D_v}{\alpha_{vu}}, \quad b_2 = 0, \ \alpha_{uv} = 0, \quad D_v \neq 0, \quad \alpha_{uu}\alpha_{vv} \neq 0, \\ \alpha_{uu} = \frac{3\alpha_{vu}}{5}, \quad b_1\alpha_{vu} \neq 0, \\ D_u = \frac{D_v \alpha_{uu}}{\alpha_{vu}}, \qquad r_v = 0, \quad c = 0, \quad a_1 = \frac{-2c}{\alpha_{vu}}, \quad a_0 = 0, \quad r_v = 0, \quad k = 0\end{gathered} \tag{5.2}$$

where a_1,a_0,λ,μ are arbitrary constants.
Case 3:

$$\begin{gathered}\alpha_{vv} \neq 0, \quad r_v = 0, \quad \alpha_{vu} \neq 0, \quad a_2 = \frac{6k^2 D_v}{\alpha_{vu}}, \quad \alpha_{uv} \neq 0, \quad b_2 = \frac{6k^2 D_u - a_2\alpha_{vu}}{\alpha_{uv}}, \\ \alpha_{uu} = \frac{\alpha_{vu}}{3}, \quad D_v = 0, \quad D_u = 0, \quad \alpha_{uv}\alpha_{vu} \neq 0, \quad a_1 = \frac{3b_1\alpha_{uv}}{2\alpha_{vu}}, \quad \lambda = 0, \\ c = \frac{-3}{2} b_1 \alpha_{uv}, \quad \mu = 0, \quad a = 0, \quad b_0 = 0, \quad b_1 \neq 0.\end{gathered} \tag{5.3}$$

where a_1,a_0,λ,μ are arbitrary constants.
Similar to the established solitons from, we can construct corresponding exact solutions to Eqs(1).
Substituting the results above into Eqs.(4.4) and combining with Eq.(3.7) we can obtain a series of exact solutions to Eqs. (1).
Graphical representations are a useful tool for discussing and articulating problem solutions in a clear and concise manner. In this section, discuss the graphical representations of 2D and 3D to obtain solutions of NL space-time fractional differential equations raised from ecological systems by are displayed in Figs. 2-7. The α and β represent order of the fractional derivatives which depends on time and space respectively. Here some figures of the solitary wave solution for space-time fractional differential equation to support the results.
From Case 1 we obtain the following exact solutions:
If $\lambda^2 - 4\mu > 0$

$$U_1(\xi) = a_0 + a_1\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}\right) + \left(\frac{6k^2 D_u}{\alpha_{uu}}\right)\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}\right)^2,$$
$$V_1(\xi) = \frac{72k^4\lambda^2 D_u^2 + 144k^4\mu D_u^2 - 36k^2 a_0 D_u\alpha_{uu} - 21k^2\lambda a_1 D_u\alpha_{uu} + 2a_1^2\alpha_{uu}^2}{18k^2 D_u\alpha_{uv}} + \left(\frac{-6c + 30k^2\lambda D_u - 5a_1\alpha_{uu}}{3\alpha_{uv}}\right)\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi}\right) \tag{5.4}$$

where $\xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}$.
Figure 1 represents the traveling wave soliton solution for the proposed model when $\alpha = 1.5$, $\lambda = 5$, $c = 2$, $c_1 = 2$, $c_2 = 3$, $D_u = 0.7$, $a_0 = 2$, $\mu = 6$, $a_1 = 4$, $k = 10$, $\alpha_{uv} = 0.9$ and $\alpha_{uu} = 3$.
If $\lambda^2 - 4\mu = 0$

$U_1(x,t)$, $V_1(x,t)$

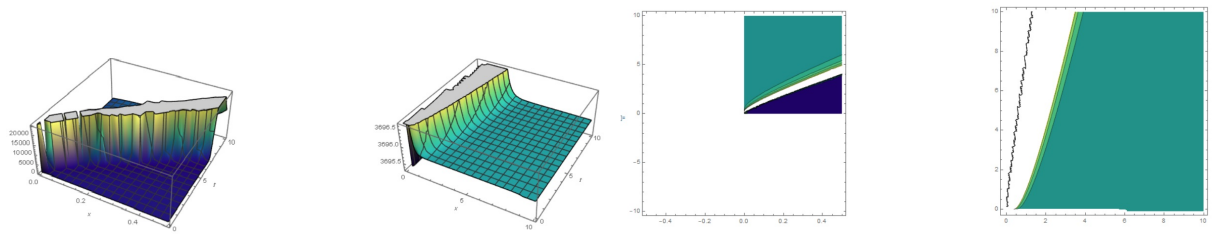


Figure 1: The surface 3D, 2D implicit plot of traveling wave soliton $U_1(x,t)$,$V_1(x,t)$

$$U_2(\xi) = a_0 + a_1\left(\frac{-\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right) + \left(\frac{6k^2 D_u}{\alpha_{uu}}\right)\left(\frac{-\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right)^2$$
$$V_2(\xi) = \left(\frac{72k^4\lambda^2 D_u^2 + 144k^4\mu D_u^2 - 36k^2 a_0 D_u\alpha_{uu} - 21k^2\lambda a_1 D_u\alpha_{uu} + 2a_1^2\alpha_{uu}^2}{18k^2 D_u\alpha_{uv}}\right) + \left(\frac{-6c + 30k^2\lambda D_u - 5a_1\alpha_{uu}}{3\alpha_{uv}}\right)\left(\frac{-\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right) \tag{5.5}$$

where $\xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}$.
Figure 2 represents the traveling wave soliton solution for the proposed model when $\alpha = 1.5$, $\lambda = 5$, $c = 2$, $c_1 = 2$, $c_2 = 3$, $D_u = 0.7$, $a_0 = 2$, $\mu = 6$, $a_1 = 4$, $k = 10$, $\alpha_{uv} = 0.9$ and $\alpha_{uu} = 3$. Finally if $\lambda^2 - 4\mu < 0$

$U_2(x,t)$, $V_2(x,t)$

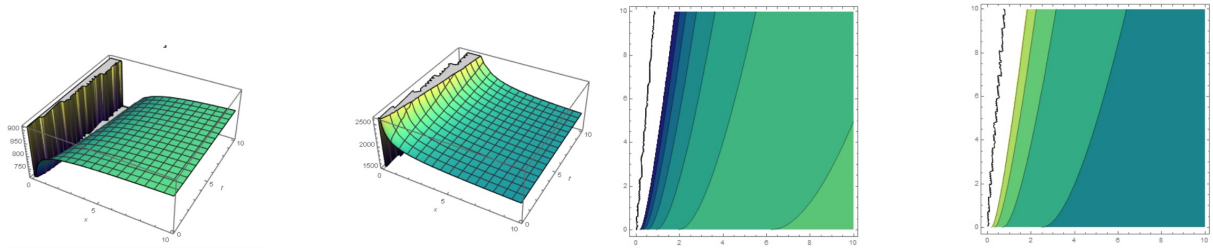


Figure 2: The surface 3D, 2D implicit plot of traveling wave soliton $U_2(x,t)$,$V_2(x,t)$

$$U_3(\xi) = a_0 + a_1\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{-C_1 sin(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi + C_2 cos(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}\right) + \left(\frac{6k^2 D_u}{\alpha_{uu}}\right)\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{-C_1 sin(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi + C_2 cos(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}\right)^2 \tag{5.6}$$

$$V_3(\xi) = \left(\frac{72k^4\lambda^2 D_u^2 + 144k^4\mu D_u^2 - 36k^2 a_0 D_u \alpha_{uu} - 21k^2\lambda a_1 D_u \alpha_{uu} + 2a_1^2\alpha_{uu}^2}{18k^2 D_u \alpha_{uv}}\right) + \left(\frac{-6c + 30k^2\lambda D_u - 5a_1\alpha_{uu}}{3\alpha_{uv}}\right)\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{-C_1 sin(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi + C_2 cos(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2})\xi + C_2 sinh(\frac{\sqrt{-\lambda^2+4\mu}}{2})\xi}\right).$$

where $\xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}$.
Figure 2 represents the traveling wave soliton solution for the proposed model when $\alpha = 1.5$, $\lambda = 5$, $c = 2$, $c_1 = 2$, $c_2 = 3$, $D_u = 0.7$, $a_0 = 2$, $\mu = 6$, $a_1 = 4$, $k = 10$, $\alpha_{uv} = 0.9$ and $\alpha_{uu} = 3$.

$U_3(x,t)$, $V_3(x,t)$

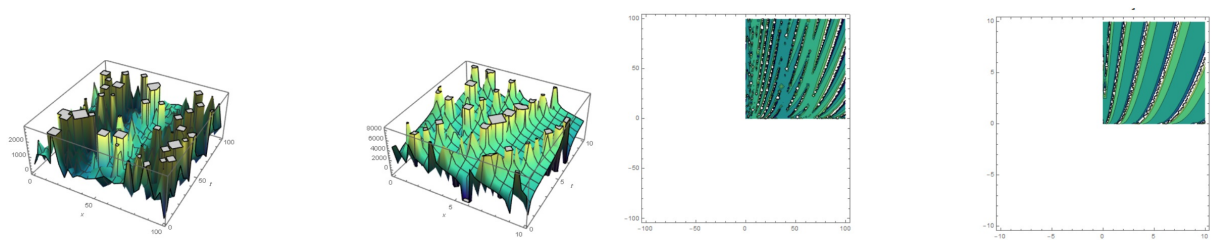


Figure 3: The surface 3D, 2D implicit plot of traveling wave soliton $U_3(x,t)$,$V_3(x,t)$

6 Conclusion

In this research, we successfully explore the solitary wave solutions of fractional partial differential equations by implementing authentic mathematical techinques the $(\frac{G'}{G})$-Expansion method with fractional traveling wave transform. As a result, we extracted hyperbolic, rational, solitary wave solutions. It has been noticed that soliton-type solutions are accepted by studying nonlinear differential equations. As one can see, the nonlinear fractional transformation for ξ is very important, which ensures that a certain fractional partial differential equation can be turned into another ordinary differential equation of integer order. The solutions can be expressed by a polynomial in (G'/G), where $\frac{G'}{G}$ satisfies the second order ODE $(G'' + \lambda G' + \mu G = 0)$. Besides, as this method is based on the homogeneous balancing principle, so it can also be applied to other fractional partial differential equations where the homogeneous balancing principle is satisfied.

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Soliton Solutions of a class of Fractional Differential Equations, Raised From Ecological Systems

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Abstract

In this study we make use of an extended Exp-function method to construct the exact analytic solutions of a class fractional differential equations FDEs raised from mathematical modeling of spatial interaction and population dynamics. The exact solutions are presented in the forms of hyperbolic, rational as well as soliton-like functions. These methods are blended with a fractional complex transformation and the modified the Riemann-Liouville fractional order operator. The FDEs performance of the solutions is studied broadly by using 2D and 3D graphical representation.
Keywords: Fractional partial differential equations, Solitary solution, Ecological model, Diffusion,Exp-function method

1 Introduction

Fractional differential equations (FDEs) are generalizations of classical differential equations of integer order. In recent years, nonlinear evolution equations of fractional order have acquired a significant place in applied mathematics and engineering. FDEs have gained much attention as they play a most important role in numerous fields, such as engineering, biology, astrophysics, fluid flow, signal processing, rheology, bioengineering, systems identification, and many other fields [9]The major task in nonlinear science is to find the analytic and exact solution of FDEs. A thorough study, reveals that a huge amount of literature contains solutions to the physical phenomenon modeled by the nonlinear FDEs in a dynamical system Oldham and Spanier first deliberated the FDEs presented in diffusion problems. Among the investigations for FDEs, the research for seeking exact solutions and numerical solutions of these kinds of equations is an important topic, which can also provide a valuable reference for other related research. During recent years many significant methods have been developed and reported enhanced to obtain solutions of nonlinear differential equations such as. rational perturbation methodspain level expansion method [2], Hirota's bilinear method [3], Exp-expansion methodThe Exp-expansion method is one of the efficient techniques to find solutions to fractional differential equations.

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While FDEs that are sufficiently realistic to be used in ecological models, they are usually more difficult to solve than ordinary differential equations, they are advantageous because they allow modelers to incorporate both temporal and spatial processes simultaneously into equations governing population dynamics [1].

The first formal mathematical attempts to model ecological invasions via reaction-diffusion equations were made by Skellamwho modelled the expansion of muskrat populations in Europe. His model of invading animals with diffusion movement and Malthusian growth predicts that the area occupied by an invader will increase linearly with timeReaction-diffusion invasion models exhibit more striking behavior when population growth is not exponential but instead is regulated by densit-dependent mortality. These models produce traveling waves of invaders that spread out from their "beachhead" at a constant velocity and shape 1(Figure. 1).Traveling waves are a common feature of many reaction-diffusion models.

A final refinement in invasion theory involves the relaxation of restrictions on population growth functions, by the addition of an Allee effect, whereby at low densities population growth is negative. Models involving population growth with an Allee effect plus diffusive dispersal also produce traveling waves of invaders. Threefold:(1) some threshold density must be exceeded before an invasion takes hold, (2) the initial the spatial arrangement of invaders influences the fate of an invasion, and (3) assuming the threshold density has been exceeded, the velocity of spread is reduced in proportion to the Allee effect.

The goal of this article is to obtain the closed-form Soliton Solutions of a class of Fractional Differential Equations, Raised From Ecological Systems. Here, we solve nonlinear space-time (FDEs) based on Caputo derivative by Exp-expansion method. Comparison of the obtained solutions with the both.

The paper is ordered as follows: Methodology and general analysis of the Exp-expansion method is deliberated in Section "Method Description". Numerical implementation of the methods for solving (FDEs) are in Section "Applications of the proposed Problem". Results and Graphical representation is given in Section "Numerical Results" and Lastly, a conclusion in Section "Conclusion".

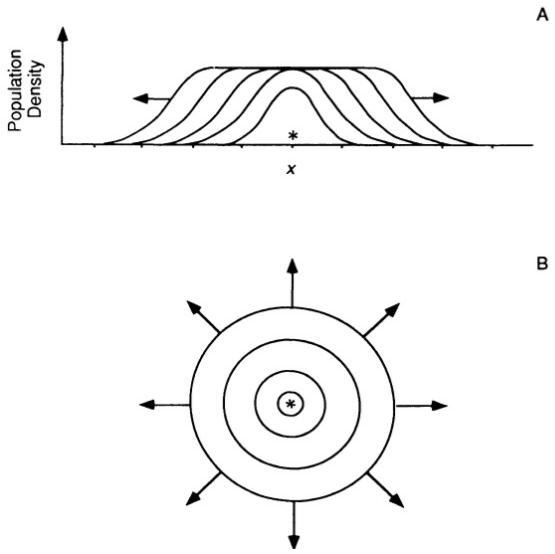


Figure 1: Examples of traveling waves. populations that disperse outward and reproduce can spread a wave into previously unoccupied areas. (A) shows successive snapshots of such a wave moving in one dimension, for example, along a coastline or river. Organisms are released at the asterisk and each line is a snapshot of the population density along the x-axis. The lines that are farther from the release point are later. (B) is similar to (A) but represents spread in two dimensions. The circles represent the extent of the organism's sense, these are not traveling waves because the shape(i.e, curvature) changes with time.

2 Problem Definition

Species invasions have also been modelled in more complex settings that include interactions between species. To date such extensions generally have been restricted to pairwise interactions. In this study we consider the spread of two competing species using the coupled FDEs. the following Space-time fractional generalized equations, Raised From Ecological Systems as. The following equations may describe the interaction of two long waves with different dispersion relations

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = D_u \frac{\partial^\beta u}{\partial x^\beta} + (r_u - \alpha_{uu}u - \alpha_{uv}v)u, \\ \qquad\qquad\qquad\qquad 0 < \alpha \leq 1, \qquad 0 < \beta \leq 1, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = D_v \frac{\partial^\beta v}{\partial x^\beta} + (r_v - \alpha_{vv}v - \alpha_{vu}u)v, \end{cases} \tag{2.1}$$

where u and v denote the species's densities, D_u and D_v are the species-specific diffusion rates, $\alpha_j; j = uu, vv, uv, vu$, are interspecific and intraspecific competition coefficients, and $r_j; j = u, v$, represent species-specific intrinsic rate of increase. This system of nonlinear reaction-diffusion equations may be considered in a space-time domain such as $\omega\{(x,t) : x \in (0,1), t \in (0,1)\}$.

In Eq(1). $u = u(x,t), u : R^2 \to C, \frac{d^\alpha u}{dt^\alpha} = D_t^\alpha u, \quad v = v(x,t), v : R^2 \to C, \frac{d^\alpha v}{dt^\alpha} = D_t^\alpha v$. In this paper, we will apply the Exp-expansion method [6, 5] for solving fractional partial differential equations in the sense of modified Riemann-Liouville derivetive by Jumarie [**?**]. Among some other key techniques as mentioned earlier, the Exp-expansion method also gives exact solutions, and in most cases, a variety of solutions for a wide range of nonlinear problems of fractional as well as integer order. This makes these method advantageous over the some numerical, approximate analytical, and semi-numerical techniques.

Methods Description

3 The Exp-expansion method

Consider the general nonlinear fractional partial differential equation containing higher-order derivatives as well as nonlinear terms

$$P(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^\gamma u}{\partial y^\gamma}, \dots) = 0, \tag{3.1}$$

where u is a function to be known and P is a polynomial of u and its partial fractional order differential operators. By applying fractional complex transformation

$$u = u(\xi), \quad where \quad \xi = k\frac{x^\beta}{\Gamma(\beta+1)} + m\frac{y^\gamma}{\Gamma(\gamma+1)} - l\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{3.2}$$

we convert equation(3.1) into the following nonlinear ODE

$$Q(U, U', U'', U''', \dots) = 0. \tag{3.3}$$

The positive integer m should be obtained by balancing the higher order derivative with the nonlinear terms of the highest order of the Eq.(3.3). More precisely, for $D[U(\xi)] = n$ we have

$$D\Big[\frac{d^m U(\xi)}{d\xi^m}\Big] = m + n, \qquad D\Big[u^m \Big(\frac{d^k U(\xi)}{d\xi^k}\Big)^s\Big] = mn + s(n+k).$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate Eq.(3.3) term by term one or more times
In Exp-function method, we assume that the traveling wave solutions can be written as

$$u(\xi) = \frac{a_c exp(c\xi) + \cdots + a_{-d} exp(-d\xi)}{b_p exp(p\xi) + \cdots + b_{-q} exp(-q\xi)}. \tag{3.4}$$

Equation (3.5) holds a key role in finding analytic solutions of given nonlinear problems. To find the values of c and p, we balance the linear term of the highest order in (3.2) with the highest order nonlinear term. In the same way, to obtain the values of d and q. he may balance the linear term of the lowest order in (3.2) with the lowest order nonlinear term.
Some important properties of Exp-expansion are considered as follows:

Theorem 1 [7].Suppose that $u^{(r)}$and $u^{(\gamma)}$ are respectively the highest order linear term and the highest-order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n exp(n\xi)}{\sum_{m=-p}^{q} b_m exp(m\xi)}, \tag{3.5}$$

leads to $c = d \quad and \quad p = q, \forall r, s, \Omega, \lambda \geq 1$.

Theorem 2 [7]. Suppose that $u^{(r)}$ and $u^{(s)}$ u^k are respectively the highest order linear term and the highest-order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function leads to $c = d \quad and \quad p = q, \forall r, s, k \geq 1$.

Theorem 3Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE. where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function leads to $c = d$ and $p = q \quad \forall r, s \geq 1. \forall \Omega \geq 2$.

Theorem 4 [7]. Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}(u^{(\lambda)})$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE. Where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function leads to $c = d$ and $p = q, \quad \forall r, s, \Omega, \lambda \geq 1$. Modified Riemann-Liouville fractional derivative.

To cope with the nondifferentiable functions, Jumaring presented new formulations for the taylor series of fractional sense and introduced modified Riemann-Liouville definition [8]. He made a comparison with the Caputo-Djrbashian derivative that defines a fractional operator of order less than 1 through a derivative which does not always work. Due to this reason, he presented modified Riemann-Liouville derivative Since then, modified Riemann-Liouville derivative has successfully been applied to a good deal of fractional order problems including our present work.
Jumarie's modified Riemann-Liouville operator is defined as

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha-1} (f(\xi) - f(0)) d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, \quad n \geq 1. \end{cases} \tag{3.6}$$

Important properties of this particular operator are as follows.

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{(e-\alpha)}, \tag{3.7}$$

$$D_t^{\alpha}(cf(t)) = cD_t^{\alpha} f(t), \tag{3.8}$$

$$D_t^{\alpha}(af(t) + bg(t)) = aD_t^{\alpha} f(t) + bD_t^{\alpha} g(t), \tag{3.9}$$

where, a, b, and c are constants.

4 Closed Form solutions of the main problem using Exp-expansion method

By using the following complex fractional transformation

$$U(\xi) = u(x,t), \quad V(\xi) = v(x,t), \quad \xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} - \frac{lt^{\alpha}}{\Gamma(1+\alpha)} \tag{4.1}$$

and modified Riemann-Liouville derivative, equation (1) is converted into the following ODE

$$\begin{cases} (l + kD_u)U' + r_u U - \alpha_{uu} U^2 - \alpha_{uv} VU = 0 \\ (l + kD_v)V' + r_v \alpha_{vv} V^2 - \alpha_{vu} UV = 0. \end{cases} \tag{4.2}$$

The values of c and d can be freely assigned because the resulting solution does not strongly depend on the choices of c and d So, we opt $p = c = 1$ and $q = d = 1$ so that (3.5) becomes

$$U(\xi) = \frac{a_{-1}e^{-\xi} + a_0 + a_1 e^{\xi}}{b_{-1}e^{-\xi} + b_0 + b_1 e^{\xi}}, \quad V(\xi) = \frac{c_{-1}e^{-\xi} + c_0 + c_1 e^{\xi}}{d_{-1}e^{-\xi} + d_0 + d_1 e^{\xi}}. \tag{4.3}$$

Substituting (4.3) into (4.2)gives $r_u a_{-1}b_{-1}d_{-1} - \alpha_{uu}a_{-1}^2 d_{-1} - \alpha_{uv}a_{-1}c_{-1}b_{-1} = 0$,

$$\begin{aligned}
&r_u a_{-1}b_{-1}d_0 + r_u a_{-1}b_0 d_{-1} + r_u a_0 b_{-1}d_{-1} - \alpha_{uu}a_{-1}^2 d_0 - 2\alpha_{uu}a_{-1}a_0 d_{-1} - 2\alpha_{uu}a_{-1}a_0 d_{-1} \\
&-\alpha_{uv}a_{-1}c_{-1}b_0 - \alpha_{uv}a_0 c_{-1}b_{-1} - \alpha_{uv}a_{-1}c_0 b_{-1} - (l + D_u k)a_{-1}b_0 d_{-1} - (l + D_u k)a_{-1}b_0 d_{-1} \\
&+(l + D_u k)a_0 b_{-1}d_{-1} = 0, \\
&-(l + D_u k)a_{-1}b_0 d_0 + (l + D_u k)a_0 b_{-1}d_0 - 2(l + D_u k)a_{-1}b_1 d_{-1} + 2(l + D_u k)a_1 b_{-1}d_{-1} \\
&+r_u a_{-1}b_{-1}d_1 + r_u a_{-1}b_0 d_0 + r_u a_{-1}b_1 d_{-1} + r_u a_0 b_{-1}d_0 + r_u a_0 b_0 d_{-1} + r_u a_1 b_{-1}d_{-1} \\
&-\alpha_{uu}a_1^2 d_1 - 2\alpha_{uu}a_{-1}a_0 d_0 - 2\alpha_{uu}a_{-1}a_1 d_{-1} = 0 \\
&-(l + D_u k)a_{-1}b_0 d_1 + (l + D_u k)a_0 b_{-1}d_1 - 2(l + D_u k)a_{-1}b_1 d_0 + 2(l + D_u k)a_1 b_{-1}d_0 \\
&+(l + D_u k)a_1 b_0 d_{-1} - (l + D_u k)a_0 b_1 d_{-1} + r_u a_{-1}b_0 d_1 + r_u a_{-1}b_1 d_0 + r_u a_0 b_{-1}d_1 \\
&-\alpha_{uv}a_{-1}c_0 b_1 + r_u a_1 b_{-1}d_0 + r_u a_0 b_0 d_0 + r_u a_0 b_1 d_{-1} - 2\alpha_{uu}a_0 a_1 d_{-1} - \alpha_{uv}a_0 c_{-1}b_1 \\
&+r_u a_1 b_0 d_{-1} - 2\alpha_{uu}a_{-1}a_0 d_1 - 2\alpha_{uu}a_{-1}a_1 d_0 - \alpha_{uu}a_0^2 d_0 - \alpha_{uv}a_1 c_{-1}b_0 - \alpha_{uv}a_0 c_0 b_0 \\
&-2\alpha_{uu}a_0 a_1 d_{-1} - \alpha_{uv}a_{-1}c_1 b_0 - \alpha_{uv}a_1 c_0 b_{-1} - \alpha_{uv}a_0 c_1 b_{-1} = 0
\end{aligned}$$

$$\begin{aligned}
&-2(l + D_u k)a_{-1}b_1 d_1 + 2(l + D_u k)a_1 b_{-1}d_1 + (l + D_u k)a_1 b_0 d_0 + (l + D_u k)a_1 b_0 d_1 \\
&-(l + D_u k)a_0 b_1 d_0 + r_u a_{-1}b_1 d_1 + r_u a_0 b_0 d_1 + r_u a_0 b_1 d_0 + r_u a_1 b_{-1}d_1 - 2\alpha_{uu}a_0 a_1 d_0 \\
&+r_u a_1 b_0 d_0 + r_u a_1 b_1 d_{-1} - 2\alpha_{uu}a_{-1}a_1 d_1 - \alpha_{uu}a_0^2 d_{-1} - \alpha_{uu}a_0^2 d_1 - \alpha_{uu}a_1^2 d_{-1} \\
&-\alpha_{uv}a_1 c_{-1}b_1 - \alpha_{uv}a_0 c_0 b_{-1} - \alpha_{uv}a_0 c_0 b_1 - \alpha_{uv}a_{-1}c_1 b_1 - \alpha_{uv}a_1 c_0 b_0 - \alpha_{uv}a_0 c_1 b_0 \\
&-\alpha_{uv}a_1 c_1 b_{-1} = 0 \\
&-(l + D_u k)a_0 b_1 d_1 + r_u a_0 b_1 d_1 + r_u a_1 b_0 d_1 + r_u a_1 b_1 d_0 - 2\alpha_{uu}a_0 a_1 d_1 - \alpha_{uu}a_1^2 d_0 \\
&-\alpha_{uv}a_1 c_0 b_1 - \alpha_{uv}a_0 c_1 b_1 - \alpha_{uv}a_1 c_1 b_0 = 0 \\
&r_u a_1 b_1 d_1 - \alpha_{uu}a_1^2 d_1 - \alpha_{uv}a_1 c_1 b_1 = 0
\end{aligned}$$

$r_v\alpha_{vv}c_{-1}^2b_{-1} - \alpha vua_{-1}c_{-1}d_{-1} = 0$

$-(l+D_vk)c_{-1}b_{-1}d_0 + (l+D_vk)c_0b_{-1}d_{-1} + r_v\alpha_{vv}c_{-1}^2b_0 + 2r_v\alpha_{vv}c_{-1}c_0b_{-1}$
$-\alpha_{vu}a_{-1}c_{-1}d_0 - \alpha_{vu}a_0c_{-1}d_{-1} - \alpha_{vu}a_{-1}c_0d_{-1} = 0$

$-(l+D_vk)c_{-1}b_0d_0 + (l+D_vk)c_0b_0d_{-1} - 2(l+D_vk)c_{-1}b_{-1}d_1 + 2(l+D_vk)c_1b_{-1}d_{-1}$
$+(l+D_vk)c_1b_{-1}d_0 + r_v\alpha_{vv}c_{-1}^2b_1 + 2r_v\alpha_{vv}c_{-1}c_0b_0 + 2r_v\alpha_{vv}c_{-1}c_1b_{-1} + r_v\alpha_{vv}c_0^2b_{-1}$
$-\alpha_{vu}a_{-1}c_{-1}d_1 - \alpha_{vu}a_0c_{-1}d_0 - \alpha_{vu}a_{-1}c_0d_0 - \alpha_{vu}a_1c_{-1}d_{-1} - \alpha_{vu}a_0c_0d_{-1}$
$-\alpha_{vu}a_{-1}c_1d_{-1} = 0$
$-(l+D_vk)c_{-1}b_1d_0 + (l+D_vk)c_0b_1d_{-1} - 2(l+D_vk)c_{-1}b_0d_1 + 2(l+D_vk)c_1b_0d_{-1}$
$+(l+D_vk)c_1b_0d_0 - (l+D_vk)c_0b_{-1}d_1 + 2r_v\alpha_{vv}c_{-1}c_0b_1 + 2r_v\alpha_{vv}c_{-1}c_1b_0 + r_v\alpha_{vv}c_0^2b_0$
$+2r_v\alpha_{vv}c_0c_1b_{-1} - \alpha_{vu}a_0c_{-1}d_1 - \alpha_{vu}a_{-1}c_0d_1 - \alpha_{vu}a_1c_{-1}d_0 - \alpha_{vu}a_0c_0d_0$
$-\alpha_{vu}a_{-1}c_1d_0 - \alpha_{vu}a_1c_0d_{-1} - \alpha_{vu}a_0c_1d_{-1} = 0$

$-2(l+D_vk)c_{-1}b_1d_1 + 2(l+D_vk)c_1b_1d_{-1} + (l+D_vk)c_1b_1d_0 - (l+D_vk)c_0b_0d_1$
$+2r_v\alpha_{vv}c_{-1}c_1b_1 + r_v\alpha_{vv}c_0^2b_1 + 2r_v\alpha_{vv}c_1c_0b_0 + r_v\alpha_{vv}c_1^2b_{-1} - \alpha_{vu}a_1c_{-1}d_1$
$-\alpha_{vu}a_0c_0d_1 - \alpha_{vu}a_{-1}c_1d_1 - \alpha_{vu}a_1c_0d_0 - \alpha_{vu}a_0c_1d_0 - \alpha_{vu}a_1c_1d_{-1} = 0$

$-(l+D_vk)c_0b_1d_1 + 2(l+D_vk)c_0c_1b_{-1} + r_v\alpha_{vv}c_1^2b_0 - \alpha_{vu}a_1c_0d_1 - \alpha_{vu}a_0c_1d_1$
$-\alpha_{vu}a_1c_1d_0 + r_v\alpha_{vv}c_1^2b_1 - \alpha_{vu}a_1c_1d_1 = 0$
k, and l, are constant.
Equating coefficients of $exp(n\xi)$ to be zero, we obtain the following system of equations and on solving these equations with the aid of the mathematical software such as mathematica, we have
Case 1: Suppose $a_{-1}c_{-1}\alpha_{uu}\alpha_{uv}\alpha_{vv} \neq 0, \quad a_{-1}\alpha_{uu} \neq 0$, then we find $a_1 = 0, \quad b_1 = 0, \quad d_{-1} = 0, \quad b_{-1} = 0, \quad c_1 = 0,$
$k = 0, \quad l = 0, \quad a_0 = 0, \quad d_0 = \frac{-b_0c_{-1}\alpha_{uv}}{a_{-1}\alpha_{uv}},$
where α_{vu}, r_u, r_v, D_u, D_v, b_0, c_0, d_1 are arbitrary constants.
The corresponding solutions for cases $(1), (2)$ *and* (3) are as follows $U_{1,1}(x,t) = a_{-1}(b_0)^{-1}$, $V_{1,1}(x,t) = (c_{-1}+c_0)(\frac{-b_0c_{-1}\alpha_{uv}}{a_{-1}\alpha_{uv}} + d_1)^{-1}$ where l=0, k=0.
Case 2: Suppose $D_u = 0, \quad a_{-1}c_{-1}\alpha_{uu}\alpha_{uv} \neq 0, \quad a_{-1}d_1\alpha_{uu} \neq 0$, then we find $d_{-1} = 0, \quad b_{-1} = 0, \quad a_1 = 0, \quad b_1 = 0, \quad D_v = 0,$
$c_1 = \frac{d_1r_u}{\alpha_{uv}}, \quad l = 0, \quad a_0 = 0, \quad d_0 = 0$
where r_u, α_{vv}, r_v, α_{vu}, b_0, c_0, k are arbitrary constants.
$U_{1,2}(x,t) = (a_{-1}e^{\kappa x^\beta})(b_0)^{-1}, \quad V_{1,2}(x,t) = (c_{-1}e^{\kappa x^\beta} + c_0 + \frac{d_1r_u}{\alpha_{uv}}e^{\kappa x^\beta})(d_1e^{\kappa x^\beta})^{-1}$ where $\kappa = \frac{k}{\Gamma(1+\beta)}, \quad l = 0.$
Case 3: Suppose $k = \frac{-1}{D_v}, \quad a_{-1}c_{-1}\alpha_{uu}\alpha_{uv} \neq 0$, then we find $a_0 = 0, \quad d_0 = 0, \quad a_{-1}d_1(-D_u + D_v)l \neq 0, \quad l = \frac{d_1D_vr_u - c_1D_v\alpha_{uv}}{d_1(-D_u+D_v)}, \quad D_v \neq 0,$
$d_{-1} = 0, \ b_{-1} = 0, \ a_1 = 0, \quad b_1 = 0,$
where α_{vu}, α_{vv}, r_u, r_v, b_0 c_0, c_1 are arbitrary constants. $U_{1,3}(x,t) = (a_{-1}e^{\omega t^\alpha - \kappa x^\beta})(b_0)^{-1}$,
$V_{1,3}(x,t) = (c_{-1}e^{\omega t^\alpha - \kappa x^\beta} + c_0 + c_1e^{\omega t^\alpha - \kappa x^\beta})(d_1e^{\omega t^\alpha - \kappa x^\beta})$
where $\omega = \frac{d_1D_vr_u - c_1D_v\alpha_{uv}}{d_1(-D_u+D_v)\Gamma(\alpha+1)}, \quad \kappa = \frac{-1}{D_v\Gamma(\beta+1)}$

Numerical Results

Graphical representations are a useful tool for discussing and articulating problem solutions in a clear and concise manner. In this section, discuss the graphical representations of 2D and 3D to obtain solutions of NL space-time fractional differential equations raised from ecological systems by are displayed in Figs. 2-7.

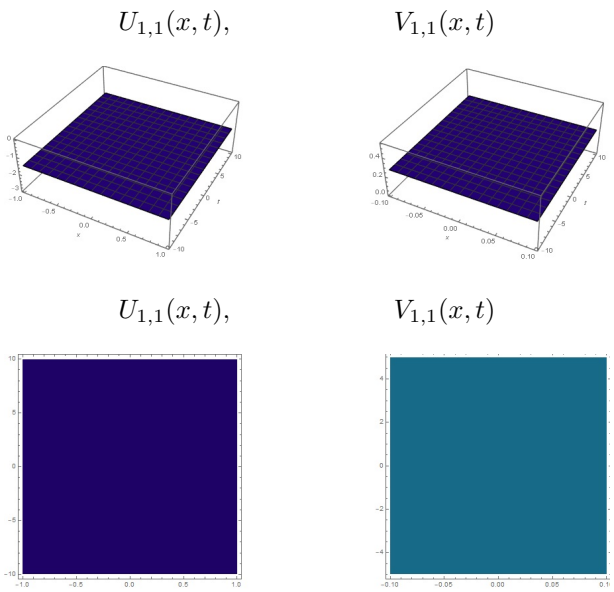


Figure 2: The surface 3D, 2D implicit plot of traveling wave soliton $U_{1,1}(x,t)$,$V_{1,1}(x,t)$ with parameters, $a_{-1} = -3, b_0 = 2, c_{-1} = -2, c_0 = 1, d_1 = 5, \alpha_{uv} = 4$.

The α and β represent order of the fractional derivatives which depends on time and space respectively. Here some figures of the solitary wave solution for space-time fractional differential equation to support the results. Such as Fig. 2 If $\lambda^2 - 4\mu > 0$ represent the traveling wave soliton solution for the proposed model at $\alpha = 0.5, \beta = 0.5, \mu = 6, \lambda = 5, c_0 = 2.4, c_1 = 2, l = 2, D_u = 0.6, r_u = 0.5, k = 5, \alpha_{uv} = 0.9, \alpha_{uu} = 0.7$. Fig. 3 If $\lambda^2 - 4\mu = 0$ shows the periodic shape soliton at $\alpha = 0.5, \beta = 0.5, \mu = 6, \lambda = 5, c_0 = 2.4, c_1 = 2, l = 2, D_u = 0.6, r_u = 0.5, k = 5, \alpha_{uv} = 0.9, \alpha_{uu} = 0.7$. Fig. 4 If $\lambda^2 - 4\mu < 0$ also shows the soliton solution at $\alpha = 0.5, \beta = 0.5, \mu = 6, \lambda = 5, c_0 = 2.4, c_1 = 2, l = 2, D_u = 0.6, r_u = 0.5, k = 5, \alpha_{uv} = 0.9, \alpha_{uu} = 0.7$. Fig. 5 shows the periodic shape soliton at $a_{-1} = -3, b_0 = 2, c_{-1} = -2, c_0 = 1, d_1 = 5, \alpha_{uv} = 4$. Fig. 6 shows the periodic shape soliton at $\beta = 0.5, a_{-1} = 2, b_0 = 3, c_{-1} = -3, c_0 = 4, d_1 = 6, r_u = 2, k = 2, \alpha_{uv} = 4$. Fig. 7 also shows the soliton solution at $\beta = 0.5, \alpha = 0.5, a_{-1} = -12, b_0 = 3.5, c_{-1} = -3.7, c_0 = 2.4, c_1 = 13d_1 = 5, D_v = 0.3, D_u = 0.7r_u = 5.7, k = 2, \alpha_{uv} = 4$. The 2D graphical representations are obtained for various values of other unknown parameters.

Conclusion

In conclusion, our study has focused on the space-time a class of Fractional Differential Equations, Raised From Ecological Systems using conformable fractional derivatives. It should be clear that FDEs can portray great variety of ecological processes and interactions.Less clear is whether or not FDEs generate useful insights or predictions of relevance to a field ecologist. On the other hand, for equations relating to invasions or spatial patterning, FDEs are ideal. For instance, by applying a wide variety of FDE models to the process of invasion, we see that rates of invasion consistently are proportional to rates of population growth at low density and dispersal distances per generation. We have applied the $Exp - expansion$ method to finding new sets of traveling wave solutions for these fractional evolution problems. The $Exp - expansion$ method

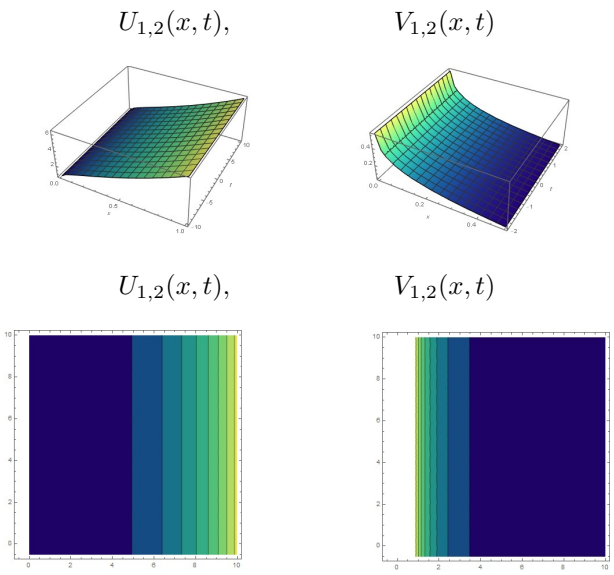


Figure 3: The surface 3D, 2D implicit plot of traveling wave soliton $U_{1,2}(x,t)$,$V_{1,2}(x,t)$ with parameters, $\beta = 0.5, a_{-1} = 2, b_0 = 3, c_{-1} = -3, c_0 = 4, d_1 = 6, r_u = 2, k = 2, \alpha_{uv} = 4$.

is a promising, powerful and can be used to solve many other unstable nonlinear problems of mathematical physics. The performance of the $Exp - expansion$ method is reliable, effective. It is worthwhile to mention that the proposed method is straightforward and concise. It has been noticed that soliton-type solutions are accepted by studying nonlinear differential equations. An important advantage of this technique is its low computational work and preserving the high exactness of the numerical result. These new results have many uses in the area of physics and various other branches of physical science. Through the acquired results, we found the stability of the method applied, and this approach can be applied to other forms of unstable nonlinear evolution equations arising in engineering, mathematical physics, and biological science. All the computational work is performed by the use of mathematical software. In addition, we have plotted graphs for some of the obtained new traveling waves. The obtained solutions are significant in the study of the dynamics and behavior of these fractional equations. Our research contributes to the development of conformable fractional calculus and also to the above two space-time fractional evolution equations. Further investigations could focus on exploring the properties of the obtained solutions and their potential applications. Our results may have practical applications in the fields of material science, optics, and acoustics.

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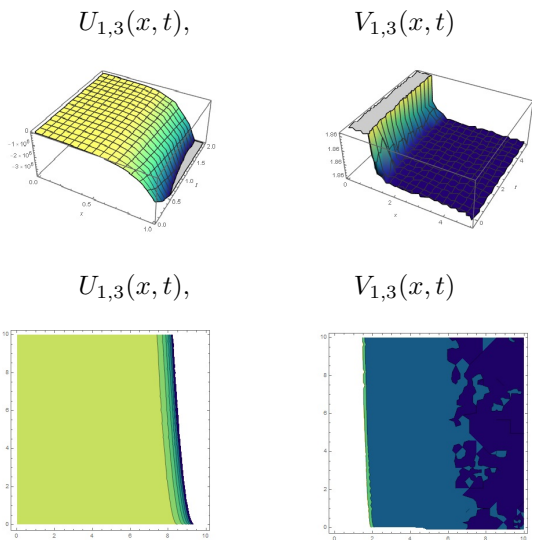


Figure 4: The surface 3D, implicit plot of traveling wave soliton $U_{1,3}(x,t)$,$V_{1,3}(x,t)$ with parameters, $\beta = 0.5, \alpha = 0.5, a_{-1} = -12, b_0 = 3.5, c_{-1} = -3.7, c_0 = 2.4, c_1 = 13 d_1 = 5, D_v = 0.3, D_u = 0.7 r_u = 5.7, k = 2, \alpha_{uv} = 4$.

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A Study of Zagreb and Rivan Indices in Recursive Fibonacci Graphs

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Abstract

In this paper, we investigate various degree-based topological indices of the Fibonacci trees FT_n, which are defined recursively by connecting the roots of two smaller Fibonacci trees FT_{n-1} and FT_{n-2}. We compute and compare several classical and modern indices, including the first and second Zagreb indices and the first and second Revan indices. These indices are calculated explicitly for Fibonacci trees up to a certain depth and analyzed to identify growth patterns and structural sensitivities. The comparative results provide insight into the topological behavior of recursive trees and highlight the potential of new indices such as Revan in distinguishing subtle graph differences.

Keywords: Recursive graphs, Fibonacci trees, Topological indices, Zagreb index, Revan index.

1 Introduction

Graphs are mathematical structures that model pairwise relationships between objects. In graph theory, a graph G=(V,E) consists of a set of vertices V and a set of edges E that connect pairs of vertices. Graphs are widely used in disciplines such as computer science, biology, chemistry, and network analysis to represent complex systems.

A key concept in graph theory is the graph invariant a property that remains unchanged under graph isomorphisms. Invariants such as degree sequences, connectivity, and topological indices provide valuable insights into the structure and behavior of graphs [2]. Topological indices are typically computed for simple and connected graphs, as these properties ensure well-defined and interpretable results. Among various types, degree-based indices are especially popular due to their computational simplicity and effectiveness in capturing local structural information. These indices have found extensive applications in cheminformatics, biology, and network science. Classical indices include the Zagreb indices [5, 9], based on vertex degrees, and the Wiener index [3, 8], based on shortest path distances.

In recent years, several novel degree-based indices have been introduced to enhance the sensitivity and discriminating power of topological analysis [4, 7]. These include Revan indices, which aim to capture both local and global structural variations of graphs more effectively. Furthermore, hybrid indices such as the index have emerged to combine the advantages of multiple descriptors [1].

A particularly interesting class of graphs in this context is the recursive graph. Recursive graphs are defined by a repeated construction rule, allowing the structure to grow in a systematic and self-similar way.

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Their hierarchical nature makes them ideal for studying scaling properties and structural evolution through graph invariants.

One notable example of a recursive graph is the Fibonacci tree FT_n, a rooted tree that grows according to the Fibonacci recurrence: each tree of order n is formed by connecting a tree of order $n-1$ and a tree of order $n-2$ to a new root. These trees exhibit rich combinatorial properties and have been studied in various contexts, including data structures and discrete mathematics [6].

Despite their recursive and well-defined structure, modern degree-based indices such as the F-Sombor index, and the first and second Revan indices have not yet been computed for Fibonacci trees.

In this work, we aim to calculate these topological indices on the Fibonacci trees FT_n, defined recursively with their root connected to two subtrees FT_{n-1} and FT_{n-2}, thereby contributing to the understanding of how such indices behave on recursive structures. Moreover, we present a graphical comparison of the computed indices across different orders of the Fibonacci tree, which allows for a clearer visualization of their growth behavior and relationships. This comparative analysis highlights the distinctive characteristics and sensitivity of each index to structural changes, underscoring their potential value in practical applications such as molecular structure analysis, network resilience, and algorithmic graph classification.

The Zagreb indices of Type I and Type II are among the most well-known and widely used topological indices in graph theory and they are defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d(v)^2, \qquad M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$$

where $d(v)$ is the degree of vertex v in the graph G.

The Revan indices are designed to measure the structural characteristics of graphs by considering the degree distribution of the vertices and their recursive relationships.The first and second Revan indices (denoted R_1) and (denoted R_2) are defined as follows:

$$R_1(G) = \sum_{uv \in E(G)} (r_u + r_v)$$

$$R_2(G) = \sum_{uv \in E(G)} r_u r_v$$

where $V(G)$ is the vertex set and $E(G)$ is the edge set. The *Revan degree* of a vertex $u \in V(G)$ is defined as:

$$r_u = \Delta + \delta - d_u,$$

where:

- Δ: Maximum degree among all vertices in G,
- δ: Minimum degree among all vertices in G,
- d_u: Degree of vertex u (number of edges incident to u).

In the next section, we will calculate these indices for the Fibonacci tree graph and provide closed-form expressions for them. Finally, we will compare their values both numerically and graphically.

Figure 1: Fibonacci trees.

2 Main results

Definition 1. *Fibonacci Trees Fibonacci Trees, represented by FT_n, are defined recursively as follows:*

- *FT_1 is a simple tree consisting of a single vertex.*
- *FT_2 is the path graph P_2, which consists of two vertices connected by an edge.*

For $n > 4$, $FT_n = (V_n, E_n)$ is a binary tree with root r_n, where: The left child is the tree FT_{n-1}, The right child is the tree FT_{n-2}.

The number of vertices in FT_n follows a specific relationship with Fibonacci numbers. In fact, the total number of vertices in FT_n is $F_{n+2} - 1$, where F_n denotes the n-th Fibonacci number (Figure1).

Theorem 2.1. *Let FT_n be the n-th Fibonacci tree. Then, the first and second Zagreb indices of $FT_n, (n > 4)$ are given by:*

$$(i).M_1(FT_n) = 4F_{n+2} + 2F_n - 14,$$
$$(ii).M_2(FT_n) = 12F_n + 5F_{n-1} - 24.$$

where $d(v)$ denotes the degree of vertex v in FT_n.

Proof. (i). The vertex set $V(FT_n)$ can be partitioned into mutually disjoint subsets based on the degrees of the vertices, as follows:

(a) $V_1 = \{x \in V(FT_n) | deg(x) = 1\}$, (leaves)

(b) $V_2 = \{x \in V(FT_n) | deg(x) = 2\}$, (internal vertices with two connections),

(c) $V_3 = \{x \in V(FT_n) | deg(x) = 3\}$, (branching vertices with three connections).

Based on the above partitioning of the vertex set $V(FT_n)$, we compute the first Zagreb index $M_1(FT_n)$ as follows. Since the Zagreb index is defined as the sum of the squares of vertex degrees, we consider each subset separately:

$$M_1(FT_n) = \sum_{v \in V(FT_n)} \deg(v)^2 = \sum_{v \in V_1} \deg(v)^2 + \sum_{v \in V_2} \deg(v)^2 + \sum_{v \in V_3} \deg(v)^2.$$

Given that the degrees in each subset are constant, the above expression simplifies to:

$$M_1(FT_n) = |V_1| \cdot 1^2 + |V_2| \cdot 2^2 + |V_3| \cdot 3^2 = |V_1| + 4|V_2| + 9|V_3|.$$

Due to the recursive structure of the Fibonacci tree, the number of vertices of degree one, i.e., the size of the set V_1, is equal to F_n, where F_n denotes the n-th Fibonacci number. Similarly, the number of vertices of degree three, i.e., the size of the set V_3, is given by $F_n - 2$. Since the total number of vertices in FT_n is $F_{n+2} - 1$, the number of vertices of degree two, i.e., the size of the set V_2, can be determined by subtracting the sizes of V_1 and V_3 from the total number of vertices:

$$|V_2| = (F_{n+2} - 1) - F_n - (F_n - 2) = F_{n+1} - F_n + 1.$$

Substituting these into the Zagreb index expression, we obtain:

$$\begin{aligned} M_1(FT_n) &= |V_1| \cdot 1^2 + |V_2| \cdot 2^2 + |V_3| \cdot 3^2 \\ &= F_n + 4\left(F_{n+1} - F_n + 1\right) + 9(F_n - 2) = 4F_{n+2} + 2F_n - 14. \end{aligned}$$

(ii). To compute the second Zagreb index, the edges are partitioned as follows: Here is the translation in a more formal, impersonal tone:

$$E(FT_n) = E_{1,2} \cup E_{2,2} \cup E_{2,3} \cup E_{1,3} \cup E_{3,3}$$

where each $E_{i,j}$ is defined as:

$$\begin{aligned} E_{1,2} &= \{uv \in E(FT_n) \mid \deg(u) = 1, \deg(v) = 2\} \\ E_{1,3} &= \{uv \in E(FT_n) \mid \deg(u) = 1, \deg(v) = 3\} \\ E_{2,2} &= \{uv \in E(FT_n) \mid \deg(u) = 2, \deg(v) = 2\} \\ E_{2,3} &= \{uv \in E(FT_n) \mid \deg(u) = 2, \deg(v) = 3\} \\ E_{3,3} &= \{uv \in E(FT_n) \mid \deg(u) = 3, \deg(v) = 3\} \end{aligned}$$

$$\begin{aligned} M_2(FT_n) &= |E_{1,2}| \cdot (1 \cdot 2) + |E_{1,3}| \cdot (1 \cdot 3) + |E_{2,2}| \cdot (2 \cdot 2) \\ &\quad + |E_{2,3}| \cdot (2 \cdot 3) + |E_{3,3}| \cdot (3 \cdot 3) \\ M_2(FT_n) &= 2|E_{1,2}| + 3|E_{1,3}| + 4|E_{2,2}| + 6|E_{2,3}| + 9|E_{3,3}| \end{aligned}$$

Thus,

$$|E_{1,2}| = F_{n-1}, |E_{1,3}| = F_{n-2}, |E_{2,2}| = 0, |E_{2,3}| = F_{n-1} + 2, |E_{3,3}| = F_n - 4.$$

Using the Fibonacci recurrence relation, $F_{n-2} = F_n - F_{n-1}$:

$$\begin{aligned} M_2(FT_n) &= 2F_{n-1} + 3F_{n-2} + 4.0 + 6(F_{n-1} + 2) + 9(F_n - 4) \\ &= 8F_{n-1} + 3F_{n-2} + 9F_n - 24 = 12F_n + 5F_{n-1} - 24. \end{aligned}$$

□

Lemma 2.2. *Let FT_n be the n-th Fibonacci tree, The first and second Revan indices of $FT_n, (n > 4)$ is defined as follows:*

$$(i). \boxed{\mathrm{R_1(G)}(FT_n) = 6F_n + 4F_{n-1} - 2}$$

$$(ii). \boxed{\mathrm{R_2(G)}(FT_n) = 5F_{n-1} + 4F_n}$$

Proof. In view of Theorem 2.1, the result follows directly. □

In the following, we will compare these indices and present their graphical representations.

conclusion In this paper, we investigated two important degree-based topological indices,the Zagreb indices and the Revan index on Fibonacci trees. By utilizing the recursive structure of Fibonacci trees, we derived exact expressions for both indices as functions of the tree order. The results reveal how the values of these indices evolve with the growth of the tree, reflecting the underlying complexity and hierarchical nature of Fibonacci tree structures. These findings not only contribute to the theoretical understanding of degree-based indices on special graph families but also provide a basis for potential applications in areas such as mathematical chemistry, data structures, and network analysis

Table 1: Topological indices of FT_n, $(4 < n < 11)$

n	$\|V(FT_n)\| = F_{n+2} - 1$	$M_1(FT_n)$	$M_2(FT_n)$	$R_1(FT_n)$	$R_2(FT_n)$
5	12	48	51	40	35
6	20	86	97	66	57
7	33	148	172	108	92
8	54	248	293	176	149
9	88	410	489	286	241
10	143	672	806	464	390

Figure 2: Fibonacci trees.

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Orthogonally Lie (i,i)-(Homomorphisms, Derivations)-System on Orthogonally Lie Algebras

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Abstract

In this paper, we introduce the concept of orthogonally Lie (i,i)-(homomorphisms, derivations)-system between orthogonally Lie algebras and we investigate its a classes of additive, quadratic, cubic, quartic orthogonally Lie (homomorphisms, derivations)-system for each $i = 1,2,3,4$. After that, we prove the Hyers-Ulam stability of orthogonally Lie triple (i,i)-(homomorphisms, derivations)-system on orthogonally Lie algebras by using orthogonally fixed point theorem. Finally, we show orthogonally Lie (i,i)-(homomorphisms, derivations)-system can be superstable by control functions of Găvruta and Rassias.

Keywords: (i,i)-(homomorphisms, derivations)-system, stability, superstability, Lie algebras.

1 Introduction

In 1940, Ulam introduced the concept of stability in functional equations (FEs) which raised the question of whether functions that are approximately solutions to a FE are also close to being true solutions of that equation. This inquiry sparked a new area of study focused on investigating the stability properties of FEs, delving into the nuances in the input of a FE that impact the proximity of the output to the true solution. In 1941, Hyers became the first to provide a positive response to Ulam's query within the context of B-spaces for additive functions. Subsequently, in 1978, Rassias expanded on Hyers' work by introducing the generalized stability of HU (Hyers-Ulam) and a novel stability concept utilizing a control function of the form $\varepsilon(\|u\|^t + \|v\|^t)$, where $\varepsilon > 0$ and $t < 1$, for additive mappings. Later, in 1994, Găvruta modified Rassias's control function to $\varphi(u,v)$ and demonstrated its stability. The investigation into the stability of FEs, fixed point (FP) theory, fractional differential equations (FDEs) Fibonacci numbers, and optimization theory across different spaces, has been pursued by numerous scholars [2, 4, 5].

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Definition 1 ([3]). *Let $\mathfrak{B}$ be a non-empty set, we say $\perp \subseteq \mathfrak{A} \times \mathfrak{A}$ have property orthogonally set whenever for each $t_1 \in \mathfrak{A}$ there is $t_0 \in \mathfrak{A}$ s.t., $t_1 \perp t_0$ or $t_0 \perp t_1$ and this orthogonally set denotes by $(\mathfrak{A}, \perp)$. For each $(\mathfrak{A}, \perp)$ and mapping $\urcorner : \mathfrak{A} \to \mathfrak{A}$, are said to have property orthogonally generalized metric space, orthogonally sequence, $\perp-$continuous, orthogonally complete and orthogonally preserving, respectively if,*

(a) $(\mathfrak{A}, d)$ be a generalized metric space, then $(\mathfrak{A}, \perp, d)$.
(b) $\{t_n\}_{n\in\mathbb{N}}$ be a sequence, then for each n, $t_n \perp t_{n+1}$ or $t_{n+1} \perp t_n$.
(c) For each orthogonally sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{A}$ with $t_n \to t$.
(d) Every Cauchy orthogonally sequence is convergent to a point in $\mathfrak{A}$.
(e) For every $t_0 \perp t_1$, we have $\urcorner(t_0) \perp \urcorner(t_1)$.

Lie algebras (LAs) are a fundamental concept in mathematics that are named after the Norwegian mathematician Sophus Lie (1842-1899). In 1893, the work of Scheffers compiled much of the foundational knowledge of Lie algebras based on Lie's own lectures in Leipzig. An orthogonally Banach algebra (BA) can be defined as a orthogonally LA with the product $[t_1, t_2] := \frac{1}{2}(t_1 t_2 - t_2 t_1)$.

Definition 2. *Let $\mathfrak{A}$ be an orthogonally LA, A mapping $\mathfrak{g} : \mathfrak{A} \to \mathfrak{A}$ be a Lie homomorphism and $\mathfrak{h} : \mathfrak{A} \to \mathfrak{A}$ be an orthogonally $i-$mapping, $\forall\, u, v \in \mathfrak{A}$,*
(1) $\mathbb{C}$-linear mapping $\zeta : \mathfrak{A} \to \mathfrak{A}$ is called an orthogonally Lie $(1,1)$-(homomorphism, derivation)-system if ζ is an orthogonally additive mapping satisfies,

$$\zeta([u, v]) = [\mathfrak{g}(u), \mathfrak{h}(v)] + [\mathfrak{h}(u), \mathfrak{g}(v)].$$

(2) quadratic mapping $\zeta : \mathfrak{A} \to \mathfrak{A}$ is called an orthogonally Lie $(2,2)$-(homomorphism, derivation)-system if ζ is an orthogonally quadratic mapping satisfies,

$$\zeta([u, v]) = [\mathfrak{g}(u), \mathfrak{h}(v)] + [\mathfrak{h}(u), \mathfrak{g}(v)].$$

(3) cubic mapping $\zeta : \mathfrak{A} \to \mathfrak{A}$ is called an orthogonally cubic Lie $(3,3)$-(homomorphism, derivation)-system if ζ is an orthogonally cubic mapping satisfies,

$$\zeta([u, v]) = [\mathfrak{g}(u), \mathfrak{h}(v)] + [\mathfrak{h}(u), \mathfrak{g}(v)].$$

(4) quartic mapping $\zeta : \mathfrak{A} \to \mathfrak{A}$ is called an orthogonally Lie $(4,4)$-(homomorphism, derivation)-system if ζ is an orthogonally quartic mapping satisfies,

$$\zeta([u, v]) = [\mathfrak{g}(u), \mathfrak{h}(v)] + [\mathfrak{h}(u), \mathfrak{g}(v)].$$

The monomial $\mathfrak{J}(u) = wu^i$ $(u \in \mathbb{R},\ i = 1, 2, 3, 4)$ is a solution of the orthogonally FE,

$$\begin{aligned} &\mathfrak{J}(wu + v) + \mathfrak{J}(wu - v) \\ &\qquad = w^{i-2}[\mathfrak{J}(u + v) + \mathfrak{J}(u - v)] + 2(w^2 - 1)[w^{i-2}\mathfrak{J}(u) + \tfrac{(i-2)(1-(i-2)^2)}{6}\mathfrak{J}(v)], \end{aligned} \tag{1.1}$$

for each $u, v \in \mathfrak{A}$ and any fixed integers $w \neq 0, \pm 1$.

Theorem 1.1 ([1]). *Consider an O-complete generalized metric space $(\mathfrak{A}, d, \perp)$ and a $\perp$-preserving, $\perp$-continuous and $\perp$-γ-contraction $\mathfrak{J}_{tf} : \mathfrak{A} \to \mathfrak{A}$ with $0 < \gamma < 1$. Assume that $s_0 \in \mathfrak{A}$ satisfies for all $t_1 \in \mathfrak{A}$, $t_0 \perp t_1$ or for all $st_1 \in \mathfrak{A}$, $t_1 \perp t_0$, and consider the "O-sequence of successive approximations with initial element t_0":*

$$t_0,\ \mathfrak{J}_{tf}(t_0),\ \mathfrak{J}_{tf}^2(t_0),\ \ldots,\ \mathfrak{J}_{tf}^n(t_0), \ldots .$$

Then, either $d\big(\mathfrak{J}_{tf}^n(t_0), \mathfrak{J}_{tf}^{n+1}(t_0)\big) = \infty$, for $n \geq 0$, or there is $n_0 \in \mathbb{N}$ s.t.,
$d\left(\mathfrak{J}_{tf}^n(t_0), \mathfrak{J}_{tf}^{n+1}(t_0)\right) < \infty$ for every $n > n_0$. If the second alternative holds, then
i) The O-sequence of $\{\mathfrak{J}_{tf}^n(t_0)\}$ is convergent to a FP, t^* of $\mathfrak{J}_{tf}$.
ii) t^* is the unique FP of $\mathfrak{J}_{tf}$ in $\mathfrak{A}^* = \big\{t_1 \in \mathfrak{A} : d(\mathfrak{J}_{tf}^n(t_0), t_1) < \infty\big\}$.
iii) If $t^* \in \mathfrak{A}$, then $d(t_1, t^*) \leq \frac{1}{1-\gamma} d(t_1, \mathfrak{J}_{tf}(t_1))$.

2 Stability and superstability

Throughout this section we suppose that $\mathfrak{A}$ is an orthogonally LA, any fixed integers $w \neq 0, \pm 1$ and i be a positive integer less than 5.

Next, for the sake of making our work easier, we investigate the ideas related to the control function of Găvruta. Let $\varrho_i, \vartheta_i : \mathfrak{A}^2 \to [0, \infty)$ be functions, such that there exists a constant $0 < \gamma < 1$ such that,

$$\varrho_i \left(\tfrac{u}{w}, \tfrac{v}{w}\right) \leq \tfrac{L}{w^i} \varrho_i(u, v), \tag{2.1}$$

and

$$\varrho_i \left(\tfrac{u}{w}, \tfrac{v}{w}\right) \leq \tfrac{L}{w^{2i}} \vartheta_i(u, v), \tag{2.2}$$

for all elements $u, v \in \mathfrak{A}$.

In the following theorem, we apply the FP method to analyze i-mapping that exhibits stability and superstability according to the HU theorem with Găvruta's control function on LA.

Theorem 2.1. *Let $\mathfrak{J} : \mathfrak{A} \to \mathfrak{A}$ be a mapping for which there exist functions $\varrho_i : \mathfrak{A}^2 \to [0, \infty)$ such that*
(i)

$$\begin{aligned} \Big\| \mathfrak{J}(wu + v) + \mathfrak{J}(wu - v) - w^{i-2}\big[\mathfrak{J}(u + v) + \mathfrak{J}(u - v)\big] \\ + 2(w^2 - 1)\Big[w^{i-2}\mathfrak{J}(u) + \tfrac{(i-2)(1-(i-2)^2)}{6}\mathfrak{J}(v)\Big] \Big\| \leq \varrho_i(u, v). \end{aligned} \tag{2.3}$$

Then there exists a unique orthogonally $i-$mapping $\mathfrak{T}_i : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|\mathfrak{J}(u) - \mathfrak{T}_i(u)\| \leq \tfrac{1}{2w^i(\gamma^{-1} - 1)} \varrho_i(u, 0), \tag{2.4}$$

(ii) Let $\lim_{n\to\infty} w^{in} \varrho_i(0, \frac{v}{w^n}) = 0$ and

$$\begin{aligned} \Big\| \mathfrak{J}(wu + v) + \mathfrak{J}(wu - v) - w^{i-2}\big[\mathfrak{J}(u + v) + \mathfrak{J}(u - v)\big] \\ + 2(w^2 - 1)\Big[w^{i-2}\mathfrak{J}(u) + \tfrac{(i-2)(1-(i-2)^2)}{6}\mathfrak{J}(v)\Big] \Big\| \leq \varrho_i(0, v). \end{aligned} \tag{2.5}$$

Then $\mathfrak{J}$ is an $i-$mapping.

In the next theorem, we apply the FP method to an (i, i)–(homomorphisms, derivations)-system that demonstrates stability and superstability per the HU theorem, using Găvruta's control function on LA.

Theorem 2.2. *Let $\mathfrak{g}, \mathfrak{h}, \zeta : \mathfrak{A} \to \mathfrak{A}$ are mappings for which there exist functions $\vartheta_i : \mathfrak{A}^2 \to [0, \infty)$ such that*
(i) satisfying in (2.3), (2.1), and $\|\mathfrak{g}([u, v]) - [\mathfrak{g}(u), \mathfrak{g}(v)]\| \leq \vartheta_i(u, v)$,

$$\|\zeta([u, v]) - [\mathfrak{g}(u), \mathfrak{h}(v)] - [\mathfrak{h}(u), \mathfrak{g}(v)]\| \leq \vartheta_i(u, v).$$

Then there exists a unique orthogonally $i-$mappings $\mathfrak{T}_i, \mathfrak{F}_i, \mathfrak{D}_i : \mathfrak{A} \to \mathfrak{A}$ satisfying (2.4) such that $\mathfrak{F}_i$ is an orthogonally Lie i-homomorphisms and $\mathfrak{D}_i$ an orthogonally Lie (i, i)-(homomorphisms, derivations)-system.
(ii) Let $\lim_{n\to\infty} w^{2in} \vartheta_i(0, \frac{v}{w^n}) = 0$, and $\|\mathfrak{g}([u, v]) - [\mathfrak{g}(u), \mathfrak{g}(v)]\| \leq \vartheta_i(0, v)$,

$$\|\zeta([u, v]) - [\mathfrak{g}(u), \mathfrak{h}(v)] - [\mathfrak{h}(u), \mathfrak{g}(v)]\| \leq \vartheta_i(0, v).$$

Then $\mathfrak{T}_i, \mathfrak{F}_i, \mathfrak{D}_i : \mathfrak{A} \to \mathfrak{A}$ are orthogonally $i-$mappings and $\mathfrak{F}_i$ is an orthogonally i-homomorphisms and $\mathfrak{D}_i$ an orthogonally (i, i)-(homomorphisms, derivations)-system.

In the following corollary, we explore how Theorem 2.1 can be made stable superstable by substituting the control function of Rassias on LA.

Corollary 2.3. *Let θ, r be non–negative real numbers with and $r > i$. Suppose that $\mathfrak{J} : \mathfrak{A} \to \mathfrak{A}$ is an orthogonally mapping such that,*

(i)

$$\Big\|\mathfrak{J}(wu+v)+\mathfrak{J}(wu-v)-w^{i-2}\big[\mathfrak{J}(u+v)+\mathfrak{J}(u-v)\big] + 2(w^2-1)\Big[w^{i-2}\mathfrak{J}(u)+\tfrac{(i-2)(1-(i-2)^2)}{6}\mathfrak{J}(v)\Big]\Big\| \leq \theta(\|u\|^r+\|u\|^r). \tag{2.6}$$

Then there exists a unique orthogonally $i-$mapping $\mathfrak{T}_i$ satisfying

$$\|\mathfrak{J}(u)-\mathfrak{T}_i(u)\| \leq \tfrac{\theta}{2(w^r-w^i)}\|u\|^r. \tag{2.7}$$

(ii) Let

$$\Big\|\mathfrak{J}(wu+v)+\mathfrak{J}(wu-v)-w^{i-2}\big[\mathfrak{J}(u+v)+\mathfrak{J}(u-v)\big] + 2(w^2-1)\Big[w^{i-2}\mathfrak{J}(u)+\tfrac{(i-2)(1-(i-2)^2)}{6}\mathfrak{J}(v)\Big]\| \leq \theta(\|v\|^r). \tag{2.8}$$

Then $\mathfrak{J}$ is an $i-$mapping.

In the next corollary, we achieve stability and superstability by substituting Rassias' control function on LAs using Theorem 2.2.

Corollary 2.4. *Let θ, r be non–negative real numbers with $r > i$. Suppose that $\mathfrak{g}, \mathfrak{h}, \zeta : \mathfrak{A} \to \mathfrak{A}$ are mappings such that*

(i) $\|\mathfrak{g}([u,v]) - [\mathfrak{g}(u), \mathfrak{g}(v)]\| \leq \theta(\|u\|^r + \|v\|^r)$ *and*

$$\|\zeta([u,v]) - [\mathfrak{g}(u), \mathfrak{h}(v)] - [\mathfrak{h}(u), \mathfrak{g}(v)]\| \leq \theta(\|u\|^r + \|v\|^r).$$

Then there exists a unique $\mathfrak{T}_i, \mathfrak{F}_i, \mathfrak{D}_i : \mathfrak{A} \to \mathfrak{A}$ satisfying (2.7) such that $\mathfrak{F}_i$ is an orthogonally Lie i-homomorphisms and $\mathfrak{D}_i$ an orthogonally Lie (i,i)-(homomorphisms, derivations)-system.

(ii) Let $\lim_{n\to\infty} w^{2in}\vartheta_i(0, \frac{v}{w^n}) = 0$, $\|\mathfrak{g}([u,v]) - [\mathfrak{g}(u), \mathfrak{g}(v)]\| \leq \theta(\|v\|^r)$, *and*

$$\|\zeta([u,v]) - [\mathfrak{g}(u), \mathfrak{h}(v)] - [\mathfrak{h}(u), \mathfrak{g}(v)]\| \leq \theta(\|v\|^r).$$

Then homomorphisms $\mathfrak{T}_i$, $\mathfrak{F}_i$, $\mathfrak{D}_i : \mathfrak{A} \to \mathfrak{A}$ are orthogonally $i-$mappings and $\mathfrak{F}_i$ is an orthogonally Lie i-homomorphisms and $\mathfrak{D}_i$ an orthogonally Lie (i,i)-(homomorphisms, derivations)-system.

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A local meshless finite difference method for acoustic wave scattering

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Abstract

This article presents a novel application of the Radial Basis Function-Finite Difference (RBF-FD) method to solve acoustic scattering problems. The approach combines the flexibility of radial basis functions with the simplicity of finite difference schemes to discretize the Helmholtz equation governing acoustic wave propagation. A radiation boundary condition is implemented to simulate the Sommerfeld condition on open-domain scattering scenarios effectively. The study investigates the computational performance, shape parameter effect and method's convergence through a numerical example that has an analytical solution. The study demonstrates the RBF-FD method has good performance at low wave numbers, producing accurate results with sparser node arrangements than conventional discretization techniques. The approach accurately handles boundary conditions and complex boundary shapes, making it suitable for practical applications such as sonar, noise control, and underwater acoustics.

Keywords: Acoustic wave scattering, Radial basis function-finite difference, Radiation boundary condition.

1 Acoustic scattering problem

Acoustic scattering problems involve studying how sound waves interact with objects in their propagation path. These problems are fundamental to numerous engineering applications including sonar systems, noise control, architectural acoustics, and medical ultrasound imaging. Acoustic scattering problems are typically modeled using the Helmholtz equation, which describes time-harmonic wave propagation in the presence of scatterers. For a linear time-harmonic problem, we assume a time dependence of the form $\exp(-i\omega t)$, where ω represents the angular frequency of the wave and $i^2 = -1$ is the imaginary unit. The acoustic pressure $p(\mathbf{x})$ in an unbounded domain Ω, scattered by an obstacle with boundary Γ, satisfies the Helmholtz equation [1]:

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{1.1}$$

where ∇^2 denotes the Laplace operator, and $k = \omega/c$ represents the wavenumber with sound speed c.

The boundary condition on Γ, derived from the incident wave $p_{\text{in}}(\mathbf{x})$, is given by:

$$\frac{\partial p_{\text{in}}(\mathbf{x})}{\partial n_b} + \frac{\partial p(\mathbf{x})}{\partial n_b} = ik\rho c v(\mathbf{x}) \quad \text{on } \Gamma, \tag{1.2}$$

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where n_b denotes the outward normal vector at the boundary surface point, $v(\mathbf{x})$ is the normal vibration velocity, and ρ represents the density of the homogeneous isotropic medium.

At infinity (Γ_∞), the Sommerfeld radiation condition is imposed:

$$\lim_{r\to\infty} r^{\frac{(d-1)}{2}} \left(\frac{\partial p}{\partial r} - ikp\right) = 0, \tag{1.3}$$

where d is the spatial dimension and $r = |\mathbf{x}|$. To create a finite computational domain, an artificial boundary Γ_R is introduced, typically chosen as a circle (2D) or sphere (3D) of radius R, which truncates the original unbounded domain as illustrated in Figure 1. To suppress spurious reflections of outgoing acoustic waves at the truncation boundary and ensure solution uniqueness, a Radiation Boundary Condition (RBC) is imposed on the artificial boundary [2]. The first order RBC, expressed as

$$\frac{\partial p(\mathbf{x})}{\partial n} - (jk - \frac{1}{2r_a})p(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_R \tag{1.4}$$

enforces the Sommerfeld radiation condition, where n denotes the outward normal direction.

So, the acoustic scattering problem is governed by equations (1.1) with boundary conditions (1.2) and (1.4). Since the analytical solution for this problem with complicated domain is unavailable, necessitating an accurate and stable numerical approach. This paper presents a local meshless method based on radial basis function-generated finite differences (RBF-FD) for solving this problem efficiently.

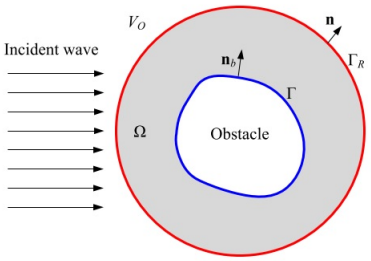


Figure 1: Schematic drawing of finite computational domain and artificial boundary.

2 Formulation of RBF-FD method

Consider the elliptic boundary value problem:

$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = u_0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where f and u_0 are given functions.

To discretize this problem using a meshless method, we distribute N nodes across the domain, with N_i nodes in the interior and N_b nodes on the boundary $\partial\Omega$ (where $N = N_i + N_b$). For any node $\mathbf{x}^*$, its neighborhood $\mathcal{N}(\mathbf{x}^*)$ is defined as the set of n nearest nodes:

$$\mathcal{N}(\mathbf{x}^*) = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \mid \|\mathbf{x}_i - \mathbf{x}^*\| < \tau\}, \tag{2.3}$$

where τ is a prescribed radius of influence.

The radial basis function-finite difference (RBF-FD) method extends the classical finite difference approach [3]. While both methods approximate a differential operator $\mathcal{L}$ at a point using a weighted combination of nearby function values, they differ fundamentally in how weights are determined.

Classical Finite Difference Method

For a uniform 1D grid with spacing h, the classical approach uses predetermined weights. For example, the second derivative approximation at x_i employs the stencil:

$$\mathcal{D}^{\in} u(x_i) \approx \frac{1}{h^2}\big[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})\big] \tag{2.4}$$

These weights are fixed and lead to a sparse differentiation matrix when assembled across the domain.

RBF-FD Method

The RBF-FD method computes weights dynamically by solving a local interpolation problem. For any $\mathbf{x}^* \in \Omega \subset \mathbb{R}^d$, the approximation uses n neighboring nodes $\{\mathbf{x}_j\}_{j=1}^n \in \mathcal{N}(\mathbf{x}^*)$:

$$\mathcal{L}u(\mathbf{x}^*) \approx \sum_{j=1}^{n} w_j u(\mathbf{x}_j) \tag{2.5}$$

The weights $\mathbf{w} = [w_1, \ldots, w_n]^\top$ are determined by enforcing exactness for radial basis functions centered at the neighboring nodes as follows:

$$(\mathcal{L}\varphi_k)(x^*) = \sum_{x_j \in \mathcal{N}(x^*)} w_j^* \varphi_k(x_j), \quad x_k \in \mathcal{N}(x^*). \tag{2.6}$$

This results in a linear system:

$$\mathbf{A}\mathbf{w} = \mathbf{b} \tag{2.7}$$

where $\mathbf{A}_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ is the RBF interpolation matrix and $\mathbf{b}_i = \mathcal{L}\phi(\|\mathbf{x}^* - \mathbf{x}_i\|)$.

The matrix $\mathbf{A}$ is symmetric. Boundary conditions involving differential operators—such as Dirichlet, Neumann, or radiation conditions—are discretized using the RBF-FD method, similar to the treatment of the operator $\mathcal{L}$. Once the weight vectors $\mathbf{w}$ for all nodes in the computational domain Ω are computed, they are assembled into an $N \times N$ sparse matrix. The functions f and u_0 are then incorporated into the right-hand side of the resulting linear system. Finally, solving this system yields an approximate solution for u.

3 Numerical Examples

The scattering of acoustic waves by infinitely long rigid cylinders has served as a benchmark problem in numerous numerical studies. For our analysis, we consider a cylinder with cross-sectional radius $a = 0.5\,\mathrm{m}$. The incident plane wave propagates along the positive x-axis, given by $p_{\mathrm{in}} = \exp(ikx)$.

The normalized analytical solution for the scattered pressure field in polar coordinates (r, θ) is given by [1]:

$$p(r,\theta) = -\sum_{n=0}^{\infty} i^n \epsilon_n \frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} H_n^{(1)}(kr)\cos(n\theta), \tag{3.1}$$

where:

- $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ otherwise
- $J_n(x)$ denotes the Bessel function of the first kind of order n

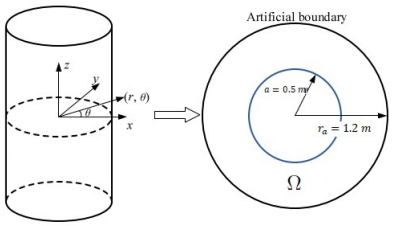


Figure 2: The infinitely long rigid circular cylinder with radius $a = 0.5$m.

- $H_n^{(1)}(x)$ represents the Hankel function of the first kind of order n
- The prime notation indicates differentiation with respect to the argument ka.

The computational domain is bounded by a circular truncation boundary Γ_R with radius $R = 1.2\,\mathrm{m}$ centered at the origin, as illustrated in Figure 2. In this work, we employ generalized multiquadric radial basis functions (GMQ) [4] for approximating the scattered field. The GMQ basis function is defined as:

$$\varphi(r) = \left(r^2 + \varepsilon^2\right)^q, \tag{3.2}$$

where:

- r represents the radial distance,
- ε is the shape parameter that controls the function's flatness,
- q is the exponent that determines the function's growth rate.

These parameters play a crucial role in enhancing both the numerical stability and approximation accuracy of the method. Through numerical experimentation and based on previous optimization studies [5], we adopt $q = 2.5$ as the optimal exponent value for our implementation. The shape parameter ε is carefully selected through a systematic sensitivity analysis to achieve optimal performance.

Accuracy of the RBF-FD method

To validate the accuracy of the proposed method, the real and imaginary components of scattered field across the artificial boundary for three different wave numbers, $k = 2\pi, 4\pi$ and 6π are computed via the proposed RBF-FD method and compared with the analytical solution. As shown in Figures 1-2, the RBF-FD solutions are in good agreement with the exact solutions at $k = 2\pi$ and 4π, whereas a deviation exists when $k = 6\pi$.

Shape parameter effect

The shape parameter ε in GMQ-RBFs significantly influences both accuracy and stability. For our numerical implementation, the shape parameter follows the scaling relation:

$$\varepsilon = \alpha_c d_c \tag{3.3}$$

where α_c is the dimensionless support domain scaling factor and d_c represents the characteristic nodal spacing. For two-dimensional problems, the characteristic nodal spacing d_c is computed as:

$$d_c = \frac{\sqrt{S}}{\sqrt{S n_S} - 1} \tag{3.4}$$

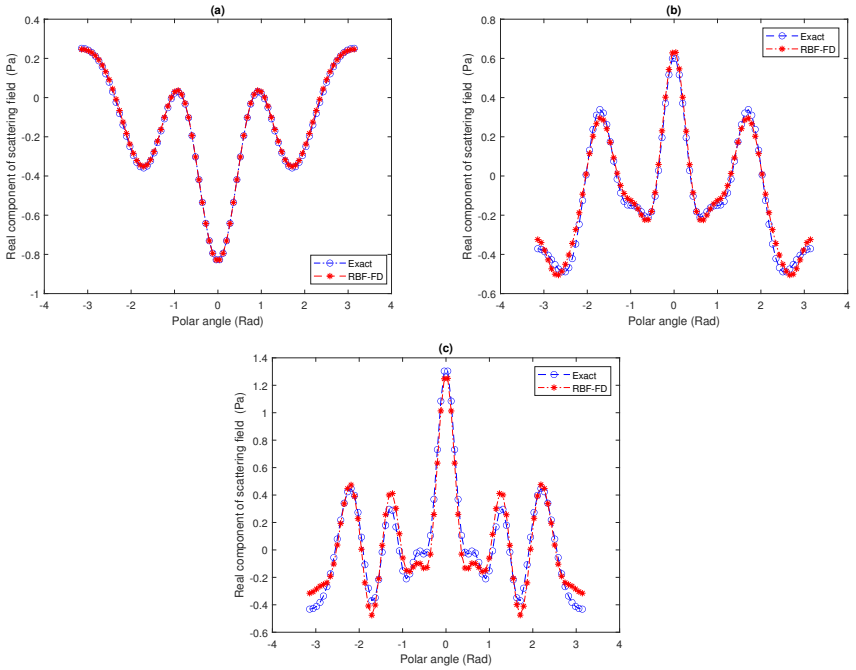


Figure 3: Real components of the scattered field along Γ_R for circular cylinder, a) $k = 2\pi$, b) $k = 4\pi$ and c) $k = 6\pi$.

where S denotes the area of the support domain and n_S is the number of nodes within the support domain. To investigate the impact of the shape parameter α_c on the RBF-FD method's accuracy, we conduct a systematic numerical study with $N = 1320$ uniformly distributed nodes across the computational domain, average nodal spacing $d_c = 0.0274$ and acoustic wavenumber $k = 2\pi$. The shape parameter varies in range $[0, 10]$ with $\Delta\alpha_c = 0.5$ increments.

Figure 3 presents the root-mean-square (RMS) errors of imaginary (ϵ_{imag}) components of the scattering field, evaluated on the artificial boundary. The error metrics are defined as:

$$\epsilon_{\text{imag}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\Im\{p_i^{\text{exact}} - p_i^{\text{approximate}}\}\right)^2} \tag{3.5}$$

Analysis of result demonstrates remarkable stability for $\alpha_c > 4$. Consequently, $\alpha_c = 6$ is considered for all simulations in this study.

convergence analysis of RBF-FD

The convergence properties of the RBF-FD method are numerically investigated by varying the total number of nodes distributed across the computational domain at $k = 2\pi$. The root-mean-square (RMS) prediction errors along the artificial boundary, for both the real (ϵ_{real}) and imaginary (ϵ_{imag}) components, are summarized in Table 1.

As shown in the table, both ϵ_{real} and ϵ_{imag} decrease as the number of nodes increases, confirming the convergence of the numerical solution toward the analytical solution.

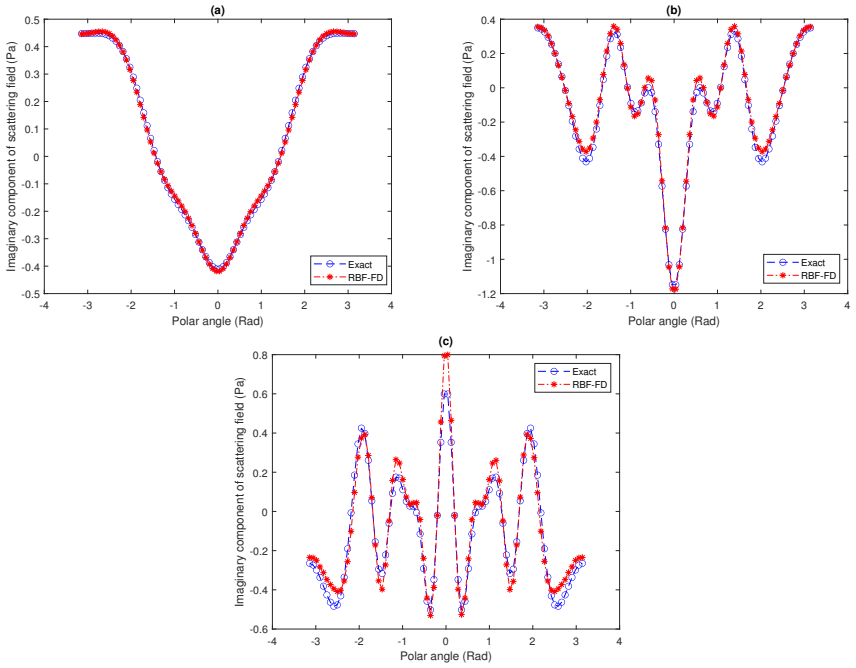


Figure 4: Imaginary components of the scattered field along Γ_R for circular cylinder, a) $k = 2\pi$, b) $k = 4\pi$ and c) $k = 6\pi$.

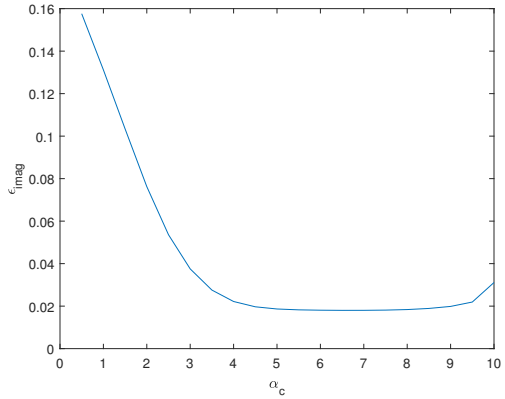


Figure 5: Comparison of RMS-error against the shape parameter α_c.

4 Conclusion

This study establishes the RBF-FD method as an accurate and computationally efficient solution for acoustic scattering problems, particularly in low-wavenumber regimes. By leveraging the geometric flexibility of radial basis functions, the method achieves high precision with sparser node distributions compared to conventional finite difference or finite element approaches. Numerical experiments validate the method's convergence and offer practical guidelines for parameter optimization. RBF-FD emerges as a promising tool for advancing

Table 1: RMS errors for different numbers of nodes in the RBF-FD method.

Number of Nodes	$\epsilon_{\mathbf{real}}$	$\epsilon_{\mathbf{imag}}$
436	5.7e-03	6.9e-03
620	5.2e-03	4.8e-03
820	4.8e-03	4.4e-03
1060	4.5e-03	3.4e-03
1320	3.5e-03	2.9e-03

computational acoustics such as sonar systems and noise control.

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Functional Linear Regression for Predicting Disease Spread: The Role of Environmental Factors

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Abstract

This study explores the application of functional linear regression (FLR) in predicting the spread of infectious diseases by modeling the influence of environmental factors. Specifically, temperature is examined as a key determinant of disease prevalence. Through rigorous statistical analysis, and the usage of real-world data, the findings demonstrate that FLR provides a robust framework for understanding these relationships and improving prediction accuracy in epidemiological studies.

Keywords: functional linear regression, infectious diseases, environmental factors, prediction Accuracy.

1 Introduction

The spread of infectious diseases is significantly influenced by environmental conditions, such as temperature and humidity. Understanding these relationships is crucial for effective public health interventions. Functional linear regression (FLR) methodologies allow for the analysis of complex interactions between time-varying environmental factors and disease outcomes (see e.g. [1] and [2]). This paper investigates the role of environmental factors in predicting disease prevalence using FLR, considering the mathematical formulation and statistical techniques necessary for this analysis.

The relationship between environmental factors and the spread of infectious diseases has been the focus of numerous studies. Research shows that temperature and humidity levels play a significant role in the transmission dynamics of diseases like influenza, dengue, and malaria (see [4] and [5]). For instance, [6] emphasized the effect of climate variability on the epidemiology of infectious diseases, suggesting that shifting patterns could alter transmission dynamics. Similarly, [3] utilized functional regression models to demonstrate strong correlations between climate variables and disease prevalence, advocating for more nuanced statistical approaches.

Functional data analysis (FDA) has emerged as a powerful tool in epidemiological studies, enabling the modeling of complex data structures that arise when observations are collected over time or space. [8] introduced techniques that allow for the fusion of time series observations into functional forms, thereby capturing temporal dynamics absent in traditional regression techniques.

Functional Linear Regression (FLR) combines the benefits of regression analysis and functional data approaches. This method enables researchers to examine how changes in functional predictors, like environmental patterns, impact scalar responses, such as disease prevalence (see e.g. [1]). Studies have shown that FLR can significantly enhance predictive capabilities in epidemiological models when compared to standard linear regression methods (refer to [7]).

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2 Methodology

The FLR model can be mathematically represented as:

$$Y_i = \int_a^b X_i(t)\beta(t)dt + \epsilon_i, \tag{2.1}$$

where Y_i represents the disease prevalence at location i, $X_i(t)$ are the functional environmental predictors (e.g., temperature, humidity), $\beta(t)$ is the functional coefficient describing the relationship, and ϵ_i is the error term.

3 Case Study

For this study, data regarding the disease prevalence of dengue fever across several regions in Brazil were collected from the Ministry of Health between 2015 and 2020. Corresponding temperature data were obtained from local meteorological stations.

Functional coefficients were estimated using the kernel smoothing technique, which allows the estimation of $\beta(t)$ without assuming a specific parametric form. The kernel estimator is given by:

$$\hat{\beta}(t) = \frac{\sum_{i=1}^{n} K_h(t - t_i)Y_i}{\sum_{i=1}^{n} K_h(t - t_i)},$$

where K_h is a kernel function (such as the Gaussian kernel) and h is the bandwidth parameter.

Using real data, functional coefficients $\hat{\beta}(t)$ were estimated, and predictive performance evaluated through cross-validation methods. Mean Squared Error (MSE) and R-squared values were used as metrics for model assessment.

Model	**MSE**	**R-squared**
FLR	0.025	0.85
Parametric Linear Regression	0.043	0.75

Table 1: Model Evaluation Metrics

The results indicated that the FLR model significantly outperformed traditional parametric linear regression in terms of both MSE and R-squared values. This demonstrates the effectiveness of FLR in capturing the intricate relationships between environmental factors and disease prevalence.

The analysis revealed several statistically significant relationships between environmental factors and dengue fever rates. Notably, the functional coefficient $\hat{\beta}(t)$ indicated that increases in temperature corresponded to higher rates of disease incidence.

4 Discussion

The findings underscore the critical role that environmental factors play in the epidemiology of infectious diseases. By employing functional linear regression, this study has illustrated how fluctuations in temperature can be quantitatively assessed, leading to improved predictions of disease prevalence. These insights can inform public health strategies, particularly in anticipating outbreaks and deploying resources effectively.

Additionally, the flexibility of FLR can be beneficial for analyzing other diseases influenced by environmental variables, such as malaria and influenza. Future research should consider integrating additional covariates, such as socio-economic factors or mobility data, to further enhance predictive accuracy.

5 Conclusion

This study demonstrates the utility of functional linear regression in modeling the relationship between environmental factors and disease prevalence. The application of FLR provides a robust framework for understanding the dynamics of infectious disease spread and highlights the importance of accurate environmental data in public health decision-making.

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Weaving continuous k-g-frames in hilbert spaces

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Abstract

In this paper, we introduce the concept of weaving continuous K-g-frames in Hilbert spaces which are the generalization of weaving K-g frames and weaving c-g-frames. We prove some new results for these frames.

Keywords: c-K-g-frame, continuous weaving frame, c-K-woven, c-K-g-woven.

1 Introduction

Nowadays, frames have a significant role in both pure and applied mathematics, so that these are a fundamental research area in mathematics, computer science and engineering, but technical advances and measure amounts of data which cannot be handled with a signal processing system have been increased using the various frames as g-frame, K-frame and weaving frames and etc. c-K-g-frames was introduced in [1], for more information on c-K-g-frames, see [2, 7, 4]. Recently, a new concept of weaving frames was introduced by Bemrose et al. [6] which is motivated by a question in distributed signal processing. By their work, two frames $\{f_j\}_{j\in\mathbb{J}}$ and $\{g_j\}_{j\in\mathbb{J}}$ for a Hilbert space H are (weakly) woven if for each subset $\sigma \in \mathbb{J}$, the family $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is a frame for H. Also, the notation of continuous g-woven in Hilbert spaces and discuss some of their properties had been introduced in [3]. Also, we introduced the concept of weaving continuous K-g-frames in Hilbert spaces and obtained some new results for these frames, see[5]. Throughout this paper, (Ω, μ) is a measure space with positive measure μ, H and $\{H_\omega\}_{\omega\in\Omega}$ are Hilbert spaces and a family of Hilbert spaces, respectively, and $\mathcal{B}(H, K)$ is the set of all bounded and linear operators from H to K. If $H = K$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$ and $K \in \mathcal{B}(H)$. For each $m > 1$ where $m \in \mathbb{N}$, we define $[m] := \{1, 2, \cdots, m\}$ and $[m]^c = \{m+1, m+2, \cdots\}$. The continuous version of K-g-frames have been introduced in [1] in the following way.

Definition 1. *[1] Let $K \in \mathcal{B}(H)$. A family $\{\Lambda_\omega \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega}$ is called a continuous K-g-frame (or c-K-g-frame) for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if*

(i) the mapping

$$\Omega \longmapsto \mathbb{C},$$
$$\omega \longmapsto \|\Lambda_\omega f\|,$$

is measurable for any $f \in H$.

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(ii) there exist constants $0 < A \leq B < \infty$ *such that for each* $f \in H$,

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \leq B\|f\|^2. \tag{1.1}$$

A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-g-Bessel family if the right hand inequality (1.1) holds and the number B is called the Bessel constant. If $K = Id_H$, then the family$\{\Lambda_\omega\}_{\omega\in\Omega}$ is called c-g-frame. If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and if $A = B = 1$, it is called Parseval c-K-g-frame, and we have

$$\int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) = \|K^*f\|^2.$$

Definition 2. ([8]). A family of c-frames $\{F_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ for H with respect to μ is said to be c-woven if there exist universal same positive constants $0 < A \leq B < \infty$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of Ω, the family $\{F_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ is a c-frame for H with bounds A and B. Each family $\{F_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ is called a weaving.

2 Weaving Continuous K-g-Frames

In this section, we introduce the notation of continuous K-g-woven in Hilbert spaces and some of their properties will discuss.

Definition 3. *A family of c-*K*-g-frames* $\{\Lambda_{\omega i} \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega, i\in[m]}$ *for* H *is said to be continuous* K*-g-woven (or c-*K*-g-woven) if there exist universal constants* $0 < A \leq B$ *such that for each partition* $\{\sigma_i\}_{i\in[m]}$ *of* Ω*, the family* $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ *is a c-*K*-g-frame for* H *with bounds* A *and* B*. Each family* $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ *is called a weaving.*

In the above definition, A and B is called universal c-K-g-frame bounds. It is easy to show that every c-K-g-woven has an universal upper c-K-g-frame bound. Indeed, let $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ be a c-g-Bessel family for H with bound B_i for each $i \in [m]$. Then, for any partition $\{\sigma_i\}_{i\in[m]}$ of Ω and $f \in H$,we have

$$\sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu \leq \sum_{i\in[m]} \int_\Omega \|\Lambda_{\omega i} f\|^2 \, d\mu \leq \Big(\sum_{i\in[m]} B_i \Big) \|f\|^2.$$

The next proposition shows that it is enough to check c-K-g-weaving on smaller measurable space than the original which this is an extension of Proposition 3.10 in [8].

Theorem 2.1. *For each* $i \in [m]$*, let* $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ *be a c-*K*-g-frame for* H *with frame bounds* A_i *and* B_i*. If there exists a measurable subset* $\Sigma \subset \Omega$ *such that the family of c-*K*-g-frame* $\{\Lambda_{\omega i}\}_{\omega\in\Sigma, i\in[m]}$ *is a c-*K*-g-woven for* H *with universal frame bounds* A *and* B*, then* $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *is a c-*K*-g-woven for* H *with universal bounds* A *and* $\sum_{i\in[m]} B_i$*.*

Proof. Suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Ω and $f \in H$. we have

$$\sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \leq \sum_{i\in[m]} B_i \|f\|^2.$$

For the lower bound, it is clear that $\{\sigma_i \cap \Sigma\}_{i\in[m]}$ is a partition of Σ. Thus, $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i\cap\Sigma, i\in[m]}$ is a c-K-g-frame for H with the lower frame bound A. Hence, for each $f \in H$

$$\begin{aligned} \sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) &\geq \sum_{i\in[m]} \int_{\sigma_i\cap\Sigma} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\ &\geq A\|K^*f\|^2. \end{aligned}$$

□

Casazza and Lynch showed that It is possible to remove vectors from woven frames and still be left with woven frames. After, this topic was been presented in [8]. Now, we study it for c-K-g-woven in the following Theorem.

Theorem 2.2. *Let $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ be a c-K-g-woven for H with universal bounds A and B. If there exists $0 < D < A$ and a measurable subset $\Sigma \subset \Omega$ and $n \in [m]$ such that for each $f \in H$,*

$$\sum_{i\in[m]\setminus\{n\}} \int_{\Omega\setminus\Sigma} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \leq D\|f\|^2, \qquad \forall f \in H,$$

Then the family $\{\Lambda_{\omega i}\}_{\omega\in\Sigma, i\in[m]}$ is a c-K-g-woven for H with frame bounds $A - D$ and B.

The next result provides a necessary and sufficient condition for c-K-g-woven which connects to c-K-woven.

Theorem 2.3. *Let $\Omega_i \subseteq \Omega$ be measurable subsets for all $i \in [m]$, and let F_i and G_i be c-frame mappings on Ω_i for H_ω with the pair frame bounds (A_{F_i}, B_{F_i}) and (A_{G_i}, B_{G_i}), respectively, for each $\omega \in \Omega$. Assume that $\Lambda_{\omega i}, \Theta_{\omega i} \in \mathcal{B}(H, \Omega_i)$ for any $i \in [m]$ such that $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ are strongly measurable. Then the following assertions are equivalent.*

(I) $\{\Lambda^*_{\omega i} F_i\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta^*_{\omega i} G_i\}_{\omega\in\Omega, i\in[m]}$ are c-K-woven for H.

(II) $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ are c-K-g-woven for H.

In next theorem, a sufficient condition for c-K-g-woven is given .

Theorem 2.4. *Let $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ be a c-K-g-frame for H with frame bounds A_i and B_i for each $i \in [m]$. Suppose that there exists $M > 0$ such that for all $f \in H$, $i \neq k \in [m]$ and all measurable subset $\Delta \subset \Omega$,*

$$\int_\Delta \|(\Lambda_{\omega i} - \Lambda_{\omega k}) f\|^2 \, d\mu(\omega) \leq M \min\Big\{ \int_\Delta \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega), \int_\Delta \|\Lambda_{\omega k} f\|^2 \, d\mu(\omega) \Big\}.$$

Then, the family $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ is a c-K-g-woven with universal bounds

$$\frac{A}{(m-1)(M+1)+1} \quad \text{and} \quad B,$$

where, $A := \sum_{i\in[m]} A_i$ and $B = \sum_{i\in[m]} B_i$.

3 Main Results

For a given c-K-g-woven $\{\Lambda_{\omega i} \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega, i\in[m]}$ of H, we will obtain another c-K-g-woven for the space. One approach is to construct a family $\{\Lambda_{\omega i} \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega, i\in[m]}$, where $U \in \mathcal{B}(H)$. The following theorem gives us necessary and sufficient conditions for $\{\Lambda_{\omega i} U^*\}_{\omega\in\Omega, i\in[m]}$ to be a c-K-g-woven of H.

Theorem 3.1. *Let $K \in \mathcal{B}(H)$ be closed range, $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ be a c-K-g-woven for H with universal bounds A, B and $U \in \mathcal{B}(H)$ with $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Then $\{\Lambda_{\omega i} U^*\}_{\omega\in\Omega, i\in[m]}$ is a c-K-g-woven for H if and only if there exists a constant $\delta > 0$ such that for all $f \in H$*

$$\|U^* f\| \geq \delta \|K^* f\|.$$

Corollary 3.2. *Let K be closed range, $\{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ be a c-K-g-woven for H with universal bounds A, B. If $U \in \mathcal{B}(H)$ with $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Then $\{\Lambda_{\omega i}U^*\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H if and only if U is surjective.*

Theorem 3.3. *Suppose that $\Lambda = \{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H with the lower bound A and $U \in \mathcal{B}(H)$ with $\mathcal{R}(U) \subseteq \mathcal{R}(K)$. Then Λ is a c-U-g-woven for H .*

Theorem 3.4. *Suppose that $K \in \mathcal{B}(H)$ is with dense range, $\Lambda = \{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H and $U \in \mathcal{B}(H)$ is with closed range. If $\{\Lambda_{\omega i}U^*\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H then U is surjective.*

Corollary 3.5. *Let $K \in \mathcal{B}(H)$ be dense range, $\Lambda = \{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ be a c-K-g-woven for H and $U \in \mathcal{B}(H)$ is closed range with $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-woven for H if and only if U is surjective.*

Theorem 3.6. *Suppose that $K \in \mathcal{B}(H)$ and $\Lambda = \{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H with the frame bounds A, B. If $U \in \mathcal{B}(H)$ has closed range with $UK = KU$ and $\mathcal{R}(K^*) \subset \mathcal{R}(U)$, then $\{\Lambda_{\omega i}U^*\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H with frame bounds $A\|U^\dagger\|^{-2}$ and $B\|U\|^2$.*

Proof. As U has closed range, so it has the pseudo-inverse $U^\dagger$ such that $UU^\dagger = I$. Now, for each $f \in H$, we obtain

$$\begin{aligned}\|K^*f\|^2 &= \|UU^\dagger K^*f\|^2 = \|(U^\dagger)^*U^*K^*f\|^2\\ &= \|(U^\dagger)^*K^*U^*f\|^2 \le \|U^\dagger\|^2\|K^*U^*f\|^2,\end{aligned}$$

from which we arrive at

$$\begin{aligned}\sum_{i\in[m]}\int_{\sigma_i}\|\Lambda_{\omega i}U^*f\|^2\,d\mu(\omega) &\ge A\|K^*U^*f\|^2\\ &= A\|U^*K^*f\|^2\\ &\ge A\|U^\dagger\|^{-2}\|K^*f\|^2.\end{aligned}$$

The upper bound is obvious. □

Theorem 3.7. *Let $K \in \mathcal{B}(H)$ be dense range, $\Lambda = \{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ be a c-K-g-woven for H and $U \in \mathcal{B}(H)$ be closed range. If $\{\Lambda_{\omega i}U^*\}_{\omega\in\Omega,i\in[m]}$ and $\{\Lambda_{\omega i}U\}_{\omega\in\Omega,i\in[m]}$ are c-K-g-woven for H, then U is invertible.*

Corollary 3.8. *Suppose that $K \in \mathcal{B}(H)$ has a dense range, that $U \in \mathcal{B}(H)$ has closed range with $UK = KU$, and $\Lambda_{\omega i} \in \mathcal{B}(H, H_\omega)$ for any $\omega \in \Omega, i \in [m]$. If $\{\Lambda_{\omega i}U\}_{\omega\in\Omega,i\in[m]}$ and $\{\Lambda_{\omega i}U^*\}_{\omega\in\Omega,i\in[m]}$ are both c-K-g-woven for H, then $\{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H.*

Proof. By theorem 3.7, U is invertible on H. Since $UK = KU, U^{-1}K = KU^{-1}, \mathcal{R}(K^*) \subset \mathcal{R}(U^{-1}) = H$ and $\{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]} = \{\Lambda_{\omega i}U^*(U^{-1})^*\}_{\omega\in\Omega,i\in[m]}$. Hence by theorem 3.6, $\{\Lambda_{\omega i}\}_{\omega\in\Omega,i\in[m]}$ is a c-K-g-woven for H. □

Bemrose et al. in [6] proved sufficient conditions for weaving frames by means of perturbation and diagonal dominance. perturbation of c-K-g-frames and weaving c-g-frames have been investigated in[1, 7]. We study a Paley-Wiener type perturbation for c-K-g-woven.

Theorem 3.9. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ be two c-K-g-frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively. Suppose that there exist non-negative scalers η and $0 \le \lambda < \frac{1}{2}$ such that $(\frac{1}{2} - \lambda)A_1 > \gamma$ and for each $f \in H$,*

$$int_\Omega\|(\Lambda_\omega - \Gamma_\omega)f\|^2\,d\mu(\omega) \le \lambda\int_\Omega\|\Lambda_\omega f\|^2\,d\mu(\omega) + \gamma\|K^*f\|^2.$$

Then, $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ are c-K-g-woven for H with universal frame bounds $(\frac{1}{2}-\lambda)A_1-\gamma$ and B_1+B_2.

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On φ-Splitting of the Short Exact Sequences of Dual Banach Algebras

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Abstract

Let φ and ψ be w^*-continuous non-zero linear functionals on dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. In this paper, the relation between the notions of φ-splitting of the short exact sequences, are given. In addition to this, some examples of dual Banach algebras which confirm the mentioned properties are presented.

Keywords: φ-σwc virtual diagonal, dual Banach algebra, φ-splitting, the short exact sequences.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. Then E is called dual if there exists a closed submodule $E_* \subseteq E^*$, such that $E = (E_*)^*$. A dual Banach $\mathcal{A}$-bimodule E is called normal if the module actions of $\mathcal{A}$ on E are w^*-continuous. If a Banach algebra as a Banach $\mathcal{A}$-bimodule is dual, then it is called dual. If $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$, then we write $\mathcal{A} = (\mathcal{A}_*)^*$, see [5, 6].

Let $\mathcal{A}$ and E be as above. A bounded, linear map $D : \mathcal{A} \to E$ satisfying $D(ab) = a.D(b) + D(a).b$ for every $a, b \in \mathcal{A}$ is called derivation. A derivation D is inner if there exists $x \in E$ such that for every $a \in \mathcal{A}$, we have $D_x(a) = a.x - x.a$, see [14].

Splitting and admissibility of the short exact sequences are significant notions in homology theory of Banach algebras. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq m \in \mathbb{N}$. A sequence

$$\mathcal{S}_1 \overset{\varphi_1}{\to} \mathcal{S}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{m-1}}{\to} \mathcal{S}_m \tag{1.1}$$

of $\mathcal{A}$-bimodules $S_1, S_2, \cdots, S_m$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_j : \mathcal{S}_j \to \mathcal{S}_{j+1}$ for $j \in \{1, \cdots, m-1\}$ is called exact at position $j = 2, \cdots, m-1$ if $Im\, \varphi_{j-1} = \ker \varphi_j$. If the sequence in (1.1), is exact at every position $j \in \{2, \cdots, m-1\}$, then it is called exact. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{S}_1 \overset{\varphi_1}{\to} \mathcal{S}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{m-1}}{\to} \mathcal{S}_m \to 0 \tag{1.2}$$

of Banach $\mathcal{A}$-bimodules $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_m$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_j : \mathcal{S}_j \to \mathcal{S}_{j+1}$ for $j \in \{1, \cdots, m-1\}$, is admissible, if there exists a bounded linear map $\rho_j : \mathcal{S}_{j+1} \to \mathcal{S}_j$ such that $\rho_j \circ \varphi_j$ on $\mathcal{S}_j$ for $j = 1, 2, \cdots, m-1$, is the identity map. Further we say Θ, splits if we may choose ρ_j to be an $\mathcal{A}$-bimodule homomorphism. Note that if $\mathcal{A}$ is a Banach algebra then the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach

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$\mathcal{A}$-bimodule in the canonical way. In this paper, we consider the following $\mathcal{A}$-bimodule homomorphism (projection induced product mapping)

$$\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}; \qquad a \otimes b \mapsto ab, \qquad (a, b \in \mathcal{A}). \tag{1.3}$$

Assume now that $\mathcal{A}$ is a unital Banach algebra, then the short exact sequence of Banach $\mathcal{A}$-bimodules, $0 \to \ker \pi \to \mathcal{A}\widehat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$, is admissible. It is known that the splitting of the short exact sequences is playing important role in harmonic analysis; in particular, in the notion of Connes amenability of Banach algebras. For instance, in [2, Proposition 4.4], Daws showed that the notion of Connes amenability of a Banach algebra is equivalent to the splitting of the related short exact sequence. Also, Chen investigated the splitting of Rees short exact sequences in [1]. Indeed, a Banach algebra $\mathcal{A}$ is called Connes amenable, if for every normal Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \to E$ is inner, see [14, Definition 4.4.7]. Connes amenability of l^1- Munn algebras is studied by Ghaffari et al. [7]. Runde showed that if G is a locally compact group, then Connes amenability of measure algebra $M(G)$, the existence of a normal virtual diagonal for $M(G)$ and amenability of G are all equivalent, (see [14, Definition 4.4.14] and [13]).
We remark some standard notations and definitions that we shall need in this paper. Throughout the paper, $\Delta(\mathcal{A})$ and $\Delta_{w^*}(\mathcal{A})$ will denote the character space of all continuous and w^*-continuous non-zero complex linear functionals on Banach algebra $\mathcal{A}$, respectively. If $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra and E is a Banach $\mathcal{A}$-bimodule, then $\sigma wc(E)$, a closed submodule of E, denotes the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \to E; \qquad a \mapsto a.x, \qquad a \mapsto x.a, \tag{1.4}$$

are w^*-w continuous. The Banach $\mathcal{A}$-bimodules E that are related to such E are called Banach φ-bimodules, where $\varphi \in \Delta_{w^*}(\mathcal{A})$.

Remark 1.1. Suppose that $\mathcal{A}$ is a dual Banach algebra. Then by above notations, $\Delta_{w^*}(\mathcal{A}) = \Delta(\mathcal{A}) \cap \sigma wc(\mathcal{A}^*)$, see [11, Lemma 2.3]. Indeed, throughout of the paper we use $\Delta_{w^*}(\mathcal{A})$ instead $\Delta(\mathcal{A}) \cap \sigma wc(\mathcal{A}^*)$.

Runde characterized Connes amenability of dual Banach algebras via the existence of normal virtual diagonals, see [14, Theorem 4.4.15]. Also, module Connes amenability for projective tensor product and Θ-Lau product of Banach algebras, χ-module normal virtual diagonal for semigroup algebras and $\chi \otimes \eta$-strong Connes amenability of certain dual Banach algebras, where χ and η are bounded w^*-continuous module homomorphisms from a semigroup algebra to itself is studied in [17, 18, 19].

Dual Banach algebra $\mathcal{A}$ is called φ-Connes amenable if for every normal φ-bimodule E, every bounded w^*-continuous derivation $D : \mathcal{A} \to E$ is inner, where $\varphi \in \Delta_{w^*}(\mathcal{A})$, see [4, Definition 2.1]. The notion of φ-Connes amenability of dual Banach algebras is investigated by Ghaffari and Javadi in [4]. Also, Connes amenability for certain product of Banach algebras and φ-Connes module amenability of dual Banach algebras, where φ is a w^*-continuous bounded module homomorphism from a Banach algebra on itself is investigated by Ghaffari et al., see [5, 6].

Helemskii investigated the homological properties of Banach algebras. He presented the concepts of biflat Banach algebras. Indeed, let $\mathcal{A}$ be a Banach algebra and $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ be the projection induced product map in (2.7). A Banach algebra $\mathcal{A}$ is biflat, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that

$$\pi^{**} \circ \rho(a) = a, \qquad (a \in \mathcal{A}) \tag{1.5}$$

is the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$, see [18]. For example let G be an amenable locally compact group and S be a left zero semigroup (i.e $st = s$ for each $s, t \in S$). Then by Johnson's theorem $L^1(G)$ is amenable. So $L^1(G)$ is biflat, see [9, Proposition 4.5]. One can easily see that $l^1(S)$ is biflat and so does $L^1(G) \otimes_w l^1(S)$, see [12, Proposition 2.5]. For more informations about biflatness, see [8].

Definition 1. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Suppose that the short exact sequence*

$$\Theta : 0 \to \mathcal{A}_* \underset{\tau}{\overset{\pi^*}{\rightleftharpoons}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \frac{\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*}{\pi^*(\mathcal{A}_*)} \to 0 \tag{1.6}$$

of $\mathcal{A}$-bimodules is admissible. Then we say that in (3) sequence Θ, φ-splits if there exists a bounded linear map $\tau : \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}_*$ such that

$$\tau \circ \pi^*(\varphi) = \varphi, \qquad \tau(\Omega.a) = \varphi(a)\tau(\Omega)$$

for every $a \in \mathcal{A}$ and $\Omega \in \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$, see [10]. Note that $\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ is a closed submodule of $(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$.

Definition 2. *Let $\mathcal{A}$ be a dual Banach algebra, and $\varphi \in \Delta_{w^*}(\mathcal{A})$. An element $M \in \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ is called a φ-σwc virtual diagonal for $\mathcal{A}$ if*

$$a.M = \varphi(a)M, \qquad \langle \varphi \otimes \varphi, M\rangle = 1 \tag{1.7}$$

for every $a \in \mathcal{A}$, see [10, Definition 3.1].

Note that the normal virtual diagonal [14, Definition 4.4.15] and the φ-σwc virtual diagonal are equivalent, (see [3, 4, 13, 15]).

Now by above notations, for two unital dual Banach algebras $\mathcal{A} = (\mathcal{A}_*)^*$ and $\mathcal{B} = (\mathcal{B}_*)^*$ we consider and apply certain three short exact sequences, which have three non-zero terms as follow;

$$\Theta_\varphi : 0 \to \mathcal{A}_* \underset{\tau_{\mathcal{A}}}{\overset{\pi^*_{\mathcal{A}}}{\rightleftharpoons}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \frac{\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*}{\pi^*_{\mathcal{A}}(\mathcal{A}_*)} \to 0 \tag{1.8}$$

$$\Theta_\psi : 0 \to \mathcal{B}_* \underset{\tau_{\mathcal{B}}}{\overset{\pi^*_{\mathcal{B}}}{\rightleftharpoons}} \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \frac{\sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^*}{\pi^*_{\mathcal{B}}(\mathcal{B}_*)} \to 0 \tag{1.9}$$

In this paper, we present some characterizations for recent mentioned notions. In fact, the relation between the notions of φ-splitting of the short exact sequences and φ-σwc virtual diagonals of the Bnach algebras, are given. Also, we present some examples in this direction.

2 main results

The aim of this section is to investigate of the relationship between the notions of φ-σwc virtual diagonals of dual Banach algebras and φ-splitting of the related short exact sequences. Also, we investigate some hereditary properties of mentioned notions.

Remark 2.1. Note that in the sequel for the sake brevity, for instance instead the sentence "the short exact sequence Θ_φ, φ-splits" we say "Θ_φ, φ-splits".

In the following, we investigate an equivalence relationship between concepts of φ-σwc virtual diagonal and φ-splitting of the short exact sequence of dual Banach algebras.

Theorem 2.2. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then if Θ_φ, φ-splits. Then the Banach algebra $\mathcal{A}$ has a φ-σwc virtual diagonal.*

Proof. Suppose that Θ_φ, φ-splits. Then, there exists a linear map $\tau : \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}_*$ such that

$$\tau \circ \pi^*(\varphi) = \varphi, \qquad \tau(\Omega.a) = \varphi(a)\tau(\Omega) \tag{2.1}$$

for every $a \in \mathcal{A}, \varphi \in \Delta_{w^*}(\mathcal{A})$ and $\Omega \in \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$. We show that there exists a φ-σwc virtual diagonal for $\mathcal{A}$. Without loss of the generality, suppose that $\mathcal{A}$ is unital, say $e_{\mathcal{A}}$. Then $\varphi(e_{\mathcal{A}}) = 1$. From (2.1), we have

$$\begin{aligned}\langle \varphi \otimes \varphi, \tau^*(e_{\mathcal{A}})\rangle &= \langle \pi^*(\varphi), \tau^*(e_{\mathcal{A}})\rangle \\ &= \varphi(e_{\mathcal{A}}) = 1\end{aligned}$$

Also, by (2.1) we have

$$\begin{aligned}\langle a.\tau^*(e_{\mathcal{A}}), \Omega\rangle &= \langle e_{\mathcal{A}}, \tau(\Omega.a)\rangle \\ &= \langle e_{\mathcal{A}}, \varphi(a)\tau(\Omega)\rangle \\ &= \varphi(a)\langle \tau^*(e_{\mathcal{A}}), \Omega\rangle\end{aligned} \tag{2.2}$$

for all $a \in \mathcal{A}$. Now we see that $\tau^*(e_{\mathcal{A}})$ is desirable. □

Example 2.3. Let $\mathcal{A}$ be an amenable C^*-algebra and let $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then $\mathcal{A}^{**}$ is φ-Connes amenable. Hence, Θ_φ, φ-splits. Then $\mathcal{A}$ has a φ-σwc virtual diagonal. The von Neumann algebras are hold in Theorem 2.2.

Before giving an example let us that remind of the following note.

Remark 2.4. Let $\mathcal{A}$ be a Banach algebra, $(I, \leq)$ be a totally ordered set and let $\Delta(\mathcal{A})$ denotes the character space of all continuous linear functionals on Banach algebra $\mathcal{A}$. By $UP_I(\mathcal{A})$ we denote the set of $I \times I$ upper triangular matrices which its entries belong to $\mathcal{A}$ and we set

$$\| (a_{i,j})_{i,j\in I} \| = \sum_{i,j\in I} \| a_{i,j} \| < \infty.$$

where $a_{i,j}$ are the entries of $\mathcal{A}$ for every $i, j \in I$. With matrix actions and $\| \cdot \|$ as a norm, $UP_I(\mathcal{A})$ turns into a Banach algebra. These algebras are similar (in properties) to the l^1-Munn algebras that thiers Connes amenability is studied in [7]. As above let $\mathcal{A}$ be a Banach algebra with a right identity that $\Delta(\mathcal{A}) \neq \emptyset$ and also let $(I, \leq)$ has smallest element. Then $UP_I(\mathcal{A})$ is biflat if and only if $\mathcal{A}$ is biflat and I is singleton, see [16, Theorem 5.1].

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras, $\psi \in \Delta_{w^*}(\mathcal{B})$ and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then if $\Theta_{\varphi\otimes\varphi}$, $\varphi \otimes \psi$-splits. Then the Banach algebra $\mathcal{A}\widehat{\otimes}\mathcal{B}$ has a $\varphi \otimes \psi$-σwc virtual diagonal.*

The next example confirms the equivalence of conditions in Corollary 2.5.

Example 2.6. Let X be a compact space. Considering $\mathcal{A} = \begin{pmatrix} 0 & 0 \\ C(X) & C(X) \end{pmatrix}$ equipped with usual matrix multiplication and with respect to the norm topology. Let $\mathcal{A}$ be a biflat Banach algebra, $I = \{1, 2\}$ be singleton and $\mathcal{B} = UP_I(\mathcal{A})$. Define $\varphi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathcal{B} \to \mathbb{C}$, via $\varphi \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} = \int_X (f + g)(x)dx$ for all $f, g \in C(X)$ and $\psi(\mathbb{A}) = \sum_{i,j\in I} \| a_{i,j} \| < \infty$ for all $i, j \in I = \{1, 2\}, a_{i,j} \in \mathcal{A}$ and $\mathbb{A} \in UP_I(\mathcal{A})$, respectively. One can see that φ and ψ are linear. Now, put $\varphi \otimes \psi\Big[\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \otimes \mathbb{A}\Big] = \varphi \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \psi(\mathbb{A})$. We claim that φ is w^*-continuous linear functionals. For this reason, suppose that $U = \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix}, U_\alpha = \begin{pmatrix} 0 & 0 \\ f_\alpha & g_\alpha \end{pmatrix} \in \mathcal{A}$ and

$U_\alpha \overset{w^*-topology}{\longrightarrow} U(\alpha \in I)$. Since ranges of φ and ψ are finite-dimensional and we known that in such spaces all topologies are coincide. It is clear that $g_\alpha \overset{w^*-topology}{\longrightarrow} g$ and $f_\alpha \overset{w^*-topology}{\longrightarrow} f$. Therefore φ is w^*-continuous. As the same way ψ is norm continuous and w^*-continuous. Then by Lemma 2.1, $\varphi \otimes \psi \in \Delta_{w^*}(\mathcal{A}\widehat{\otimes}\mathcal{B})$. Now, define

$$\Big[\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \otimes \mathbb{A}\Big]\Big[\begin{pmatrix} 0 & 0 \\ f' & g' \end{pmatrix} \otimes \mathbb{B}\Big] = \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix}\begin{pmatrix} 0 & 0 \\ f' & g' \end{pmatrix} \otimes \mathbb{A}\mathbb{B} = \begin{pmatrix} 0 & 0 \\ gf' & gg' \end{pmatrix} \otimes \mathbb{A}\mathbb{B}$$

for every $f, g \in C(X)$ and $\mathbb{A}, \mathbb{B} \in \mathcal{B}$. Then by [**?**, Chapter 6], $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a biflat Banach algebra. We can see that $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\varphi \otimes \psi$-Connes amenable and thus has a $\varphi \otimes \psi$-σwc virtual diagonal.

Corollary 2.7. *Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ are unital dual Banach algebras. Suppose that $\varphi \in \Delta_{w^*}(\mathcal{A})$ and $\psi \in \Delta_{w^*}(\mathcal{B})$. If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ has a $\varphi \otimes \psi$-σwc virtual diagonal. Then $\mathcal{A}$ has a φ-σwc virtual diagonal and $\mathcal{B}$ has a ψ-σwc virtual diagonal.*

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The time-fractional Pennes bioheat transfer model and analysis of its numerical solution

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Abstract

In this article, we consider the fractional Pennes bioheat transfer model. The numerical solution of this equation is studied. Transforming fractional equations into optimization problem and using fractional-order Bernoulli functions, we obtain the system of algebraic equation. Then, we obtain an approximation of the solution to the equation by solving the nonlinear algebraic system.

Keywords: Bioheat transfer model, optimization problem, Fractional-order Bernoulli Functions.

1 Introduction

The Pennes bioheat transfer equation describes the thermal behavior of living tissues and is mathematically governed by a second-order partial differential equation [1].

$$\rho c\frac{\partial U}{\partial t} = K\frac{\partial^2 U}{\partial r^2} + \rho_b W_b C_b(U_a - U) + q_m + q_r, \tag{1.1}$$

where $\rho_b, W_b, C_b, \rho, c, K, U_a$, and U represent the blood density, blood perfusion rate, specific heat of the blood, tissue density, specific heat of the tissue, thermal conductivity of the tissue, arterial blood temperature, and tissue temperature at time t and space position r, respectively. Recently, models of differential equations with fractional derivatives have attracted the attention of researchers. Then, we consider the fractional type of the bioheat transfer simulation (1.1) by replacing the classical time derivative with the fractional derivative in Caputo sense so that the governing equation takes the form

$$\rho c\frac{\partial^\beta U}{\partial t^\beta} = K\frac{\partial^2 U}{\partial r^2} + \rho_b W_b C_b(U_a - U) + q_m + q_r, \quad 0 < \beta \leq 1. \tag{1.2}$$

The Pennes bioheat transfer model (1.1) has been solved by different analytical and numerical methods, such as in [2] a numerical scheme based on the Laplace transform is proposed for solving the present problems. New non-orthogonal wavelet collocation method based on Fibonacci wavelets is proposed for the solution of the time-fractional Pennes bioheat transfer model in [3].

Finally, we rewrite the fractional Pennes bioheat transfer equation(FPBTE) with the following conditions and analyze and investigate this equation.

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{\partial^2 u(x,t)}{\partial x^2} + Bu(x,t) + g(x,t), \quad 0 < \beta \leq 1, \tag{1.3}$$

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subjected to the initial and boundary conditions

$$u(x,0)=v_0(x), \qquad \frac{\partial u(0,t)}{\partial x}=v_1(t), \qquad u(1,t)=v_2(t). \tag{1.4}$$

The Caputo time–fractional derivative operator of order $\beta>0$ is define

$$D_t^{\beta}f(x,t)=\begin{cases}\frac{1}{\Gamma(k-\beta)}\int_0^t\frac{1}{(t-s)^{\beta-k+1}}\frac{\partial^k f(x,s)}{\partial s^k}\,ds & (\beta>0,\ k-1<\beta<k),\\ \frac{\partial^k f(x,t)}{\partial t^k} & \beta=k.\end{cases} \tag{1.5}$$

2 Fractional-order Bernoulli functions

The classical Bernoulli polynomials of order m are defined on the interval $[0,1]$ as

$$b_m(t)=\sum_{i=0}^{m}\binom{m}{i}b_{m-i}t^i, \tag{2.1}$$

where $b_i:=b_i(0), i=0,1,\ldots,m$, are Bernoulli numbers. The fractional-order Bernoulli functions (FBFs) are constructed by change of variable t to $x^{\alpha},(\alpha>0)$, on the Bernoulli polynomials. By using (1.3) the analytic form of $B_m^{\alpha}(x)$ of order $m\alpha$, is given by

$$B_m^{\alpha}(x)=\sum_{i=0}^{m}\binom{m}{i}B_{m-i}^{\alpha}x^{i\alpha}, \tag{2.2}$$

where B_i, are Bernoulli numbers. The first few fractional-order Bernoulli functions are

$$\begin{cases}B_0^{\alpha}(x)=1 & B_1^{\alpha}(x)=x^{\alpha}-\frac{1}{2}\\ B_2^{\alpha}(x)=x^{2\alpha}-x^{\alpha}+\frac{1}{6} & B_3^{\alpha}(x)=x^{3\alpha}-\frac{3}{2}x^{2\alpha}+\frac{1}{2}x^{\alpha}\end{cases} \tag{2.3}$$

3 Solution of FPBTE

In this section at first, we express the functions approximation with the help of fractional-order Bernoulli functions and we describe the method for solution of FPBTE.

We consider $u(x)$ be an arbitrary continues function on $[0,1]$, then we can express it in the fractional-order Bernoulli functions form,

$$u(x)\simeq B_n^{\alpha}(x)=\sum_{j=0}^{n}c_jB_j^{\alpha}(x), \tag{3.1}$$

where the coefficients c_j, for fractional-order Bernoulli functions are obtained from following

$$c_j=\int_0^1 u(x)\,B_j^{\alpha}(x)\,x^{\alpha-1}dx. \tag{3.2}$$

Now, we approximate $u(x,t)$ as:

$$u(x,t)\cong u_{nm}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{ij}tx^2(x-1)B_i^{\alpha}(x)\,B_j^{\alpha}(t)+H(x,t), \tag{3.3}$$

where $B_i^{\alpha}(x)$ and $B_j^{\alpha}(t)$ are fractional-order Bernoulli functions and c_{ij} are unknown coefficients. In following, we determine $H(x,t)$ which satisfies the problem conditions (1.4), is called satisfier function. In general case, $H(x,t)$ can be chosen as interpolating function:

$$H(x,t) = v_0(x) + (x-1)(v_1(t) - v_1(0)) + x^2(v_2(t) - v_2(0)). \tag{3.4}$$

Substituting (3.3) in (1.3) and define:

$$J(c_{00}, \ldots, c_{0m}, \ldots, c_{n0} \ldots c_{nm}) = \int_0^1 \int_0^1 F(x,t,u_{nm}(x,t))\, dxdt, \tag{3.5}$$

where

$$F(x,t,u(x,t)) = \left(\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} - \frac{\partial^2 u(x,t)}{\partial x^2} - Bu(x,t) - g(x,t) \right)^2 \tag{3.6}$$

If c_{ij} are determined by the minimizing function J, then by (3.3) we achieve functions which approximate the minimum value of J in (3.5) and also satisfy all initial and boundary conditions. According to the necessary conditions of minimization for the function (3.5), we have

$$\frac{\partial J}{\partial c_{ij}} = 0, \qquad i = 0, \ldots, n, \qquad j = 0, ldots, m. \tag{3.7}$$

Then with solving this problem by Maple software, we obtain c_{ij}.

4 Numerical example

In this Section, we discuss the simulation results for an examples.

Example 4.1. Consider the following time-fractional bioheat transfer equation:

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) + g(x,t), \quad 0 < \beta \leq 1, \tag{4.1}$$

subjected to the initial and boundary conditions

$$u(x,0) = 0, \qquad \frac{\partial u(0,t)}{\partial x} = 0, \qquad u(1,t) = \frac{t^{\frac{3}{2}}}{2}, \tag{4.2}$$

where

$$g(x,t) = t^{\frac{3}{2}} x^2 (\frac{3}{2} - x) - 3t^{\frac{3}{2}}(1 - 3x^2) - \frac{3\sqrt{\pi} t^{\frac{3}{2} - \beta} x^2 (2x - 3)}{8\Gamma(\frac{3}{2} - \beta)} + 2xt^{\frac{3}{2}}(x^2 - 6x + 3). \tag{4.3}$$

The exact solution of this problem is

$$u(x,t) = t^{\frac{3}{2}} x^2 (\frac{3}{2} - x). \tag{4.4}$$

According to the method presented in this paper, we let

$$u_{nm}(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} t x^2 (x-1) B_i^{\alpha}(x) B_j^{\alpha}(t) + \frac{x^2 t^{\frac{3}{2}}}{2}, \tag{4.5}$$

now we consider $n = m = 1$. We calculate the proposed method (PE) for this problem with several values of β and $\alpha = 0.5$. The absolute errors (AE) of the present method are given in Tables 1, for $(x_i, t_i) = (\frac{i}{10}, \frac{i}{10})$ with $i = 1, 2, \ldots, 9$ at different fractional values β.

Table 1: AE of the present method and different values of (x,t) and β for Example 1

(x,t)	$\beta=0.2$	$\beta=0.4$	$\beta=0.6$	$\beta=0.8$
(0.1,0.1)	$2.7e{-}14$	$3.0e{-}15$	$1.4e{-}14$	$9.0e{-}15$
(0.2,0.2)	$1.5e{-}13$	$1.0e{-}14$	$8.0e{-}14$	$4.0e{-}14$
(0.3,0.3)	$2.0e{-}13$	$1.0e{-}13$	$1.0e{-}13$	0.0
(0.4,0.4)	$5.0e{-}13$	$1.0e{-}13$	$5.0e{-}13$	$3.0e{-}13$
(0.5,0.5)	$4.0e{-}13$	$1.0e{-}13$	$5.0e{-}13$	$2.0e{-}13$
(0.6,0.6)	0.0	$1.0e{-}13$	$4.0e{-}13$	$1.0e{-}13$
(0.7,0.7)	$3.0e{-}13$	$4.0e{-}13$	$8.0e{-}13$	$3.0e{-}13$
(0.8,0.8)	$8.0e{-}13$	$3.0e{-}13$	$4.0e{-}13$	$1.0e{-}13$
(0.9,0.9)	$9.0e{-}13$	$3.0e{-}13$	$2.0e{-}13$	$1.0e{-}13$

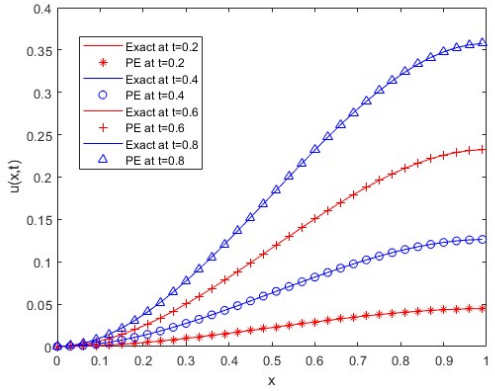


Figure 1: Graph for different time parameters for Example 1.

5 Conclusion

In the present study, we apply new technique for fractional Pennes bioheat transfer model. One of the advantages of this method is that it only requires a few number of basis functions.

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Orthogonally Triple Fuzzy Hom-Derivation: Hyers-Ulam Stability

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Abstract

In this work, we define the orthogonally generalized additive s-functional equation and orthogonally triple Fuzzy hom-derivations on orthogonally triple Fuzzy algebras. Finally, we prove Hyers-Ulam stability the orthogonally generalized additive s-functional equation and orthogonally triple Fuzzy hom-derivations.

Keywords: Hyers-Ulam stability, orthogonally triple Fuzzy hom-derivations, orthogonally triple Fuzzy algebras.

1 Introduction

Stability of functional equations is key in mathematics, especially functional analysis. Ulam introduced it in 1940, asking when approximate solutions are near exact ones. Hyers solved this for Banach spaces in 1941. Rassias extended it in 1978 using a norm-based control function with $\delta(\|u\|^p + \|v\|^p)$, with $\delta > 0$ and $p < 1$. In 1994, Găvruta replaced this with a general function $\varphi(u, v)$, proving stability.

In 2017, Eshaghi Gordji and colleagues [4] introduced the concept of an orthogonal set, which extends the idea of orthogonality found in normed spaces. Building on this concept, Bahraini and colleagues [2] established a generalized version of Diaz-Margolis' fixed point theorem within the framework of orthogonal sets in 2018.

Let $\mathcal{X}$ be nonempty and $\perp \subseteq \mathcal{X} \times \mathcal{X}$ an orthogonality relation if for each $u_1 \in \mathcal{X}$, there exists $u_0 \in \mathcal{X}$ such that $u_1 \perp u_0$ or $u_0 \perp u_1$. Denote $(\mathcal{X}, \perp)$ is orthogonally set. For $(\mathcal{X}, \perp)$ and $L : \mathcal{X} \to \mathcal{X}$, define:

a) If $(\mathcal{X}, d)$ is a generalized metric space, then $(\mathcal{X}, \perp, d)$ is an orthogonally generalized metric space.
b) A sequence $\{u_n\}$ is orthogonally sequence if $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for all n.
c) L is $\perp$-continuous if $u_n \to u$ for each orthogonally sequence $\{u_n\}$.
d) $(\mathcal{X}, \perp)$ is orthogonally complete if every Cauchy orthogonal sequence converges in $\mathcal{X}$.
e) L is orthogonally preserving if $u_0 \perp u_1$ implies $L(u_0) \perp L(u_1)$.
f) L is a $\perp$-contraction with constant $0 < \gamma < 1$ if

$$d(L(u_0), L(u_1)) \leq \gamma d(u_0, u_1), \quad u_0 \perp u_1.$$

In the following, we discuss a complete example of the orthogonally concepts.

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Example 1.1. [3] Let $\mathcal{X} = [0, 1)$ and let the metric on $\mathcal{X}$ be Euclidean metric. For all $u, v \in \mathcal{X}$; $u \perp v$ be defined by $uv \leq max\{\frac{u}{n}, \frac{v}{n}\}$. Let $L : \mathcal{X} \to \mathcal{X}$ be a mapping defined by:

$$L(u) = \begin{cases} \frac{u}{n}, & u \leq \frac{1}{n} \\ 0, & u > \frac{1}{n}. \end{cases}$$

Where $n = 2, 3, \ldots$. Since $u \perp v$ and $uv \leq max\{\frac{u}{n}, \frac{v}{n}\}$. So we have the following cases:
(1) if $u = 0, \quad b \leq \frac{1}{n} \quad \Rightarrow \quad L(u) = 0, \quad L(v) = \frac{v}{n}$;
(2) if $u = 0, \quad v > \frac{1}{n} \quad \Rightarrow \quad L(u) = L(v) = 0$;
(3) if $v \leq \frac{1}{n}, \quad u \leq \frac{1}{n} \quad \Rightarrow \quad L(v) = \frac{v}{n}, \quad L(u) = \frac{u}{n}$;
(4) if $v \leq \frac{1}{n}, \quad u > \frac{1}{n} \quad \Rightarrow \quad L(v) = \frac{v}{n}, \quad L(u) = 0$;

Hence, it's clear that $\mathcal{X}$ isn't complete but it is easy to see that O-complete, L isn't continuous and contraction on $\mathcal{X}$ but it is easy to see that $\perp$-continuous and $\perp$-λ-contraction for $\lambda = \frac{1}{n}$ and in the end L is $\perp$-preserving on $\mathcal{X}$.

In the next, we discuss about orthogonally fuzzy normed algebras.

Definition 1. *Let $\mathcal{X}$ be a vector space. A function $L : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on $\mathcal{X}$ if it satisfies the following conditions:*

(1) For every $u \in \mathcal{X}$ and all $t \leq 0$, $N(u, t) = 0$;

(2) $u = 0$ if and only if for all $t \geq 0$, $N(u, t) = 1$;

(3) For any $u \in \mathcal{X}$ and any nonzero scalar c, $N(cu, t) = N\left(u, \frac{t}{|c|}\right)$;

(4) For all $u, v \in \mathcal{X}$ and $s, t \in \mathbb{R}$, it holds that

$$N(u + v, s + t) \geq \min\{N(u, s), N(v, t)\};$$

(5) For each fixed $u \in \mathcal{X}$, the function $N(u, \cdot)$ is non-increasing on $\mathbb{R}$ and

$$\lim_{t \to \infty} N(u, t) = 1.$$

Definition 2. *Let $(\mathcal{X}, N)$ be a fuzzy normed vector space. A sequence $\{u_n\}$ in $\mathcal{X}$ is said to converge to a point $u \in \mathcal{X}$ if*

$$\lim_{t \to \infty} N(u_n - u, t) = 1$$

for every $t > 0$. In this situation, u is called the limit of the sequence $\{u_n\}$, and we write

$$N\text{-}\lim_{n \to \infty} u_n = u.$$

Definition 3. *Let $(\mathcal{X}, N)$ be a fuzzy normed vector space. A sequence $\{u_n\}$ in X is called Cauchy if, for every $\varepsilon > 0$ and $t > 0$, there exists an integer n_0 such that for all $n \geq n_0$ and all $p > 0$,*

$$N(u_{n+p} - u_n, t) > 1 - \varepsilon.$$

It is well established that every convergent sequence in a fuzzy normed vector space is also a Cauchy sequence. When every Cauchy sequence converges, the fuzzy norm is considered complete, and the fuzzy normed vector space is known as a fuzzy Banach space.

A mapping $L : \mathcal{X} \to \mathcal{X}$ is said to be continuous at a point $u_0 \in \mathcal{X}$ if, for every sequence $\{u_n\}$ that converges to u_0, the sequence $\{L(u_n)\}$ converges to $L(u_0)$. If the mapping f is continuous at every point

$u \in \mathcal{X}$, then L is called continuous on $\mathcal{X}$. In the next, we define the orthogonally generalized additive s-functional equation

$$\begin{aligned} &\|L(u+v+w)+L(u)-L(u+v)-L(v+w)\| \leq \\ &\Big\|s\Big(L(u-v-w)+L(u)-L(u-v)-L(-v-w)\Big)\Big\| \end{aligned} \tag{1.1}$$

for all $u, v, w \in \mathcal{X}$ with $u \perp v$, $u \perp w$ and $v \perp w$ and s is a fixed nonzero complex number with $|s| < 1$.

Definition 4. *Let $(\mathcal{X}, N)$ be an orthogonally triple fuzzy algebra. Let the mapping $\zeta : (\mathcal{X}, N) \to (\mathcal{X}, N)$ is orthogonally triple fuzzy homomorphism, then a $\mathbb{C}$-linear mapping $\theta : (\mathcal{X}, N) \to (\mathcal{X}, N)$ is called a orthogonally triple fuzzy hom-derivation if*

$$\theta\Big([u, v, w]\Big) = [\theta(u), \zeta(v), \zeta(w)] + [\zeta(u), \theta(v), \zeta(w)] + [\zeta(u)\zeta(v), \theta(w)]$$

for all $u, v, w \in \mathcal{X}$ with $u \perp v$, $u \perp w$ and $v \perp w$

In the next section, We investigate the orthogonally generalized $\mathbb{C}$-linear s-functional equation is $\mathbb{C}-$linear mapping. After that, by using [[2], Theorem 3.1]), we prove can the generalized $\mathbb{C}$-linear s-functional equation and orthogonally triple fuzzy hom-derivation be the stable hyers-Ulam on orthogonally triple fuzzy algebras.

2 Main results

In this section, let $\mathcal{X}$ be an orthogonally triple fuzzy algebra and $\lambda \in T^1_{\frac{1}{n_0}}$ is defined as the set of all $e^{i\theta}$ such that θ ranges from 0 to $\frac{2\pi}{n_0}$. Next, for the sake of making our work easier, we define

$$\begin{aligned} \Lambda_\perp(L_\lambda)(u, v, w) := &\Big\|L(\lambda(u+v+w)) + \lambda L(u) - L(\lambda(u+v)) - L(\lambda(v+w))\| \leq \\ &\Big\|s\Big(L(\lambda(u-v-w)) + \lambda f(u) - L(\lambda(u-v)) - L(\lambda(-v-w))\Big)\Big\|, \end{aligned} \tag{2.1}$$

$$\Theta_\perp(\zeta)([u, v, w]) := \theta\Big([u, v, w]\Big) - [\zeta(u), \zeta(v), \zeta(w)]$$

and

$$\begin{aligned} \Psi_\perp(\theta)([u, v, w]) :=& \theta\Big([u, v, w]\Big) - [\theta(u), H(v), H(w)] - [H(u), \theta(v), H(w)] \\ &- [H(u)H(v), \theta(w)] \end{aligned}$$

for all $u, v, w \in \mathcal{X}$.
In the following lemma, we can show the orthogonally generalized additive s-functional equation is additive.

Lemma 2.1. *Let $L : \mathcal{X} \to \mathcal{X}$ be mapping satisfies in (2.1),then L is an additive mapping.*

In the next lemma, we investigate the orthogonally generalized additive s-functional equation can be $\mathbb{C}-$linear mapping.

Lemma 2.2. *If a mapping $L : \mathcal{X} \to \mathcal{X}$ satisfies in (2.2), then the mapping L is $\mathbb{C}-$linear mapping.*

We proceed to examine the principal theorem of this section.

Theorem 2.3. *Let $L, \zeta, \theta : \mathcal{X} \to \mathcal{X}$ be a mapping for which there exist functions $\tau : \mathcal{X}^3 \to [0, \infty)$ such that there exists an $0 < \lambda < 1$ with*

$$\tau(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}) \leq \frac{\lambda}{2}\tau(u, v, w)$$

$$N_{\mathcal{X}}\Big(\Lambda_{\perp}(f_{\lambda})(u, v, w), t\Big) \geq \frac{t}{t + \tau(u, v, w)}.$$

Suppose that

$$N_{\mathcal{X}}\Big(\Theta_{\perp}(\zeta)([u, v, w]), t\Big) \geq \frac{t}{t + \tau(u, v, w)}$$

and

$$N_{\mathcal{X}}\Big(\Psi_{\perp}(\theta)([u, v, w]), t\Big) \geq \frac{t}{t + \tau(u, v, w)},$$

then there exists unique orthogonally triple fuzzy homomorphisms $H : \mathcal{X} \to \mathcal{X}$ and unique orthogonally triple fuzzy hom-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N_{\mathcal{X}}\Big(\zeta(u) - H(u), t\Big) \geq \frac{(6 - 6\lambda)t}{(6 - 6\lambda)t + \tau(u, u, 0)}$$

$$N_{\mathcal{X}}\Big(L(u) - D(u), t\Big) \geq \frac{(6 - 6\lambda)t}{(6 - 6\lambda)t + \tau(u, u, 0)}$$

In the following corollary, we prove Hyers-Ulam-Rassias stability of orthogonally triple fuzzy hom-derivation with condition $0 < p < 1$.

Corollary 2.4. *Let $\delta > 0$ and $L, \zeta, \theta : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$N_{\mathcal{X}}\Big(\Lambda_{\perp}(L_{\lambda})(u, v, w), t\Big) \geq \frac{t}{t + \delta(\|u\|^p + \|u\|^p + \|u\|^p)}.$$

Suppose that

$$N_{\mathcal{X}}\Big(\Theta_{\perp}(\zeta)([u, v, w]), t\Big) \geq \frac{t}{t + \delta(\|u\|^p \|u\|^p \|u\|^p)}$$

and

$$N_{\mathcal{X}}\Big(\Psi_{\perp}(\theta)([u, v, w]), t\Big) \geq \frac{t}{t + \delta(\|u\|^p \|u\|^p \|u\|^p)},$$

then there exists unique orthogonally triple fuzzy homomorphisms $H : \mathcal{X} \to \mathcal{X}$ and unique orthogonally triple fuzzy hom-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N_{\mathcal{X}}\Big(\zeta(u) - H(u), t\Big) \geq \frac{t}{t + \delta(\|u\|^p)} \quad \textit{and} \quad N_{\mathcal{X}}\Big(\theta(u) - D(u), t\Big) \geq \frac{t}{t + \delta(\|u\|^p)}$$

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Results and Inclusion Properties for Poisson distribution

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Abstract

The main object of this paper is to establish connections between various subclasses of analytic univalent functions and Poisson distribution series by applying certain convolution operator involving Poisson distribution series.

Keywords: Analytic functions, Poisson distribution series, Coefficient bounds.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$.

The following subclasses of analytic functions are considered by Ali et al. [1] and Murugusundaramoorthy et al. [3].

For some $\mu_1(0 \leq \mu_1 < 1)$, $\mu_2(\mu_2 \geq 0)$ and $\mu_3(0 \leq \mu_3 \leq 1)$, let the class $\mathcal{P}^*_{\mu_3}(\mu_1, \mu_2)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\frac{zf'(z)}{(1-\mu_3)z + \mu_3 f(z)} - \mu_1\right) > \mu_2 \left|\frac{zf'(z)}{(1-\mu_3)z + \mu_3 f(z)} - 1\right| \quad , z \in \Delta$$

and the class $C^*_{\mu_3}(\mu_1, \mu_2)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\frac{zf'(z) + z^2 f''(z)}{(1-\mu_3)z + \mu_3 z f'(z)} - \mu_1\right) > \mu_2 \left|\frac{zf'(z) + z^2 f''(z)}{(1-\mu_3)z + \mu_3 z f'(z)} - 1\right| \quad , z \in \Delta.$$

Remark 1.1. [2, 6] For some $\mu_1(0 \leq \mu_1 < 1), \mu_2(\mu_2 \geq 0)$ and choosing $\mu_3 = 1$ and f of the form (1.2), let the subfamily $P^*_1(\mu_1, \mu_2)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\frac{zf'(z)}{f(z)} - \mu_1\right) > \mu_2 \left|\frac{zf'(z)}{f(z)} - 1\right| \quad , z \in \Delta$$

and the class $C^*_1(\mu_1, \mu_2)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1 - \mu_1\right) > \mu_2 \left|\frac{zf''(z)}{f'(z)}\right| \quad , z \in \Delta.$$

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Remark 1.2. [3] For some $\mu_1(0 \leq \mu_1 < 1), \mu_2(\mu_2 \geq 0)$ and choosing $\mu_3 = 0$ and f of the form (1.2), let the subfamily $P_0^*(\mu_1, \mu_2)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(f'(z) - \mu_1\right) > \mu_2 \left|f'(z) - 1\right| \quad , z \in \Delta$$

and the class $C_0^*(\mu_1, \mu_2)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\left(zf'(z)\right)' - \mu_1\right) > \mu_2 \left|\left(zf'(z)\right)' - 1\right| \quad , z \in \Delta.$$

Remark 1.3. [7] For some $\mu_1(0 \leq \mu_1 < 1), \mu_2(\mu_2 \geq 0)$ and choosing $\mu_2 = 0, \mu_3 = 1$, the subfamily $P_1^*(\mu_1, 0)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \mu_1 \quad , z \in \Delta$$

and the class $C_1^*(\mu_1, 0)$ consists of functions in $\mathcal{A}$ satisfying the criteria

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \mu_1 \quad , z \in \Delta.$$

Remark 1.4. Both subclasses $P_1^*(\mu_1, 0) \equiv S^*(\mu_1)$ and $C_1^*(\mu_1, 0) \equiv K(\mu_1)$ are well known subclasses of starlike and convex functions of order μ_1, respectively (see [7]).

A variable X is said to have Poisson distribution, if it takes the values $0, 1, 2, 3, \cdots$ with probabilities e^{-m}, $\frac{me^{-m}}{1!}$, $\frac{m^2e^{-m}}{2!}$, $\frac{m^3e^{-m}}{3!}, \cdots$, respectively, where m is called the parameter. Thus

$$P(X = k) = \frac{e^{-m}m^k}{k!}, \quad k = 0, 1, 2, 3, \cdots, \quad m > 0.$$

Recently, Porwal [4] introduce a power series whose coefficients are probabilities of Poisson distribution

$$\varphi_m(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (m > 0). \tag{1.2}$$

It is easy to see that the radius of convergence of $\varphi_m(z)$ is infinite.

Now, we introduce the series

$$\Phi_m(z) = 2z - \varphi_m(z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n. \tag{1.3}$$

Next, by the convolution $(*)$, let a linear operator $\Phi_m(z)f : \mathcal{A} \longrightarrow \mathcal{A}$ as:

$$\Phi_m f = \Phi_m(z) * f(z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n.$$

2 Main Results

To establish our main results, we recall the following Lemma according to Murugusundaramoorthy et al. [3].

Lemma 2.1. *A function f defined by (1.2) is in the class $P^*_{\mu_3}(\mu_1,\mu_2)$ if and only if*

$$\sum_{d=2}^{\infty}\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]|a_d|\leq 1-\mu_1. \tag{2.1}$$

Lemma 2.2. *A function f defined by (1.2) is in the class $C^*_{\mu_3}(\mu_1,\mu_2)$ if and only if*

$$\sum_{d=2}^{\infty}d\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]|a_d|\leq 1-\mu_1. \tag{2.2}$$

The aim of present paper is establish connections between geometric function theory and Poisson distribution by findings relations between different subclasses of analytic univalent functions. Finally, we give conditions for the integral operator $T_m(z)=\int_0^z\frac{\varphi_m(t)}{t}dt$ belonging to the these classes.

Theorem 2.3. *If $m>0$, then Φ_m is in $P^*_{\mu_3}(\mu_1,\mu_2)$, if and only if*

$$\left(\mu_2-\mu_3(\mu_1+\mu_2)+1\right)\left(e^m-1\right)\leq e^m\left(1-\mu_1-m(\mu_2+1)\right). \tag{2.3}$$

Proof. Since

$$\Phi_m(z)=2z-\varphi_m(z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n, \tag{2.4}$$

according to Lemma 1.1, we must show that

$$\sum_{d=2}^{\infty}\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]\frac{e^{-m}m^{d-1}}{(d-1)!}\leq 1-\mu_1. \tag{2.5}$$

By writting $d=(d-1)+1$, we have

$$\begin{aligned}
&\sum_{d=2}^{\infty}\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]\frac{e^{-m}m^{d-1}}{(d-1)!}\\
&\quad=(\mu_2+1)\sum_{d=2}^{\infty}\frac{e^{-m}m^{d-1}}{(d-2)!}+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right]\sum_{d=2}^{\infty}\frac{e^{-m}m^{d-1}}{(d-1)!}\\
&\quad=(\mu_2+1)\sum_{d=0}^{\infty}\frac{e^{-m}m^{d+1}}{d!}+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right]\sum_{d=1}^{\infty}\frac{e^{-m}m^{d}}{d!}\\
&\quad=m(\mu_2+1)\sum_{d=0}^{\infty}\frac{e^{-m}m^{d}}{d!}+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right]\sum_{d=1}^{\infty}\frac{e^{-m}m^{d}}{d!}\\
&\quad=m(\mu_2+1)+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right]\left(1-e^{-m}\right)\leq 1-\mu_1.
\end{aligned}$$

□

Corollary 2.4. *If $m>0$, then Φ_m is in $P^*_1(\mu_1,\mu_2)$, if and only if*

$$\left(1-\mu_2)+1\right)\left(e^m-1\right)\leq e^m\left(1-\mu_1-m(\mu_2+1)\right).$$

Corollary 2.5. *If $m>0$, then Φ_m is in $P^*_0(\mu_1,\mu_2)$, if and only if*

$$(\mu_2+1)\left(e^m-1\right)\leq e^m\left(1-\mu_1-m(\mu_2+1)\right).$$

Corollary 2.6. *If $m > 0$, then Φ_m is in $S^*(\mu_1)$, if and only if*

$$(\mu_2 - \mu_1 + 1)(e^m - 1) \leq e^m(1 - \mu_1 - m).$$

Theorem 2.7. *If $m > 0$, then Φ_m is in $C^*_{\mu_3}(\mu_1, \mu_2)$, if and only if*

$$\begin{aligned}[\mu_2 - \mu_3(\mu_1 + \mu_2) + 1](e^m - 1) \quad \leq \quad & \Big(1 - \mu_1 - m^2(\mu_2 + 1) \\ & -m[3(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)]\Big)e^m. \end{aligned} \tag{2.6}$$

Proof. Since

$$\Phi_m(z) = 2z - \varphi_m(z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \tag{2.7}$$

according to Lemma 1.2, we must show that

$$\sum_{d=2}^{\infty} d[d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)] \frac{e^{-m} m^{d-1}}{(d-1)!} \leq 1 - \mu_1. \tag{2.8}$$

By writting $d^2 = (d-1)(d-2) + 3(d-1) + 1$, we have

$$\begin{aligned}
&\sum_{d=2}^{\infty} d[d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)] \frac{e^{-m} m^{d-1}}{(d-1)!} \\
&\quad = (\mu_2 + 1) \sum_{d=3}^{\infty} \frac{e^{-m} m^{d-1}}{(d-3)!} + [3(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)] \sum_{d=2}^{\infty} \frac{e^{-m} m^{d-1}}{(d-2)!} \\
&\qquad + [\mu_2 - \mu_3(\mu_1 + \mu_2) + 1] \sum_{d=2}^{\infty} \frac{e^{-m} m^{d-1}}{(d-1)!} \\
&\quad = (\mu_2 + 1) \sum_{d=0}^{\infty} \frac{e^{-m} m^{d+2}}{d!} + [3(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)] \sum_{d=0}^{\infty} \frac{e^{-m} m^{d+1}}{d!} \\
&\qquad + [\mu_2 - \mu_3(\mu_1 + \mu_2) + 1] \sum_{d=1}^{\infty} \frac{e^{-m} m^{d}}{d!} \\
&\quad = m^2(\mu_2 + 1) \sum_{d=0}^{\infty} \frac{e^{-m} m^{d}}{d!} + m[3(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)] \sum_{d=0}^{\infty} \frac{e^{-m} m^{d}}{d!} \\
&\qquad + [\mu_2 - \mu_3(\mu_1 + \mu_2) + 1] \sum_{d=1}^{\infty} \frac{e^{-m} m^{d}}{d!} \\
&\quad = m^2(\mu_2 + 1) + m[3(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)] \\
&\qquad + [\mu_2 - \mu_3(\mu_1 + \mu_2) + 1](1 - e^{-m}) \leq 1 - \mu_1.
\end{aligned}$$

□

Corollary 2.8. *If $m > 0$, then Φ_m is in $C^*_1(\mu_1, \mu_2)$, if and only if*

$$(1 - \mu_1)(e^m - 1) \leq \Big(1 - \mu_1 - m^2(\mu_2 + 1) - m[2\mu_2 + 3 - \mu_1]\Big)e^m.$$

Corollary 2.9. *If $m > 0$, then Φ_m is in $C_0^*(\mu_1, \mu_2)$, if and only if*

$$(\mu_2 + 1)(e^m - 1) \leq \Big(1 - \mu_1 - m^2(\mu_2 + 1) - 3m(\mu_2 + 1)\Big)e^m.$$

Corollary 2.10. *If $m > 0$, then Φ_m is in $K(\mu_1)$, if and only if*

$$(1 - \mu_1)(e^m - 1) \leq \Big(1 - \mu_1 - m^2 - m(3 - \mu_1)\Big)e^m.$$

3 Inclusion Properties

For $0 < h_1 \leq 1,\ h_2 < 1,\ h_3 \in \mathbb{C}$, a function $f \in \mathcal{A}$ is said to be in the class $KC^{h_3}(h_1, h_2)$, if it satisfies the following

$$\left|\frac{(1 - h_1)\frac{f(z)}{z} + h_1 f'(z) - 1}{2h_3(1 - h_2) + (1 - h_1)\frac{f(z)}{z} + h_1 f'(z) - 1}\right| < 1, \quad z \in \Delta.$$

Lemma 3.1. *[7] If $f \in KC^{h_3}(h_1, h_2)$ is of the form (1.2), then*

$$|a_n| \leq \frac{2|h_3|(1 - h_2)}{h_1(d - 1) + 1}, \quad n \in \mathbb{N} - \{1\}. \tag{3.1}$$

Theorem 3.2. *For $f \in KC^{h_3}(h_1, h_2)$, if the inequality*

$$\Big(m(\mu_2 + 1) + [\mu_2 - \mu_3(\mu_1 + \mu_2) + 1](1 - e^{-m})\Big) \leq \frac{h_1(1 - \mu_1)}{2|h_3|(1 - h_2)} \tag{3.2}$$

is satisfied, then $\Phi_m f \in C_{\mu_3}^*(\mu_1, \mu_2)$.

Proof. Since $f \in KC^{h_3}(h_1, h_2)$ and by Lemma 1.2 it suffices to show that

$$\sum_{d=2}^{\infty} d\,[d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)]\,\frac{e^{-m}m^{d-1}}{(d - 1)!}|a_d| \leq 1 - \mu_1. \tag{3.3}$$

According to Lemma 3.1, we have

$$\begin{aligned}
&\sum_{d=2}^{\infty} d\,[d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)]\,\frac{e^{-m}m^{d-1}}{(d - 1)!}|a_d| \\
&\quad \leq (2|h_3|(1 - h_2))\sum_{d=2}^{\infty} d\,[d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)]\,\frac{e^{-m}m^{d-1}}{(d - 1)!\,(h_1(d - 1) + 1)}.
\end{aligned}$$

Since $h_1(d - 1) + 1 \geq dh_1$, we have

$$\begin{aligned}
&\sum_{d=2}^{\infty} d\,[d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)]\,\frac{e^{-m}m^{d-1}}{(d - 1)!}|a_d| \\
&\quad \leq \frac{2|h_3|(1 - h_2)}{h_1}\sum_{d=2}^{\infty} [d(\mu_2 + 1) - \mu_3(\mu_1 + \mu_2)]\,\frac{e^{-m}m^{d-1}}{(d - 1)!}.
\end{aligned}$$

Proceeding as in Theorem 2.1, we get

$$\sum_{d=2}^{\infty} d\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]\frac{e^{-m}m^{d-1}}{(d-1)!}|a_d|$$
$$\leq \frac{2|h_3|(1-h_2)}{h_1}\Big(m(\mu_2+1)+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right](1-e^{-m})\Big).$$

The last expression is bounded above by $1-\mu_1$, this concludes the proof of this Theorem. □

3.1 An integral operator

In this part, we will examine some inclusion properties of integral operator associated with the function $\Phi_m(z)$ as follow

$$T_m(z)=\int_0^z \frac{\Phi_m(t)}{t}dt, \tag{3.4}$$

where $\Phi_m(z)$ is given by (2.7).

Theorem 3.3. *If $m>0$, then T_m defined by (4.6) is in $C^*_{\mu_3}(\mu_1,\mu_2)$ if and only if*

$$(\mu_2-\mu_3(\mu_1+\mu_2)+1)(e^m-1)\leq e^m(1-\mu_1-m(\mu_2+1)). \tag{3.5}$$

Proof. Since

$$\begin{aligned} T_m(z) &= z-\sum_{d=2}^{\infty}\frac{e^{-m}m^{d-1}}{(d-1)!}\frac{z^d}{d} \\ &= z-\sum_{d=2}^{\infty}\frac{e^{-m}m^{d-1}}{d!}z^d. \end{aligned} \tag{3.6}$$

By Lemma 1.2, we need only to show that

$$\sum_{d=2}^{\infty} d\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]\frac{e^{-m}m^{d-1}}{d!}\leq 1-\mu_1. \tag{3.7}$$

Now

$$\begin{aligned} &\sum_{d=2}^{\infty} d\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]\frac{e^{-m}m^{d-1}}{d!} \\ &=\sum_{d=2}^{\infty}\left[d(\mu_2+1)-\mu_3(\mu_1+\mu_2)\right]\frac{e^{-m}m^{d-1}}{(d-1)!} \\ &=(\mu_2+1)\sum_{d=2}^{\infty}\frac{e^{-m}m^{d-1}}{(d-2)!}+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right]\sum_{d=2}^{\infty}\frac{e^{-m}m^{d-1}}{(d-1)!} \\ &=m(\mu_2+1)\sum_{d=0}^{\infty}\frac{e^{-m}m^{d}}{d!}+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right]\sum_{d=1}^{\infty}\frac{e^{-m}m^{d}}{d!} \\ &=m(\mu_2+1)+\left[\mu_2-\mu_3(\mu_1+\mu_2)+1\right](1-e^{-m}), \end{aligned}$$

which is bounded above by $1-\mu_1$, if and only if (4.6) holds. □

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Necessary and sufficient conditions for functions defined by binomial distribution to be in subfamilies of analytic functions

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Abstract

The aim of the present paper is to create connections between various subfamilies of analytic univalent functions and Binomial distribution series by applying certain convolution operator involving Binomial distribution series.

Keywords: Analytic functions, Binomial distribution series, Coefficient bounds.

1 Introduction

Let $\mathcal{A}$ be the family of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \tag{1.1}$$

such that $f(0) = f'(0) - 1 = 0$. Further, we denote by $\mathcal{S}$ the subfamily of $\mathcal{A}$ consisting of functions of the form (1.2) which are also univalent in $\mathbb{U}$ and let T be the subfamily of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

The binomial distribution $g(p, \eta)$ for any random variable is given by

$$P(X = r) = \binom{p}{r} \eta^r (1-\eta)^{p-r} = \frac{p!}{r!(p-r)!} \eta^r (1-\eta)^{p-r}, \quad r = 0, 1, 2, \cdots, p$$

when $p < r$, then $g(p, \eta) = 0$.

When η is small and p is big ($p\eta$ is moderate), the Poisson distribution and the binomial distribution are connected. Similarly, when $p = 1$, the Bernoulli distribution and the binomial distribution are related.

Recently, Nazeer et. al [6] introduced the following power series whose coefficients are probabilities of the binomial distribution

$$H_{(p,\eta)}(z) = z + \sum_{d=2}^{\infty} \frac{(p-1)!}{(p-d)!(d-1)!} \eta^{d-1} (1-\eta)^{p-d} z^d. \tag{1.3}$$

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Now, we introduce the series

$$F_{(p,\eta)}(z) = 2z - H_{(p,\eta)}(z) = z - \sum_{d=2}^{\infty} \frac{(p-1)!}{(p-d)!(d-1)!} \eta^{d-1}(1-\eta)^{p-d} z^d. \tag{1.4}$$

Next, by the convolution $(*)$, let a linear operator $F_{(p,\eta)}f : \mathcal{A} \longrightarrow \mathcal{A}$ as:

$$F_{(p,\eta)}f = F_{(p,\eta)}(z) * f(z) = z - \sum_{d=2}^{\infty} \frac{(p-1)!}{(p-d)!(d-1)!} \eta^{d-1}(1-\eta)^{p-d} a_d z^d.$$

For some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in \mathbb{U}$, let the family $T(\lambda, \alpha)$ consists of functions in T satisfying the condition

$$\Re\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha. \tag{1.5}$$

Also, for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in \mathbb{U}$, let the family $C(\lambda, \alpha)$ consists of functions in T satisfying the condition

$$\Re\left\{\frac{f'(z) + zf''(z)}{f'(z) + (1-\lambda) z f''(z)}\right\} > \alpha. \tag{1.6}$$

From (1.2) and (2.1), it is easy to verify that

$$f(z) \in C(\lambda, \alpha) \Longleftrightarrow zf'(z) \in T(\lambda, \alpha).$$

Lemma 1.1. *[1] A function f defined by (1.2) is in the family $T(\lambda, \alpha)$, if and only if*

$$\sum_{d=2}^{\infty} [d - \lambda\alpha d - \alpha + \lambda\alpha] |a_d| \leq 1 - \alpha. \tag{1.7}$$

Lemma 1.2. *[1] A function f defined by (1.2) is in the family $C(\lambda, \alpha)$, if and only if*

$$\sum_{d=2}^{\infty} d\,[d - \lambda\alpha d - \alpha + \lambda\alpha] |a_d| \leq 1 - \alpha. \tag{1.8}$$

Remark 1.3. For some $\lambda(0 \leq \lambda < 1)$ and choosing $\lambda = 0$, the subfamily $T(0, \alpha)$ consists of functions in T satisfying the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad , z \in \mathbb{U}$$

and the family $C(0, \alpha)$ consists of functions in T satisfying the condition

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \quad , \in \mathbb{U}.$$

Remark 1.4. Both subfamilies $T(0, \alpha) \equiv S^*(\alpha)$ and $C(0, \alpha) \equiv K(\alpha)$ are well known subfamilies of starlike and convex functions of order α, respectively (see [7]).

Recently, several researchers determined necessary and sufficient conditions using hypergeometric functions (see for example, [5]), generalized Bessel functions (see for example, [3]), Poisson distribution series (see for example, [2, 4]).

In this paper, we determine conditions for the functions $F_{(p,\eta)}$ to be in the subfamilies $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$. Finally, we give conditions for an integral operator $G_{(p,\eta)}(z) = \int_0^z \frac{F_{(p,\eta)}(t)}{t} dt$ to be in the family $C(\lambda, \alpha)$.

2 Main Results

In this section, we give a necessary an sufficient condition for the function $F_{(p,\eta)}$ to be in the subfamilies $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$.

Theorem 2.1. *The function $F_{(p,\eta)}$ is belongs to the family $T(\lambda,\alpha)$, if and only if*

$$\eta(1-\alpha\lambda)(p-1)\leq(1-\alpha)(1-\eta)^{p-1}. \tag{2.1}$$

Proof. Since

$$F_{(p,\eta)}(z)=z-\sum_{d=2}^{\infty}\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}z^{d}. \tag{2.2}$$

According to Lemma 1.1, we must show that

$$\sum_{d=2}^{\infty}\left[d-\lambda\alpha d-\alpha+\lambda\alpha\right]\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}\leq 1-\alpha. \tag{2.3}$$

By writting $d=(d-1)+1$, we have

$$\begin{aligned}
&\sum_{d=2}^{\infty}\left[d-\lambda\alpha d-\alpha+\lambda\alpha\right]\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}\\
&=(1-\alpha\lambda)\sum_{d=2}^{\infty}\frac{(p-1)!}{(p-d)!(d-2)!}\eta^{d-1}(1-\eta)^{p-d}\\
&\quad+(1-\alpha)\sum_{d=2}^{\infty}\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}\\
&=(1-\alpha\lambda)\sum_{d=0}^{\infty}\frac{(p-1)!}{(p-d-2)!d!}\eta^{d+1}(1-\eta)^{p-d-2}\\
&\quad+(1-\alpha)\sum_{d=1}^{\infty}\frac{(p-1)!}{(p-d-1)!d!}\eta^{d}(1-\eta)^{p-d-1}\\
&=(1-\alpha\lambda)(p-1)\sum_{d=0}^{\infty}\frac{(p-2)!}{(p-d-2)!d!}\eta^{d+1}(1-\eta)^{p-d-2}\\
&\quad+(1-\alpha)\sum_{d=1}^{\infty}\frac{(p-1)!}{(p-d-1)!d!}\eta^{d}(1-\eta)^{p-d-1}\\
&=\eta(1-\alpha\lambda)(p-1)\sum_{d=0}^{\infty}\frac{(p-2)!}{(p-d-2)!d!}\eta^{d}(1-\eta)^{p-d-2}\\
&\quad+(1-\alpha)\sum_{d=1}^{\infty}\frac{(p-1)!}{(p-d-1)!d!}\eta^{d}(1-\eta)^{p-d-1}\\
&=\eta(1-\alpha\lambda)(p-1)+(1-\alpha)\left(1-(1-\eta)^{p-1}\right).
\end{aligned}$$

□

Corollary 2.2. *The function $F_{(p,\eta)}$ is belongs to the family $T(0,\alpha)$, if and only if*

$$\eta(p-1)\leq(1-\alpha)(1-\eta)^{p-1}.$$

Theorem 2.3. *The function $F_{(p,\eta)}$ is belongs to the family $C(\lambda,\alpha)$, if and only if*

$$\eta^2(1-\alpha\lambda)(p-1)(p-2)+\eta\left(3-2\alpha\lambda-\alpha\right)(p-1)\leq(1-\alpha)\left(1-\eta\right)^{p-1}. \tag{2.4}$$

Proof. Since

$$F_{(p,\eta)}(z)=z-\sum_{d=2}^{\infty}\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}z^d. \tag{2.5}$$

According to Lemma 1.2, we must show that

$$\sum_{d=2}^{\infty}d\left[d-\lambda\alpha d-\alpha+\lambda\alpha\right]\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}\leq 1-\alpha. \tag{2.6}$$

By writting $d^2=(d-1)(d-2)+3(d-1)+1$, we have

$$\begin{aligned}
&\sum_{d=2}^{\infty}d\left[d(1-\lambda\alpha)-\alpha+\lambda\alpha\right]\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}\\
&\quad=(1-\alpha\lambda)\sum_{d=3}^{\infty}\frac{(p-1)!}{(p-d)!(d-3)!}\eta^{d-1}(1-\eta)^{p-d}\\
&\qquad+(3-2\alpha\lambda-\alpha)\sum_{d=2}^{\infty}\frac{(p-1)!}{(p-d)!(d-2)!}\eta^{d-1}(1-\eta)^{p-d}\\
&\qquad+(1-\alpha)\sum_{d=2}^{\infty}\frac{(p-1)!}{(p-d)!(d-1)!}\eta^{d-1}(1-\eta)^{p-d}\\
&\quad=(1-\alpha\lambda)\sum_{d=0}^{\infty}\frac{(p-1)!}{(p-d-3)!d!}\eta^{d+2}(1-\eta)^{p-d-3}\\
&\qquad+(3-2\alpha\lambda-\alpha)\sum_{d=0}^{\infty}\frac{(p-1)!}{(p-d-2)!d!}\eta^{d+1}(1-\eta)^{p-d-2}\\
&\qquad+(1-\alpha)\sum_{d=1}^{\infty}\frac{(p-1)!}{(p-d-1)!d!}\eta^{d}(1-\eta)^{p-d-1}\\
&\quad=\eta^2(1-\alpha\lambda)(p-1)(p-2)\sum_{d=0}^{\infty}\frac{(p-3)!}{(p-d-3)!d!}\eta^{d}(1-\eta)^{p-d-3}\\
&\qquad+\eta\left(3-2\alpha\lambda-\alpha\right)(p-1)\sum_{d=0}^{\infty}\frac{(p-1)!}{(p-d-2)!d!}\eta^{d}(1-\eta)^{p-d-2}\\
&\qquad+(1-\alpha)\sum_{d=1}^{\infty}\frac{(p-1)!}{(p-d-1)!d!}\eta^{d}(1-\eta)^{p-d-1}\\
&\quad=\eta^2(1-\alpha\lambda)(p-1)(p-2)+\eta\left(3-2\alpha\lambda-\alpha\right)(p-1)+(1-\alpha)\left(1-(1-\eta)^{p-1}\right).
\end{aligned}$$

□

Corollary 2.4. *The function $F_{(p,\eta)}$ is belongs to the family $C(0,\alpha)$, if and only if*

$$\eta^2(p-1)(p-2)+\eta\left(3-\alpha\right)(p-1)\leq(1-\alpha)\left(1-\eta\right)^{p-1}.$$

3 An integral operator

In this part, we will examine some inclusion properties of integral operator associated with the function $F_{(p,\eta)}(z)$ as follow

$$G_{(p,\eta)}(z) = \int_0^z \frac{F_{(p,\eta)}(t)}{t} dt, \tag{3.1}$$

where $F_{(p,\eta)}(z)$ is given by (1.2).

Theorem 3.1. *The function $G_{(p,\eta)}$ is belongs to the family $C(\lambda, \alpha)$, if and only if*

$$\eta(1-\alpha\lambda)(p-1) \leq (1-\alpha)(1-\eta)^{p-1}. \tag{3.2}$$

Proof. Since

$$\begin{aligned} G_{(p,\eta)}(z) &= z - \sum_{d=2}^{\infty} \frac{(p-1)!}{(p-d)!(d-1)!} \eta^{d-1}(1-\eta)^{p-d} \frac{z^d}{d} \\ &= z - \sum_{d=2}^{\infty} \frac{(p-1)!}{(p-d)!d!} \eta^{d-1}(1-\eta)^{p-d} z^d. \end{aligned} \tag{3.3}$$

By Lemma 1.2, we need only to show that

$$\sum_{d=2}^{\infty} d\left[d - \lambda\alpha d - \alpha + \lambda\alpha\right] \frac{(p-1)!}{(p-d)!d!} \eta^{d-1}(1-\eta)^{p-d} \leq (1-\alpha). \tag{3.4}$$

So, we must show that

$$\sum_{d=2}^{\infty} \left[d - \lambda\alpha d - \alpha + \lambda\alpha\right] \frac{(p-1)!}{(p-d)!(d-1)!} \eta^{d-1}(1-\eta)^{p-d} \leq (1-\alpha).$$

Similarly, the proof of Theorem 2.1, we have

$$\begin{aligned} &\sum_{d=2}^{\infty} \left[d - \lambda\alpha d - \alpha + \lambda\alpha\right] \frac{(p-1)!}{(p-d)!(d-1)!} \eta^{d-1}(1-\eta)^{p-d} \\ &\quad = \eta(1-\alpha\lambda)(p-1) + (1-\alpha)\left(1-(1-\eta)^{p-1}\right). \end{aligned}$$

□

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Legendre collocation approach for Distributed Order Fractional Bagley-Torvik Equations

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Abstract

This study presents a novel numerical framework for solving distributed order fractional Bagley-Torvik equations. The proposed methodology employs shifted Legendre polynomials to discretize the governing equation into a multi-term formulation. A collocation-based solution approach is subsequently developed for efficient numerical resolution. To validate the computational efficacy of our scheme, we provide a representative numerical example that demonstrates the robustness of the proposed method.
Keywords: Bagley-Torvik equation, Distributed order fractional differential equations, Collocation method.

1 Introduction

Distributed-order fractional derivatives constitute a generalized class of fractional differential operators characterized by integration over the order of differentiation across a specified interval. This formulation extends beyond conventional single-order and multi-term fractional differential equations, offering a more comprehensive mathematical framework for modeling complex systems with continuous order distributions. The distributed-order fractional differential operator on the interval $[0,1]$, formally denoted as[6]

$$D^{\alpha}_{\phi}u(t) = \int_0^1 \phi(\alpha) D^{\alpha}u(t)d\alpha, \tag{1.1}$$

where the weight function $\phi(\alpha)$ is non-negative and satisfies $\int_0^1 \phi(\alpha)d\alpha = c_0 > 0$ [3], and D^{α} denotes Caputo type fractional derivative by

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} f'(\tau)d\tau. \tag{1.2}$$

The distributed-order fractional Bagley-Torvik equation represents a powerful mathematical framework for modeling complex dynamic systems characterized by distributed-order fractional derivatives, with significant applications spanning viscoelastic materials, anomalous transport phenomena, and financial modeling. Building upon fundamental fractional calculus principles [1], this formulation extends the classical

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Bagley-Torvik equation to capture more sophisticated memory effects and non-local dynamics. Previous research has established various computational approaches for fractional Bagley-Torvik problems: Ali et al. [1], developed foundational numerical schemes, while Mohammadi and Mohyud-Din [7] advanced operational matrix methods for efficient solution approximation. Subsequent innovations include Eshi and Gudodagi's Haar wavelet implementation [2] applied the Haar wavelet, and Izadi et al.'s implicit hybrid-Galerkin technique [4] , each contributing distinct advantages for specific problem classes. In this work, we present a novel numerical framework for solving distributed-order Bagley-Torvik equations of the general form: [5]

$$au''(t) + bD_{\phi}^{\alpha}u(t) + cu(t) = f(t), \quad u(0) = u'(0) = 0, \tag{1.3}$$

where $f(t) \in L_1[0,1]$, the weight function takes the form $\phi(\alpha) = \sum_{j=0}^{m} q_j \delta(\alpha - \alpha_j)$, where q_j are given constants, $0 < \alpha_j < 1$ and $\delta(.)$ is Dirac delta function. $u(t)$ is unknown function such that $D^{\alpha}u(t)$ remains bounded, the parameters a, b, c are constant coefficients.

While spectral methods have been extensively explored for various classes of fractional differential equations, their application to distributed-order Bagley-Torvik equations remains notably underdeveloped in the existing literature. The present work introduces a novel spectral approach that distinguishes itself from conventional numerical methods through its exceptional computational efficiency and exponential convergence properties. Specifically, the proposed framework offers a streamlined implementation while achieving unprecedented accuracy—surpassing the performance of established techniques documented in prior studies. Given the broad relevance of the Bagley-Torvik equation in modeling viscoelastic materials, damping systems, and other complex physical phenomena, this high-precision methodology is poised to enable transformative advances in both theoretical and applied research domains.

The remainder of this paper is structured as follows: Section 2 presents a rigorous exposition of the proposed spectral methodology for distributed-order Bagley-Torvik equations, including theoretical foundations and implementation details. Section 3 then systematically validates the method's superior performance through carefully designed numerical experiments, with comparative analysis against state-of-the-art alternatives.

2 Main results

We now develop a rigorous numerical framework for solving Equation (2.2). Through careful application of the Dirac delta function's sifting property, we transform the original distributed-order fractional differential equation (2.2) into an equivalent multi-term formulation. This reduction yields the following computationally tractable representation:

$$au''(t) + b \sum_{j=0}^{m} q_j D^{\alpha_j}u(t) + cu(t) = f(t). \tag{2.1}$$

We consider Legendre collocation solution of degree N of (2.4) as follows

$$u_N(t) = \sum_{k=0}^{N} \beta_k P_k(t), \tag{2.2}$$

where $\{\beta_k\}_{k=0}^{N}$ are unknown coefficients and $P_k(t)$ is the shifted Legendre polynomial on $[0,1]$ given by [8]

$$P_k(t) = \sum_{i=0}^{k} (-1)^{k-i} \frac{(k+i)!}{i!^2(k-i)!} t^i. \tag{2.3}$$

Inserting (2.2) into (2.4) concludes

$$a\sum_{k=0}^{N}\beta_k P_k''(t)+b\sum_{j=0}^{m}\sum_{k=0}^{N}\beta_k q_i D^{\alpha_i}P_k(t)+c\sum_{k=0}^{N}\beta_k P_k(t)=f(t). \tag{2.4}$$

Utilizing (1.1) and the relation

$$D^{\alpha_j}t^i=\frac{\Gamma(i+1)}{\Gamma(i-\alpha_j+1)}t^{i-\alpha_j},$$

we can express (2.4) as follows

$$a\sum_{k=0}^{N}\beta_k P_k''(t)+b\sum_{j=0}^{m}\sum_{k=0}^{N}\sum_{i=0}^{k}\beta_k q_j\Big[\frac{(-1)^{k-i}(k+i)!}{i!(k-i)!\Gamma(i-\alpha_j+1)}t^{i-\alpha_j}\Big] \tag{2.5}$$
$$+c\sum_{k=0}^{N}\beta_k P_k(t)=f(t).$$

To complete our spectral discretization scheme, we implement a collocation procedure by evaluating the transformed equation (2.5) at the $N+1$ Gauss-Legendre nodes $\{t_l\}_{l=0}^{N}$, corresponding to the roots of the $P_{N+1}(t)$. This yields the final algebraic system:

$$\begin{cases} a\sum\limits_{k=0}^{N}\beta_k P_k''(t_l)+b\sum\limits_{j=0}^{m}\sum\limits_{k=0}^{N}\sum\limits_{i=0}^{k}\beta_k q_j\Big[\frac{(-1)^{k-i}(k+i)!}{i!(k-i)!\Gamma(i-\alpha_j+1)}t_l^{i-\alpha_j}\Big] \\ \qquad +c\sum\limits_{k=0}^{N}\beta_k P_k(t_l)=f(t_l),\ l=1,2,\ldots,N-2, \\ u_N(0)=0,\ u'N(0)=0. \end{cases} \tag{2.6}$$

Consequently, the spectral collocation solution for the distributed-order fractional Bagkey-Torvik equation (2.2) can be obtained by solving the system (2.6).

In the following, we implement the proposed strategy for solving a test problem.

Example 2.1. Consider (2.2) with $m=0, q_0=1, \alpha_0=\frac{1}{2}$, and $a=b=c=1$.

For validation purposes, we set $f(x)$ such that the exact solution is $u(x)=x^{\mu}$ for various values of μ. The numerical results generated by the proposed method are presented in Table 2. The table shows the absolute errors for various values of N and μ. When μ is an integer, we obtain the exact solution with an approximate error of 10^{-16}. However, for fractional values of μ, the error decreases as N increases, suggesting a smoother and more accurate solution.

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Table 1: Evaluation of the $\|u-u_N\|_{L^2[0,1]}$ for Example 2.9 under different values N.

N	$\mu=2$	$\mu=\frac{5}{2}$	$\mu=3$	$\mu=\frac{7}{2}$	$\mu=4$
3	9.93e − 17	3.20e − 2	4.61e − 16	5.90e − 2	-
4	4.95e − 16	1.46e − 2	6.71e − 16	8.34e − 3	2.22e − 16
5	1.09e − 15	8.02e − 3	1.38e − 15	2.54e − 3	2.22e − 15
6	1.78e − 15	4.92e − 3	3.44e − 15	1.03e − 3	2.36e − 15
7	1.58e − 15	3.25e − 3	1.30e − 14	4.92e − 4	9.21e − 15

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Numerical analysis of the fractional order Dynamics of the Covid-19 Pandemic

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Abstract

In this paper, we investigate the numerical solution of the mathematical model for the COVID-19 disease using Caputo fractional derivative. To demonstrate the efficiency of the proposed method, we simulate the results by using some available data for different fractional orders. Additionally, we compare our results with some reported real data by the World Health Organization (WHO) against confirmed infected and death cases per day in Wuhan.
Keywords: Corona virus COVID-19, Caputo fractional derivative, Real data, Numerical results.

1 Introduction

The COVID-19 is an infectious disease that is caused by the severe acute respiratory syndrome virus 2 (SARS-CoV-2) belonging to the class of SARS. Mathematical models of infectious disease transmission dynamics are now ubiquitous. Such models play an important role in helping to quantify possible infectious disease control. Different models have been proposed for COVID-19 since the start of the pandemic. Integer order derivative in the classical models is not sufficient to capture the complex nature of the virus. It has been proven both theoretically and empirically that when it comes to capturing the intrinsic properties of a complex infectious disease modeling system such as COVID-19, a more accurate fractional calculus captures the nature of the problem. Fractional calculus, which is a generalization of differentiation and integration of integer order, has been proposed to overcome many of the restrictions associated with integer order derivatives. Beyond biological systems, noninteger order derivatives have been successfully used to model physical phenomena in medicine, physics, image processing, optimization, electrodynamics, nanotechnology, biotechnology, engineering in general, and many more.

In this paper, we present a mathematical model of COVID-19 with Caputo fractional derivative of order

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$0 < \alpha \leq 1$ as follows ([33]):

$$\begin{cases} D^{\alpha}_{0,t}[S(t)] = -\beta \dfrac{I}{N} S - l\beta \dfrac{H}{N} S - \beta' \dfrac{P}{N} S, \\ D^{\alpha}_{0,t}[E(t)] = \beta \dfrac{I}{N} S + l\beta \dfrac{H}{N} S + \beta' \dfrac{P}{N} S - k'E, \\ D^{\alpha}_{0,t}[I(t)] = k'\rho_1 E - (\gamma_a + \gamma_i) I - \gamma_i I, \\ D^{\alpha}_{0,t}[P(t)] = k'\rho_2 E - (\gamma_a + \gamma_i) P - \gamma_p P, \\ D^{\alpha}_{0,t}[A(t)] = k'(1 - \rho_1 - \rho_2) E, \\ D^{\alpha}_{0,t}[H(t)] = \gamma_a (I + P) - \gamma_r H - \delta_h H, \\ D^{\alpha}_{0,t}[R(t)] = \gamma_i (I + P) - \gamma_r H, \\ D^{\alpha}_{0,t}[F(t)] = \delta_i I + \delta_p P + \delta_h H, \end{cases} \tag{1.1}$$

with the initial conditions

$$S(0) = S_0, E(0) = E_0, I(0) = I_0, P(0) = P_0,$$
$$A(0) = A_0, H(0) = H_0, R(0) = R_0, F(0) = F_0,$$

where $t \in [0, T]$ and $D^{\alpha}_{0,t}$ is the Caputo fractional derivative of order α (To study the preliminaries of fractional differential equations and types of fractional derivatives, you can refer to reference [3]). The existence and uniqueness of model 1.1 were discussed in [33]. In this model the total population N is divided into eight epidemiological classes:

$S(t)$: Susceptible individuals	$E(t)$: Exposed individuals
$A(t)$: Infectious but asymptomatic individuals	$P(t)$: Super-Spreaders individuals
$I(t)$: Symptomatic and infectious individuals	$H(t)$: Hospitalized individuals
$F(t)$: Fatality cases or Dead individuals	$R(t)$: Recovery individuals

Other parameters, are described in [33, Table 1]. In this paper, the predictor–corrector scheme is applied to solve a model 1.1.

The structure of this paper is as follows: In Section 2, we will present an approximation of the Caputo fractional derivative, and then the predictor–corrector scheme on the proposed COVID-19 model is investigated. The numerical results are reported in Section 3. Finally, concluding remarks are made in Section 3.

2 Main Results

2.1 Approximation of the Caputo fractional derivative

Let the following general differential equation:

$$\begin{cases} D^{\alpha}_{0,t}(y(t)) = f(t, y(t)), & t \in [0, T], \\ y^{(k)}(0) = y^{k}_{0}, & k = 0, 1, \ldots, m-1. \end{cases} \tag{2.1}$$

The equation 2.1 is equivalent to the Volterra integral equation ([3])

$$y(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, y(\tau)) d\tau. \tag{2.2}$$

Now, set $\Delta t = \frac{T}{N}$, $t_j = j\Delta t$, $j = 0, 1, \ldots, N$. So, the equation 2.2 can be discretized as follows:

$$y(t_{j+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{j+1}^k}{k!} y_0^{(k)} + \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} \left[f(t_{j+1}, y^p(t_{j+1})) + \sum_{k=0}^{j} a_{k,j+1} f(t_k, y(t_k)) \right],$$

where

$$a_{k,j+1} = \begin{cases} j^{\alpha+1} - (j-\alpha)(j+1)^\alpha, & k = 0, \\ (j-k+2)^{\alpha+1} + (j-k)^{\alpha+1} - 2(j-k+1)^{\alpha+1}, & 1 \le k \le j, \\ 1, & k = j+1, \end{cases}$$

and the predicted value $y^p(t_{j+1})$ is calculated by the arbitrary Adams–Bashforth technique by

$$y^p(t_{j+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{j+1}^k}{k!} y^{(k)}(0) + \frac{\Delta t^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{j} w_{k,j+1} f(t_k, y(t_k)), \tag{2.3}$$

where $w_{k,j+1} = (j-k+1)^\alpha - (j-k)^\alpha$.

2.2 Predictor–corrector scheme on proposed COVID-19 model

In this section, we utilize the above-proposed predictor–corrector numerical scheme for the fractional model of the in the COVID-19 disease system. Due to the proposed numerical scheme class $S(t)$ in system 1.1 can be discretized as follows:

$$\begin{aligned} S_{j+1} =& S_0 + \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} f_s(t_{j+1}, S_{j+1}^p, I_{j+1}^p, H_{j+1}^p, P_{j+1}^p) \\ &+ \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{j} a_{k,j+1} f_s(t_k, S_k, I_k, H_k, P_k), \end{aligned} \tag{2.4}$$

where

$$f_s(t, S, I, H, P) = -\beta \frac{I}{N} S - l\beta \frac{H}{N} S - \beta' \frac{P}{N} S,$$

and the predator form S_{j+1}^p, I_{j+1}^p, H_{j+1}^p, and P_{j+1}^p, are obtained according to equation 2.3. For instance,

$$S_{j+1}^p = S_0 + \frac{\Delta t^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{j} w_{k,j+1} f_s(t_k, S_k, I_k, H_k, P_k).$$

These relationships also hold true for other classes in similar manner

sectionNumerical results We perform numerical simulations to compare the results of our model with the real data obtained from several reports published by WHO [4] and Worldometer [5]. The starting point of our simulations is 4 January 2020, when the Chinese authorities informed us about the new virus [4], with already 6 confirmed cases in one day. On January 20, 1460 new cases were reported that day and 26 deaths. Here, we follow the data of the daily reports published by [5]. We show that the COVID-19 model describes well the real data of daily confirmed cases during the 2-month outbreak (66 days to be precise, from January 4 to March 9, 2020). The total population of Wuhan is about 11 million. During the COVID-19 outbreak, there was a restriction on movements of individuals due to quarantine in the city. As a consequence, there was a limitation on the spread of the disease. In agreement, in our model, we consider, as the total population under study, $N = 11000000/250$. This denominator was determined in the first days of the outbreak and later proved to be a correct value. According to the real data published by the WHO, it is an appropriate value for the restriction of movements of individuals. The initial conditions and parameters of the model 1.1 are considered as follows ([2]):

$S_0 = N - 6$	$E_0 = 0$	$I_0 = 1$	$P_0 = 5$	$A_0 = 0$
$H_0 = 0$	$R_0 = 0$	$F_0 = 0$		
$\beta = 2.55$	$I = 1.56$	$\beta' = 7.56$	$K = 0.25$	$\rho_1 = 0.58$
$\rho_2 = 0.001$	$\gamma_a = 0.94$	$\gamma_i = 0.27$	$\gamma_r = 0.5$	$\delta_i = \delta_p = \delta_h = 0.04$

According to the reports of the WHO, the list of the number of confirmed cases (L_C) in Wuhan per day and the list of the number of dead individuals (L_D) in Wuhan per day, are as follows:

$$L_C = \begin{cases} 6, 12, 19, 25, 31, 38, 44, 60, 80, 131, 131, 259, 467, 688, 776, 1776, 1460, 1739, 1984, 2101, 2590, \\ 2827, 3233, 3892, 3697, 3151, 3387, 2653, 2984, 2473, 2022, 1820, 1998, 1506, 1278, 2051, 1772, \\ 1891, 399, 894, 397, 650, 415, 518, 412, 439, 441, 435, 579, 206, 130, 120, 143, 146, 102, 46, 45, \\ 20, 31, 26, 11, 18, 27, 29, 39, 39 \end{cases}$$

$$L_D = \begin{cases} 0, 0, 0, 0, 0, 0, 0, 0, 4, 4, 4, 8, 15, 15, 25, 26, 26, 38, 43, 46, 45, 57, 64, 66, 73, 73, 86, 89, 97, 108, 97, \\ 254, 121, 121, 142, 106, 106, 98, 115, 118, 109, 97, 150, 71, 52, 29, 44, 37, 35, 42, 31, 38, 31, 30, 28, \\ 27, 23, 17, 22, 11, 7, 14, 10, 14, 13, 13 \end{cases}$$

Lists L_C and L_D have 66 numbers (i.e. $t \in [0, 65]$), where $L_C(0)$ represents the number of confirmed cases on 4 January 2020 (day 0) and $L_C(65)$ the number of confirmed cases on 9 March 2020 (day 65). Also, $L_D(0)$ represents the number of dead on 4 January 2020, and $L_D(65)$ the number of dead on 9 March 2020. Based on this, in Figure 1, we show the dynamics of $I(t)$: Infectious individuals, $P(t)$: Super-Spreaders individuals, and $H(t)$: Hospitalized individuals for different values of $\alpha = 0.85, 0.90, 0.95, 1$. Furthermore, the comparison between simulated and real data for the confirmed reported cases and death per day of the proposed model are shown in Figure 2.

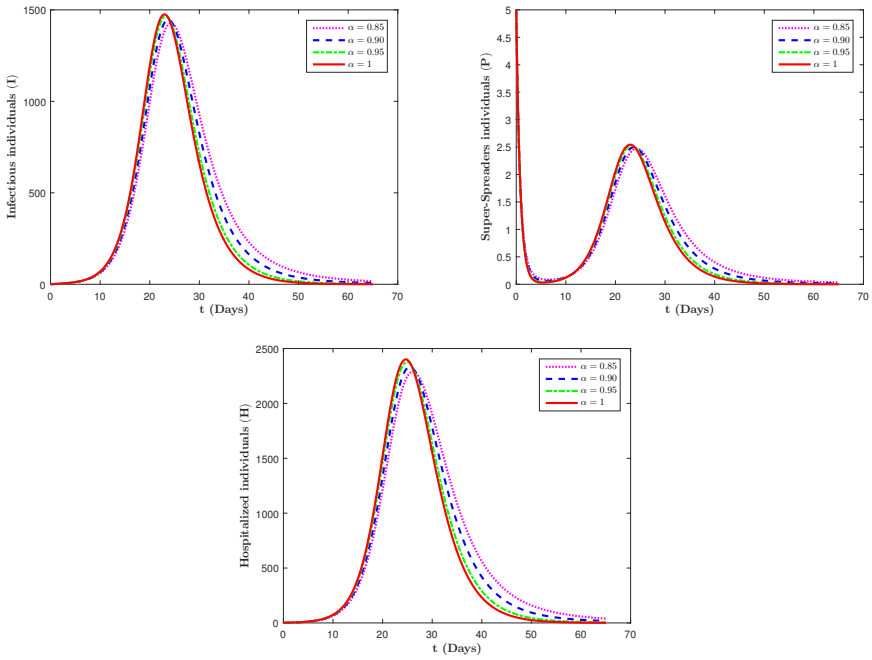


Figure 1: The dynamics of $I(t)$, $P(t)$, and $H(t)$ for different values of α.

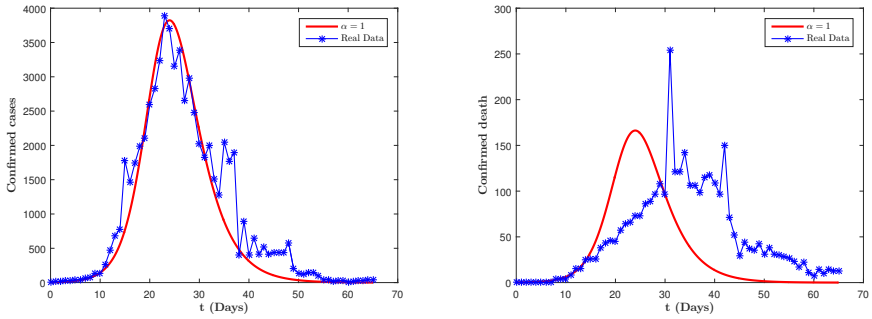


Figure 2: The comparison between simulated and real data for the confirmed reported cases and death per day of the proposed model

3 Conclusion

In this paper, we used the predictor-corrector scheme to solve the fractional order of the COVID-19 model 1.1. According to the report of the WHO, the comparison between simulated and real data for the confirmed reported cases and deaths per day of the proposed model have been investigated and graphically represented in various figures. Finally, Due to the graphical results, it can be said that the used scheme gives a good explanation of the dynamics of the model.

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How Telemedicine takes advantages of artificial intelligence: A survey

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Abstract

This article provides a comprehensive overview of telemedicine from an AI's point of view. Tracing its evolution, it highlighting its transformative impact on healthcare delivery in the digital era, and examines the diverse array of telemedicine devices and virtual visit platforms. It further explores the significant advantages and inherent challenges associated with telemedicine adoption, particularly in enhancing access for rural and underserved communities and revolutionizing patient care across various medical specialties. The article also delves into the future trajectory of telemedicine, by an emphasis on the role of artificial intelligence, remote patient monitoring, 5G connectivity, virtual reality, augmented reality, blockchain technology, and evolving regulatory landscapes. Finally, it addresses critical regulatory, legal, and ethical considerations, including informed consent, cross-state practice, online prescribing, data security, and the need for standardized guidelines to ensure equitable, high-quality, and safe virtual healthcare services.

Keywords: Telemedicine, remote healthcare, digital health, artificial intelligence, medical innovation.

1 Introduction

Telemedicine is the remote delivery of healthcare services using digital communication technologies such as video conferencing, mobile applications, and electronic health records. It allows healthcare providers to diagnose, treat, and monitor patients without requiring them to visit a physical clinic or hospital. Initially developed to support patients in rural or underserved areas, telemedicine has expanded into mainstream healthcare, especially after the COVID-19 pandemic highlighted its potential for safe and efficient medical consultations. Today, it covers a wide range of services, including virtual doctor visits, remote patient monitoring, mental health therapy, chronic disease management, and even tele-dentistry. By eliminating geographical barriers, telemedicine ensures that patients receive timely care, regardless of their location. The benefits of telemedicine are vast, including improved accessibility to healthcare, reduced medical costs, and greater convenience for both patients and doctors. It is particularly beneficial for individuals with mobility challenges, those living in remote areas, and people requiring ongoing medical supervision for chronic conditions. Additionally, telemedicine reduces hospital overcrowding and lowers the risk of infectious disease transmission, making it a valuable tool in public health crises. However, despite its advantages, telemedicine

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faces challenges such as data security risks, regulatory inconsistencies, and limitations in performing physical examinations. As technology continues to evolve, the integration of artificial intelligence, wearable health devices, and 5G connectivity is expected to enhance telemedicine, making healthcare more efficient and accessible worldwide.

2 Revolutionizing Healthcare in the Digital Age

AI is fundamentally changing how medical diagnoses are made, moving beyond simple administrative tasks to become a powerful tool for healthcare providers. By rapidly analyzing massive datasets, AI and machine learning algorithms are enhancing diagnostic accuracy, enabling the early detection of critical illnesses, and driving the development of personalized treatment plans. Specifically, AI improves diagnostic accuracy by sifting through complex medical data, minimizing human error. It also allows for earlier disease detection, particularly in conditions like cancer and heart disease, leading to faster interventions. Furthermore, AI helps to create personalized medicine by tailoring treatments to each patient's unique data, resulting in more effective care. Finally, automation through AI increases efficiency, speeding up the diagnostic process and allowing doctors to dedicate more time to direct patient care. Telemedicine 2.0 is addressing critical healthcare challenges across multiple fronts. Firstly, it's expanding access to quality care in rural areas, like in India, where systems like ReMeDi enable remote consultations using diagnostic tools and AI, connecting patients with distant doctors via trained local health workers. Secondly, it's revolutionizing mental health support by providing 24/7 access to AI-powered chatbots, such as Woebot and Tess, which deliver therapy and emotional support. These are proven to reduce anxiety and depression. Thirdly, it's transforming chronic disease management through platforms like Livongo and Omada Health, which combine smart devices, AI analysis, and personalized coaching to improve the management of conditions like diabetes and hypertension. Lastly, it's enhancing pediatric care by offering at-home exam kits for remote consultations and AI tools for early diagnoses of conditions like autism, leading to more comprehensive and accessible care for children. The future of telemedicine holds incredible promise, with several exciting possibilities on the horizon. We can anticipate advancements like advanced biometric monitoring, where everyday objects like bathroom mirrors will perform comprehensive health scans, detecting early signs of various conditions. Nanotechnology will likely play a significant role, with ingestible or injectable nanobots providing real-time internal health data. AI health assistants will become increasingly sophisticated, managing everything from appointments and medication reminders to personalized dietary recommendations and predictive health analysis. Holographic technology could bring immersive, in-person-like doctor's visits to patients' homes, enhancing the remote consultation experience. Furthermore, brain-computer interfaces could revolutionize communication for patients with severe motor impairments. Finally, as gene editing technologies mature, telemedicine platforms may offer personalized genetic analyses and remote administration of tailored treatments, marking a new era of precision medicine.

3 Telemedicine devices

Telemedicine in diagnosis allows physicians to diagnose patients remotely without being physically present on the site. Devices like Tyto, Carranza et al., Lakhe et al., Sengupta et al., Foche-Perez et al., and Krupa et al. have developed various telemedicine diagnostic devices, including the Tyto, Akibot, Lakhe et al., Sengupta et al., and Foche-Perez et al. These devices have improved diagnostic quality, reduced diagnostic failure rates, and are cost-effective. Tyto is a multifunctional pediatric tele-examination device that includes a digital stethoscope, otoscope, and tongue depressor for heart, lung, and ears diagnosis. Akibot is a telepresence robot that can run in both LAN and VPN networks. Lakhe et al. developed a digital stethoscope that can interface with mobile communication devices, while Sengupta et al. introduced

a master-slave trans-atlantic telerobotic system for ultrasonograms. Foche-Perez et al. developed an open-source, low-cost tele-stethoscopy system for remote auscultation and real-time video conferencing. Krupa et al. introduced a 4-DOF robotized medical diagnostic device for echography on remotely located patients. A 4-DOF robotized medical diagnostic device has been developed to perform echography on remotely located patients. The device uses a multitask control approach, allowing physicians to control the remote robot with a haptic device/probe. This system ensures an intersection between the image plane and the Organ of Interest (OI) and accommodates human physiological movements during the diagnosis process. Emanuele et al. presented a visuo-haptic feedback system using a virtual Ultrasonography (USG) system using Virtual Reality (VR) for training. Monfaredi et al. developed a 6-DOF compact telerobotic system for diagnosis using US imaging, providing good performance with less than 1 mm positioning error and good image quality. Mathiassen et al. proposed a tele-operated US system using a collaborative industrial robot, UR5, to reduce the physical impact of using US probes and save time for clinical approval. The system has two operating modes: force control mode and teleoperation mode. An emergency stop button is employed if discomfort or discomfort occurs.

4 Virtual Visits: Telemedicine's Role in Modern Healthcare

Telehealth is revolutionizing healthcare by enabling patients in remote areas to access quality medical services. Platforms like Practice EHR enable patients to connect with urban specialists without long-distance travel. This saves time and costs, ensuring consistent care. Telehealth also benefits elderly patients, reducing missed appointments and improving overall patient outcomes. This approach eliminates geographical barriers and ensures quality healthcare for all. The Wexford Telehealth Pilot Project in Ireland offers globally relevant insights into telemedicine's power to transform healthcare and improve access. The study revealed high patient engagement in managing chronic conditions through telehealth, leading to greater self-confidence and better clinical outcomes, including improved disease control and reduced symptoms. Operationally, it demonstrated efficiencies like fewer in-person visits and a decrease in unnecessary hospital and ER admissions, suggesting potential cost savings. The project also highlighted common challenges, such as the need for technology training for both patients and providers and the importance of integrating telehealth data with existing systems. Overall, the Wexford project underscores the worldwide movement towards digital healthcare and provides valuable experience for healthcare providers everywhere, including in the US, on how to adapt and implement successful telemedicine strategies.

5 v. Advantages and Challenges of Telemedicine

The challenges of distance and limited specialist access in rural areas are global healthcare issues. While telemedicine's benefits are becoming clearer, its full potential is often hindered by a focus on quick technological fixes rather than thorough research and development. Telemedicine, encompassing telehealth and e-health, involves delivering healthcare and exchanging information remotely for diagnosis, treatment, and education. It exists in real-time (synchronous, like video calls) and store-and-forward (asynchronous, like email) forms, each with its own advantages. Queensland's telepediatric service exemplifies successful real-time telemedicine, now a mainstream service benefiting many children. Based on a centralized referral system managed by coordinators, it guarantees specialist response within 24 hours and utilizes videoconferencing for a wide range of pediatric specialties. This model saves families significant travel and costs, with the enthusiasm of regional clinicians being crucial to its success. Contrary to common belief, telemedicine doesn't always require expensive technology. Low-cost solutions like the Swinfen Charitable Trust, which uses email and digital cameras to connect specialists with doctors in developing countries, demonstrate this. However, transitioning telemedicine beyond pilot projects is difficult. Failures often stem from

overemphasizing technology over crucial organizational and logistical considerations, such as integrating new services and ensuring adequate resources (staff, equipment, training) at all sites. Successful implementation requires focusing on the clinical problem first and understanding how telemedicine complements existing services. Health organizations should support telemedicine due to potential cost savings from reduced patient travel, as seen with travel subsidies. While patients clearly benefit from the convenience and reduced costs, specialist uptake can be influenced by reimbursement structures, which vary internationally. A major hurdle for sustainable telemedicine is the lack of investment in rigorous evaluation, leading to a lack of strong evidence supporting its long-term funding. Accessible and user-friendly telemedicine services that integrate smoothly into clinicians' workflows are essential for adoption. Learning from past successes and failures, as facilitated by conferences like SFT and educational courses, is crucial to avoid repeating costly mistakes. Ultimately, while technology is a component, successful telemedicine relies more on organizational change management and service delivery strategies. Starting with small-scale pilots and scaling gradually based on evidence is recommended. With careful planning, as demonstrated by Queensland's telepediatric service which has conducted thousands of consultations and significantly benefited families, telemedicine holds immense potential for delivering healthcare to remote populations. While telemedicine was crucial during the COVID-19 pandemic, its limitations need consideration. Pre-pandemic research highlighted implementation costs and uncertain return on investment as major obstacles to widespread adoption. Additionally, factors like older age, lower education, and lack of digital literacy can reduce the effectiveness of online solutions for vulnerable populations. Legal uncertainties regarding liability, data privacy, and confidentiality also pose challenges. Despite these weaknesses, telemedicine's importance, especially during the pandemic, remains significant. It allows for remote monitoring of symptoms and recovery for low-risk patients, enabling them to stay home.

6 Innovating Health: How Telemedicine is Revolutionizing Patient Care

For individuals managing chronic kidney disease (CKD), digital health tools aren't just a convenience – they're a lifeline for better lifestyle management. Imagine apps that are not only easy to use but also captivating, offering a personalized journey with virtual communities for support and readily available educational resources, all while seamlessly tracking their progress. By designing digital tools that truly resonate with what CKD patients want and need, we can see a real boost in their engagement and commitment, potentially leading to significant improvements in their health. The next step is to take these insights and create targeted digital solutions, then rigorously test their impact on daily health habits and overall CKD management. Telemedicine is emerging as a game-changer in the daily battle against diabetes. Picture this: digital check-ins leading to tangible results – lower HbA1c levels and weight loss, especially when combined with in-person visits. Early virtual or phone appointments are showing a remarkable increase in crucial blood sugar testing and better control. For those with more challenging blood sugar levels, a blend of clinic visits and continuous remote monitoring is proving incredibly effective. Furthermore, for the often-overlooked crisis of diabetes-related foot disease, particularly in remote Indigenous communities where access to specialists is limited, telemedicine offers a vital connection. Imagine a patient in a remote area connecting via video call with a specialist at the Royal Adelaide Hospital, eliminating arduous travel, fostering clear communication, and building trust through personalized remote care. This isn't just about managing a disease; it's about bringing expert care and reassurance directly to those who need it most. In the realm of multiple sclerosis (MS), think of telemedicine as breaking down barriers to consistent care. A long-term study comparing in-person and virtual approaches over two years paints a picture where telemedicine not only makes care more accessible and improves the patient experience but also maintains the effectiveness of clinical outcomes. This isn't just about convenience; it's about reimagining how MS care is delivered

and how clinical trials are designed, emphasizing the power of remote connection to enhance the lives and health of individuals living with MS. Digital health, including telemedicine, offers promising solutions for the increasing challenges in head and neck cancer care and healthcare shortages. Virtual multidisciplinary teams and mobile apps are improving management and rehabilitation. Telecommunication enables remote support like swallowing exercises and symptom monitoring, especially crucial for home-based care. Telehealth is transforming cancer care by empowering patients, increasing knowledge, and improving access, though disparities exist. During COVID-19, telehealth effectively maintained cancer care, with high patient adoption and satisfaction.

7 Telemedicine: A Lifeline for Rural and Remote Communities

Telemedicine is significantly improving healthcare in rural Canada. Remote presence robots in Saskatchewan and Labrador are drastically cutting down patient transport and costs. Patients and doctors prefer videoconferencing for specialist consultations like rheumatology. Tele-mental health via video is saving money and is well-accepted in remote areas like Labrador, even becoming routine for child psychiatry. Telehealth oversight is even saving lives during emergencies. Beyond these key areas, rural Canadians benefit from a wide range of telehealth services, including pre/post-surgery care, fracture management, remote procedural assistance, remote dialysis, cancer care, remote-guided ultrasound, speech therapy, dietary and physiotherapy consults, professional training, and remote learner supervision. In Glasgow, the global healthcare leaders from England, Scotland, Georgia, North Macedonia, and Sweden met to share how telemedicine is bridging the healthcare gap in rural areas. The WHO highlighted telemedicine's role in improving rural access decades ago, and this meeting showcased real-world examples. Georgia is using telemedicine to connect rural primary care doctors with specialists but needs to upskill these doctors. Sweden employs telemedicine to link nurses in remote hospitals with distant doctors, ensuring specialist access. North Macedonia faces digital infrastructure challenges but sees telemedicine as key to health equity in remote areas. Scotland shared its successful multidisciplinary approach to primary care, enabled by telemedicine. England discussed targeted recruitment to under-doctored areas and the role of AI in remote monitoring. The consensus was that telemedicine is a vital solution for rural healthcare worldwide.

8 The future of Telemedicine

AI is boosting telemedicine with precise diagnoses and virtual assistants for better patient interactions. Wearable tech is driving exponential growth in remote patient monitoring, enabling proactive care. Faster 5G will ensure seamless virtual consultations, even in remote areas. Online mental health services, including AI chatbots and virtual support, will become more accessible and destigmatized. Blockchain will enhance the security of sensitive patient data. Telemedicine will increasingly bridge healthcare gaps in rural areas with mobile clinics and AI diagnostics. VR and AR will enhance medical training, patient education, and remote surgeries. AI-powered predictive analytics will lead to personalized, proactive healthcare. Regulations and insurance coverage will expand to support telehealth. Hybrid healthcare models, combining virtual and in-person care, will offer comprehensive solutions. Xpress Health is a key player, providing both telemedicine services and healthcare staffing solutions. Overall, 2025 promises a future where technology makes healthcare more accessible, efficient, and personalized worldwide. OECD countries are all increasing their telemedicine use, but their data collection methods vary significantly both within and between countries. However, two key abilities are emerging: regularly gathering detailed data on telemedicine activity and having dedicated systems or offices to evaluate these services using this information. Both of these depend on strong health data governance and coordination for routine data collection, linking, protection, and reporting to inform future telemedicine policy. There's also an opportunity to collaborate with standards organizations to create

common coding standards for telemedicine, which would improve consistency within countries and pave the way for future cross-border telemedicine services.

9 Regulatory, Legal, and Ethical Considerations of Telemedicine

In 2005, a California man's online purchase of antidepressants from a Colorado doctor he'd never met, who wasn't licensed in California, ended in tragedy. The doctor was convicted of unlicensed practice, highlighting legal and ethical questions about telemedicine. Key issues raised include: the necessity of informed consent for virtual treatment, the legality of cross-state medical practice, the appropriateness of prescribing medication based solely on online questionnaires, regulations for virtual care, conflict of interest, malpractice coverage for telemedicine, and the protection of patient health information. The case underscores the importance of obtaining informed consent in telemedicine, with medical boards recommending specific elements to include. The lack of documented consent in the *Hageseth* case revealed a significant legal vulnerability. Ethical dilemmas for prescribers who haven't physically met their patients are also a central concern. Telemedicine informed consent is handled differently by each US state, with no federal standard. Some states mandate signed written forms, while others don't. Regardless of legal requirements, obtaining informed consent is ethically crucial at the start of telemedicine. The Federation of State Medical Boards (FSMB) recommends including: patient, provider, and credential details; the type of telemedicine used; acknowledgment that the provider can decide if telemedicine is suitable; PHI security and privacy risks; a disclaimer for technical data loss; and patient consent for sharing PHI. These are suggestions, and telemedicine providers may create their own forms with legal advice to improve transparency and reduce legal risks, as the *Hageseth v. Superior Court* case showed the danger of lacking documented consent. Dr. Hageseth had no record of the patient agreeing to his care method. Telemedicine requires the same ethical and legal standards as in-person care, aiming for equivalent results. Its growth presents challenges in regulation, data protection, and standardization. During COVID-19, proactive remote monitoring was recommended. While video consultations show promise, telehealth should currently supplement, not replace, face-to-face care. Virtual prescriptions and telepharmacy also have unresolved legal issues. Resources in the U.S. offer guidance on legal and ethical telehealth practices, emphasizing privacy and consent. Standardized regulations are still needed to ensure equitable access, quality, and data protection in telemedicine, which should complement traditional healthcare.

10 Conclusion

Telemedicine is a major positive shift in healthcare, making it easier to reach more people efficiently. By using new technologies like AI and remote monitoring, healthcare can become personalized and even predict health issues. However, to make telemedicine truly work for everyone, we need to tackle some key challenges. These include the cost of setting it up, making sure everyone can use it (regardless of age or tech skills), keeping patient data secure, and creating clear rules that everyone follows, especially when doctors treat patients in different states. We also need to carefully study how well telemedicine works and set up good systems for collecting and managing health data. Right now, telemedicine should mostly support in-person care, not completely replace it. To make the most of telemedicine, we need ongoing research, teamwork among all involved, and a strong focus on what patients need and what is ethically right. If we get this right, telemedicine can lead to a healthcare system that is more accessible, effective, and tailored to each individual, worldwide.

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A new generalization of Darbo fixed point theorem

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Abstract

The aim of this research article is to give a new nonlinear contraction for the Schauder type selections of multi-valued mappings in metric spaces, which extends Darbo theorem. Then we introduce a coupled fixed-point theorem and show the existence of a solution of a functional integral equation to demonstrate the efficiency of main results.

Keywords: measure of noncompactness, Schauder type selections, coupled fixed point, functional integral equation.

1 Introduction and preliminaries

In 1930, Schauder [8] established his famous fixed point theorem. On the other hand, since 1930, the study of measure of noncompactness (in short, MNC) has been applied in many branches (see [1, 2] and references therein). Meanwhile, Darbo [5, 1955] used the concept of an MNC to prove the existence of fixed points of condensing mappings, which generalized both Banach principle and Schauder fixed point theorem. Also, his theorem has many applications to demonstrate the existence of solutions for a big category of differential and integral equations. Moreover, in 1969, Nadler [7] expressed contraction principle for multi-valued mappings. It should be mentioned that the multi-valued mappings theorems and related selection results are useful tools in applied sciences.

In this article, we show the existence of selections for generalized multi-valued mappings on complete metric spaces using some new generalizations of Darbo theorem. Then we state a coupled fixed point theorem. Ultimately, we use our main theorem to show the existence of a solution of a functional integral equation. To do this, we need some notations and definitions which is needed in the sequel.

- $\mathcal{B}$ is a Banach space with the norm $\|.\|$;
- $A + B$ and μA with $\mu \in \mathbf{R}$ are algebraic operations on the sets A and B;
- $N(A)$ is the collection of all nonempty subsets of A;
- $M_{\mathcal{B}}$ is the collection of all nonempty and bounded subsets of $\mathcal{B}$ and $N_{\mathcal{B}}$ is its sub-collection including all relatively compact sets.

Definition 1. *[1] Assume that $\sigma : M_{\mathcal{B}} \to \mathbf{R}_+ = [0, \infty)$ a mapping satisfying the following features:*

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(i) *The family* $ker\sigma = \{A \in M_{\mathcal{B}} : \sigma(A) = 0\}$ *is a nonempty subset of* $N_{\mathcal{B}}$*, where* $ker\sigma$ *is the kernel of* σ*;*

(ii) $A \subset B$ *implies that* $\sigma(A) \leq \sigma(B)$*;*

(iii) $\sigma(\bar{A}) = \sigma(A)$*, which* $\bar{A}$ *is the closure of* A*;*

(iv) $\sigma(Conv\, A) = \sigma(A)$*, which* $Conv\, A$ *is the closed convex hull of* A*;*

(v) *For* $\mu \in [0, 1]$*,*

$$\sigma(\mu A + (1 - \mu)B) \leq \mu\sigma(A) + (1 - \mu)\sigma(B);$$

(vi) *If* (A_n) *is a nested sequence of closed sets from* $M_{\mathcal{B}}$ *so that* $\lim_{n\to\infty} \sigma(A_n) = 0$*, then* $A_\infty = \bigcap_{n=1}^{\infty} A_n \neq \emptyset$*.*

Then σ *is called an MNC in* $\mathcal{B}$*.*

Note that A_∞ in (vi) is a member of the$ker\sigma$.

Definition 2. *[4] Consider a multi-valued mapping* $\mathcal{G}$ *from* $\mathcal{B}$ *to* $N(\mathcal{B})$*.*

- *A selection from* $\mathcal{G}$ *is a function* $f : \mathcal{B} \to \mathcal{B}$*, where* $f(a) \in \mathcal{G}(a)$ *for each* $a \in \mathcal{B}$*.*
- $\mathcal{G}^{-1}(b)$ *is the set of all* a *belonging to* $\mathcal{B}$ *provided that* b *belongs to* $\mathcal{G}(a)$ *for all* $b \in \mathcal{B}$*.*

Theorem 1.1. *[4, Browder-Ky Fan Theorem] Presume* $\mathcal{G} : \mathcal{B} \to BC(\mathcal{B})$ *is a multi-valued mapping having convex values and* $\mathcal{G}^{-1}(b)$ *is open for all* b*. Then there is a continuous function* $f : \mathcal{B} \to \mathcal{B}$ *such that* $f(a) \in \mathcal{G}(a)$ *for each* a*.*

2 Main results

Suppose that $A \neq \emptyset$ is a bounded, closed and convex subset of $\mathcal{B}$. Further, assume Ψ is the class of all nondecreasing, subadditive, bounded from below and upper semi-continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n\to\infty} \psi^n(i) = 0$ for each $i \geq 0$. Furthermore, consider $\alpha : [0, +\infty) \to [0, +\infty)$ as a subadditive, continuous and nondecreasing function, where $\alpha^{-1}(0) = (0)$.

Theorem 2.1. *Assume* $\mathcal{H} : A \to N(A)$ *is a multi-valued mapping having convex values so that* $\mathcal{H}^{-1}(b)$ *is open for all* b *and*

$$\alpha(\sigma(\mathcal{H}A)) \leq \psi(\alpha(\sigma(A))) - \psi(\alpha((\sigma(\mathcal{H}A)))), \tag{2.1}$$

where $\psi \in \Psi$*. Then* $\mathcal{H}$ *possesses a fixed point.*

Theorem 2.2. *Suppose* $\mathcal{H} : A \to A$ *is a mapping satisfying*

$$\alpha(\sigma(\mathcal{H}A)) \leq \psi(\alpha(\sigma(A))) - \psi(\alpha(\sigma(\mathcal{H}A))),$$

where $\psi \in \Psi$*. Then* $\mathcal{H}$ *possesses a fixed point.*

Corollary 2.3. *Suppose* $\mathcal{H} : A \to A$ *is a mapping satisfying*

$$\alpha(||\mathcal{H}a - \mathcal{H}b||) \leq \psi(\alpha(||a - b||)) - \psi(\alpha(||\mathcal{H}a - \mathcal{H}b||)),$$

where $||\cdot||$ *is usual norm and* $\psi \in \Psi$*. Then* $\mathcal{H}$ *possesses a fixed point.*

Regarding the concept of coupled fixed-points restated by Bhaskar and Lakshmikantham [3], many researchers proved some applicable theorem in fixed point theory to show the existence of a solution of some integral and differential equations. For more details, see [5, 7] and references therein.

Theorem 2.4. *[2] Suppose that $\sigma_1, \cdots, \sigma_n$ are a family of MNC in Banach spaces $\mathcal{B}_1, \ldots, \mathcal{B}_n$, respectively. Also, assume that $\mathcal{W} : [0,\infty)^n \to [0,\infty)$ is a convex function such that $\mathcal{W}(l_1, \cdots, l_n) = 0$ if and only if $l_i = 0$ for $i = 1, \cdots, n$. Then*

$$\tilde{\sigma}(A) = \mathcal{W}(\sigma_1(A_1), \cdots, \sigma_n(A_n))$$

is an MNC in $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, where A_i are the natural projection of A into $\mathcal{B}_i$ for $i = 1, \cdots, n$.

Note that $\tilde{\sigma}(A) = \sigma(A_1) + \sigma(A_2)$ is an MNC in which A_1 and A_2 are the natural projections of A into $\mathcal{B}$ [1].

Theorem 2.5. *Presume $\mathcal{W} : A \times A \to A$ is a mapping and*

$$\alpha(\sigma(\mathcal{W}(A_1 \times A_2))) \leq \frac{1}{2}[\psi(\alpha(\sigma(A_1) + \sigma(A_2)))] - \psi(\alpha(\sigma(\mathcal{W}(A_1 \times A_2))))$$

for any subset A_1, A_2 of A, where $\psi \in \Psi$. Then $\mathcal{W}$ possesses a coupled fixed point.

3 Application

As an application of Theorem 2.2, we demonstrate the existence of solutions of a functional integral equation. To do this, presume $BC(\mathbf{R}_+)$ is the Banach space of all continuous and bounded real functions on the positive real number with $\|y\| = sup\{|y(i)| : i \geq 0\}$, $A \neq \emptyset$ is a bounded subset of $BC(\mathbf{R}_+)$ and $L > 0$. Also, for $y \in A$ and $\varrho > 0$, assume that

$$\begin{aligned}
\mathcal{N}^L(y,\varrho) &= \sup\{|y(i) - y(j)| : i, j \in [0, L], |i - j| \leq \varrho\},\\
\mathcal{N}^L(A,\varrho) &= \sup\{\mathcal{N}^L(y,\varrho) : y \in A\},\\
\mathcal{N}_0^L(A) &= \lim_{\varrho \to 0} \mathcal{N}^L(A,\varrho),\\
\mathcal{N}_0(A) &= \lim_{L \to \infty} \mathcal{N}_0^L(A).
\end{aligned}$$

Further, take $A(i) = \{y(i) : y \in A\}$ for $i \in \mathbf{R}_+$. Finally, define the mapping σ on the family $M_{BC(\mathbf{R}_+)}$ by

$$\sigma(A) = \mathcal{N}_0(A) + \limsup_{i \to \infty} diamA(i),$$

where

$$diamA(i) = \sup\{|y(i) - z(i)| : y, z \in A\}.$$

The mapping σ is an MNC in $BC(\mathbf{R}_+)$ [1]. Also, $ker\sigma$ includes nonempty and bounded sets A, functions in A are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle organized by the graphs of functions in A arrives to 0 at infinity.

Theorem 3.1. *Assume that the following conditions are held:*

(i) $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is a continuous mapping and the mapping $i \to f(i, 0)$ located in $BC(\mathbb{R}_+)$;

(ii) There is a superadditive function $\psi \in \Psi$ so that

$$|f(i,a) - f(i,b)| \leq \psi(|a-b|) \\ - \psi(|f(i,a) + \int_0^i g(i,j,a)ds - f(i,b) - \int_0^i g(i,j,b)dj|)$$

for each $i \in \mathbf{R}_+$ and any $a, b \in \mathbf{R}$.

(iii) There are continuous mappings $g : \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ and $o, h : \mathbf{R}_+ \to \mathbf{R}_+$ so that

$$\lim_{i\to\infty} o(i) \int_0^i h(j)dj = 0$$

and

$$|g(i,j,a)| \leq o(i)h(j)$$

for $i, j \in [0,\infty)$ with $j \leq i$ and for any $a \in \mathbf{R}$;

(iv) There is a positive solution r_0 for $\psi(r) + q \leq r$, where

$$q = \sup\{|f(i,0)| + o(i) \int_0^i h(j)dj : i \geq 0\}.$$

Then the functional integral equation

$$y(i) = f(i, y(i)) + \int_0^i g(i,j,y(j))dj \tag{3.1}$$

possesses a solution in $BC(\mathbf{R}_+)$.

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Coincidence Point Results in Complete Quasi-Ordered $\mathfrak{F}$-Metric Spaces

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Abstract

In the present article, we prove the existence of a coincidence point for an infinite family of mappings in complete quasi-ordered $\mathfrak{F}$-metric spaces. Then we demonstrate the existence of a coupled coincidence point for the same infinite family of mappings in complete quasi-ordered $\mathfrak{F}$-metric spaces. An example and an application of the existence of a solution for an integral equation are also considered to show the efficiency of obtained results.

Keywords: complete quasi-ordered $\mathfrak{F}$-metric spaces, coincidence point, integral equation.

1 Introduction and Preliminaries

In 2018, Jleli and Samet [33] introduced new space named $\mathfrak{F}$-metric space and proved some fixed point theorems.

Definition 1. *[2, 33] A function $f : (0,\infty) \to \mathbb{R}$ is nondecreasing if $0 < s_1 < s_2$ induces $f(s_1) \leq f(s_2)$ for each $s_1, s_2 \in (0,\infty)$. Also, f is logarithmic-like when $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} f(t_n) = -\infty$ for all positive sequence $\{t_n\}$.*

Suppose that $\mathfrak{F}$ is set of all nondecreasing and logarithmic-like functions.

Definition 2. *[33] Assume $B \in [0,\infty)$ and $f \in \mathfrak{F}$ is constant. Then $\rho : X \times X \to [0,\infty)$ is called a $\mathfrak{F}$-metric on X if, for all $x, y \in X$, the following conditions are held:*

$(D1)$ *$\rho(x,y) = 0$ iff $x = y$;*

$(D2)$ *$\rho(x,y) = \rho(y,x)$;*

$(D3)$ *for any integer $N \geq 2$,*

$$\rho(x,y) > 0 \Rightarrow f(\rho(x,y)) \leq f(\sum_{i=1}^{N-1} \rho(v_i, v_{i+1})) + B$$

for all $(v_i)_{i=1}^N \subset X$ with $(v_1, v_N) = (x,y)$.

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Then (X, ρ) is called a $\mathfrak{F}$-metric space.

Definition 3. *[2] Suppose (X, ρ) is a $\mathcal{F}$-metric space. A sequence $\{x_n\}$ in X converges to a $x \in X$ if $\lim_{n,\to\infty} \rho(x_n, x) = 0$. Also, $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.*

On the other hand, Bhaskar and Lakshmikantham [3] have proved a coupled fixed point theorem for mixed monotone mappings in partially order metric spaces to demonstrate the existence and uniqueness of a solution of a periodic boundary value problem. Then some researchers continued the study of this problem in many abstract metric spaces (see [4, 5, 8]). Moreover, the concept of a coupled coincidence point was introduced by Lakshmikantham and Ćirić [5] as follows:

Definition 4. *[5] $(x, y) \in X \times X$ is a coupled coincidence point of two mappings $F : X \times X \to X$ and $g : X \to X$ when $F(x, y) = gx$ and $F(y, x) = gy$.*

If g is an identity mapping, then $F(x, y) = x$, and $F(y, x) = y$ and (x, y) is called a coupled fixed point of F.

Definition 5. *[5] Suppose that $X \neq \emptyset$ and $F : X \times X \to X$ and $g : X \to X$ are two mappings. Then F and g are said to be commutative if $g(F(x, y)) = F(gx, gy)$ for each $x, y \in X$. Also, $\mathcal{S} : X \to X$ and $g : X \to X$ are commutative when we have $g(\mathcal{S}x) = \mathcal{S}(gx)$ for each $x \in X$.*

Definition 6. *[5] Assume that $(X, \preceq)$ is an ordered set, and $\mathcal{S} : X \to X$ and $g : X \to X$ are two mappings. Then T has nondecreasing g-monotone property if*

$$gx_1 \preceq gx_2 \Rightarrow \mathcal{S}x_1 \preceq \mathcal{S}x_2.$$

for $x_1, x_2 \in X$. Also, if $F : X \times X \to X$ and $g : X \to X$ are two mappings, then F has mixed g-monotone property when, for each $x, y \in X$, we have

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

In the sequel, presume Ψ is a family of all functions $\psi : [0, \infty) \to [0, \infty)$ fulfilling the following properties:

(1) $\psi(t) \leq t$;

(2) $\lim_{n\to\infty} \psi^n(t) = 0$.

In addition, assume Φ is a family of all functions $\phi : [0, \infty) \to [0, \infty)$ fulfilling the following properties:

(1) ϕ is lower semi-continuous;

(2) $\phi(t) = 0$ iff $t = 0$.

2 Main Results

The following is the first main theorem of this article.

Theorem 2.1. *Take $(X, \rho, \preceq)$ a complete quasi-ordered $\mathfrak{F}$-metric space, and $\{\mathcal{S}_\mu : X \to X : \mu \in \Lambda\}$ and $g : X \to X$ two mappings so that for a $\mu_0 \in \Lambda$, $\mathcal{S}_{\mu_0}$ has a nondecreasing g-monotone property and commutes with g on X. Also, suppose that there is $x_0 \in X$ so that $gx_0 \preceq \mathcal{S}_{\mu_0} x_0$ and*

$$\rho(\mathcal{S}_{\mu_0} x, \mathcal{S}_\mu u) \leq \psi(\mathcal{U}(x, u)) - \phi(\mathcal{U}(x, u)), \tag{2.1}$$

where $\psi \in \Psi$, $\phi \in \Phi$ *and* $\mathcal{U}(x,u) = \max\{\rho(gx,gu), \rho(gx,\mathcal{S}_\mu x), \rho(gu,\mathcal{S}_{\mu_0}u)\}$ *for all* $x,u \in X$ *with* $gx \preceq gu$ *and for each* $\mu \in \Lambda$. *Further, suppose that* $\mathcal{S}_{\mu_0}(X) \subset g(X)$, *and* g *and* $\mathcal{S}_{\mu_0}$ *are two continuous mappings. Then* $\{\mathcal{S}_\mu : \mu \in \Lambda\}$ *and* g *have a coincidence point in* X.

Inspiring by the method used in [4, 5, 7, 8], we give the second main result, which can be derived from 2.2. Before that, we need to a lemma as follows:

Lemma 2.2. *Assume that* $(X,\rho,\preceq)$ *is a quasi-ordered* $\mathfrak{F}$*-metric space. Then the followings are held:*

(1) $(X^n,\Omega,\preceq)$ *also is a quasi-ordered* $\mathfrak{F}$*-metric space.*

$$\Omega((u_1,\cdots,u_n),(v_1,\cdots,v_n)) = \max[\rho(u_1,v_1),\rho(u_2,v_2),\cdots,\rho(u_n,v_n)].$$

(2) $f : X^n \to X$ *and* $g : X \to X$ *have* n*-tuple coincidence point iff* $F : X^n \to X^n$ *and* $G : X^n \to X^n$ *have a coincidence point in* X^n.

$$F(u_1,u_2,\cdots,u_n) = (f(u_1,u_2,\cdots,u_n), f(u_2,\cdots,u_n,u_1),\cdots,f(u_n,u_1\cdots,u_{n-1})),$$
$$G(u_1,u_2,\cdots,u_n) = (gu_1,gu_2,\cdots,gu_n).$$

(3) $(X,\rho,\preceq)$ *is complete iff* $(X^n,\Omega,\preceq)$ *is complete.*

Theorem 2.3. *Take* $(X,\rho,\preceq)$ *a complete quasi-ordered* $\mathfrak{F}$*-metric, and* $\{\mathcal{S}_\mu : X\times X \to X : \mu \in \Lambda\}$ *and* $g : X \to X$ *two mappings so that for a* $\mu_0 \in \Lambda$, $\mathcal{S}_{\mu_0}$ *has mixed* g*-monotone and commutes with* g *on* X. *Assume that there is* $x_0,y_0 \in X$ *so that* $gx_0 \preceq \mathcal{S}_{\mu_0}(x_0,y_0)$ *and* $gy_0 \succeq \mathcal{S}_{\mu_0}(y_0,x_0)$. *In addition, assume that*

$$\rho(\mathcal{S}_{\mu_0}(x,y),\mathcal{S}_\mu(u,v)) \le \psi(\mathcal{U}(x,y,u,v)) - \phi(\mathcal{U}(x,y,u,v)),$$

where $\psi \in \Psi$ *and* $\phi \in \Phi$, *and*

$$\mathcal{U}(x,y,u,v) = \max\{\rho(gx,gu),\rho(gy,gv),\rho(gx,\mathcal{S}_\mu(x,y)),\rho(gu,\mathcal{S}_{\mu_0}(u,v)),$$
$$\rho(gy,\mathcal{S}_\mu(y,x)),\rho(gv,\mathcal{S}_{\mu_0}(v,u))\}$$

for all $x,y,u,v \in X, \mu \in \Lambda$ *with* $gx \preceq gu$ *and* $gy \succeq gv$. *If* $\mathcal{S}_{\mu_0}(X\times X) \subset g(X)$ *and* g *and* $\mathcal{S}_{\mu_0}$ *are continuous, then* $\{\mathcal{S}_\mu : \mu \in \Lambda\}$ *and* g *have a coupled coincidence point in* X.

3 Example and Application

First, we give an example to show the validity of Theorem 2.2

Example 3.1. Take the mapping $\rho : [0,1]\times[0,1] \to [0,\infty)$ by $\rho(x,y) = |x-y|$ for each $x,y \in X$. Then $([0,1],\rho)$ is a complete $\mathfrak{F}$-metric with $f(t) = \ln t$ and $B = 0$. Moreover, assume $\{\mathcal{S}_\mu : X \to X : \mu \in \mathcal{N}\}$ and $g : X \to X$ are defined by $\mathcal{S}_\mu(x) = \mu x$ and $g(x) = 2x$. It is clear that $0 = g0 \le \mathcal{S}_1(0) = 0$ for $0 \in [0,1]$, and

$$\begin{aligned}\rho(\mathcal{S}_1(x),\mathcal{S}_\mu(x^*)) &= (|x-\mu x^*|)\\ &\le \frac{1}{2}(|2x-2x^*|)\\ &\le \frac{1}{2}(\max\{\rho(gx,gx^*),\rho(gx,\mathcal{S}_\mu x),\rho(gx^*,\mathcal{S}_1 gx^*)\}).\end{aligned}$$

By setting $\phi(t) = 0$ and $\psi(t) = \frac{1}{2}$, all condition of Theorem 2.2 are fulfilled and therefore, $\mathcal{S}_\mu$ and g have a coincidence point in X.

Second, to protect Theorem 2.2, we demonstrate the existence of a solution of integral equation:

$$x(t) = \int_0^t \mathcal{H}(t, s, x(s))ds + v(t), \tag{3.1}$$

where $t \in I = [0, 1]$, $\mathcal{H} \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $v \in C(I, \mathbb{R})$. Take $C(I, \mathbb{R})$ the space of each continuous real function on I along with the norm $||x||_\infty = \max_{t\in I} |x(t)|$ in which $x \in C(I, \mathbb{R})$ and $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ the space of each continuous real function on $I \times I \times C(I, \mathbb{R})$. Also, take $C(I, \mathbb{R})$ with the norm $||x||_B = \sup_{t\in I}\{|x(t)|e^{-\kappa t}\}$ for each $x \in C(I, \mathbb{R})$ and $\kappa > 0$. Then, the metric induced by this norm on $C(I, \mathbb{R})$ is $\rho_B(x, y) = ||x - y||_B$ for each $x, y \in C(I, \mathbb{R})$. Now, assume $\rho : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to [0, \infty)$ defined by

$$\rho(x, y) = \sup_{t\in I}\{|x(t) - y(t)|e^{-\kappa t}\}$$

for each $x, y \in C(I, \mathbb{R})$. Further, take the quasi-order " $\leq$ " on $C(I, \mathbb{R})$ by

$$x \leq y \Leftrightarrow ||x||_\infty \leq ||y||_\infty$$

for each $x, y \in C(I, \mathbb{R})$. Then $(C(I, \mathbb{R}), \rho, \leq)$ will be a complete quasi-ordered $\mathfrak{F}$-metric space.

Theorem 3.2. *Assume $(C(I, \mathbb{R}), \rho, \leq)$ with $f(t) = \ln t$ is a complete quasi-ordered $\mathfrak{F}$-metric space and $\mathcal{S} : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ is an operator defined by*

$$\mathcal{S}x(t) = \int_0^t \mathcal{H}(t, s, x(s))ds + v(t), \quad v \in C(I, \mathbb{R})$$

and $gx = Ix = x$. Also, assume $\mathcal{H} \in C(I \times I \times \mathbb{R}, \mathbb{R})$ is an operator fulfilling the following conditions:

(i) $\mathcal{H}$ is continuous;

(ii) $\int_0^t \mathcal{H}(t, s, \cdot)$ is increasing for each $t, s \in I$;

(iii) there is $\kappa > 0$ so that $|\mathcal{H}(t, s, x(s)) - \mathcal{H}(t, s, y(s))| \leq \kappa|x(s) - y(s)|$ for each $x, y \in C(I, \mathbb{R})$ and $t, s \in I$.

Then (3.1) has a solution in $C(I, \mathbb{R})$.

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A projection algorithm for solving pseudo-monotone variational inequalities

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Abstract

In this article, we study a new single algorithms for solving pseudo-monotone variational inequalities involving pseudo-monotone operator in Banach spaces using Bregman distance. The algorithm is constructed around the Tseng's extragradient method. Weak convergence are presented under suitable conditions.

Keywords: Variational inequality, Pseudo-monotone operator, Tseng's extragradient method, Bregman distance, Bregman projection.

1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and its norm $\|.\|$, C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be an operator. The classical variational inequality problem (VI) for F on C is to find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote by $Sol(C, F)$ the solution set of problem (1.1). We assume that $Sol(C, F)$ is nonempty. This problem was first introduced in [24, 14] for modeling problems arising from mechanics. Variational inequality problem plays an important role in many fields such as in transportation, engineering mechanics, economics and others [2, 17, 12, 19]. There are several papers available in the literature which are devoted to this subject, most of which deal with conditions for the existence of a solution (cf. [16, 26, 28, 13, 7, 25]). Many numerical iterative methods have been constructed for solving variational inequalities and their related optimization problems (see [8, 9, 15] and the references therein). The most famous projection method for solving a monotone variational inequality problem is the extragradient method (EGM). This method was first introduced by Korpelevich [20] for solving saddle point problems. Subsequently, it was extended to problem (VI). The extragradient method is of the form:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda F(y_n)), \end{cases} \tag{1.2}$$

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where $\lambda \in (0, \frac{1}{L})$ and $F : H \to H$ is monotone and L-Lipschitz continuous.
In order to improve the performance of algorithm (1.2), Thong et al. [27], employing the inertial technique, proposed the following inertial Tseng extragradient algorithm:

$$\begin{cases} x_0, x_1 \in H, \\ u_n = x_n + \beta(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda F u_n), \\ x_{n+1} = y_n - \lambda(F y_n - F x_n), \end{cases} \tag{1.3}$$

where $F : H \to H$ is monotone and L-Lipschitz continuous on H and $\lambda \in (0, \frac{1}{L})$. Assuming some suitable conditions, they presented the weak convergence of the algorithm (3.2).
Our aim in this paper is to further improve inertial Tseng's extragradient method (3.2) for solving problem (VI) with a pseudomonotone operator in Banach spaces using Bregman distance.
The paper is organized as follows: In section 2, we recall some definitions and preliminary results for further use. Section 3 deals with our algorithms and the relevant convergence analysis.

2 Preliminaries

In this section, we recall some definitions and preliminaries. Let $f : E \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function. Also we denote by dom f, the domain of f, that is the set $\{x \in E : f(x) < \infty\}$. Let $x \in int\ dom f$. Then subdifferential of f at x is the convex set defined by:

$$\partial f(x) = \{\xi \in E^* : f(x) + \langle y - x, \xi \rangle \leq f(y), \ \ \forall y \in E\},$$

and the Fenchel conjugate of f is the convex function

$$f^* : E^* \to (-\infty, \infty], \qquad f^*(\xi) = \sup\{\langle x, \xi \rangle - f(x) : x \in E\}.$$

It is well known that $\xi \in \partial f(x)$ is equivalent to

$$f(x) + f^*(\xi) = \langle x, \xi \rangle. \tag{2.1}$$

It is not difficult to check that f^* is proper convex and lower semicontinuous function. The function f is said to be cofinite if dom $f^* = E^*$. For any convex mapping $f : E \to (-\infty, +\infty]$, we denote by $f^\circ(x, y)$ the right-hand derivative of f at $x \in int\ dom f$ in the direction y, that is

$$f^\circ(x, y) := \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}. \tag{2.2}$$

If the limit as $t \to 0$ in (2.2) exists for each y, then the function f is said to be Gâteaux differentiable at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x) \rangle := f^\circ(x, y)$ for all $y \in E$. When the limit as $t \to 0$ in (2.2) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is Fréchet differentiable at x. The function f is said to be Legendre if it satisfies the following two conditions:

(L1) $int\ dom f \neq \emptyset$ and ∂f is single-valued on its domain,

(L2) $int\ dom f^* \neq \emptyset$ and ∂f^* is single-valued on its domain.

Because the space E is assumed to be reflexive, we always have $(\partial f)^{-1} = \partial f^*$. This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f{=}(\nabla f^*)^{-1}, \quad ran\ \nabla f{=}dom\ \nabla f^*{=}int\ dom\ f^*, \quad ran\ \nabla f^*{=}dom\ \nabla f{=}int\ dom\ f.$$

Also, conditions (L1) and (L2), in conjunction with Theorem 5.4 of [4], imply that the functions f and f^* are strictly convex on the interior of their respective domains and f is Legendre if and only if f^* is Legendre.

Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the Bregman distance with respect to f is defined as [6]:

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \quad x, y \in E.$$

It should be noted that the Bregman distance is not a distance in the usual sense of the term. Clearly $D_f(x, x) = 0$ but $D_f(x, y) = 0$ may not imply $x = y$. In general, D_f is not symmetric and does not satisfy the triangle inequality. However, D_f satisfies the three point identity

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle,$$

and four point identity

$$D_f(x, y) + D_f(w, z) - D_f(x, z) - D_f(w, y) = \langle x - w, \nabla f(z) - \nabla f(y)\rangle,$$

for any $x, w \in dom f$ and $y, z \in int\ dom f$.

Following [1] and [10], we make use of the function $V_f{:}E{\times}E^*{\to}[0, +\infty]$ associated with f, which is defined by:

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \qquad \forall x \in E,\ x^*{\in}E^*.$$

Then

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \tag{2.3}$$

for all $x{\in}E$ and $x^*{\in}E^*$. Moreover, by the subdifferential inequality, we have

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \le V_f(x, x^* + y^*), \tag{2.4}$$

for all $x{\in}E$ and $x^*, y^*{\in}E^*$(see [18]).

Recall that the function f is called sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.1. *[7] If dom f contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 2.2. *[17] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

In addition if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function (see [15]).

Thus, for all $z \in E$, we have

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.5}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

The Bregman projection (see[6]) with respect to f of $x \in int\ dom f$ onto a nonempty, closed and convex set $C \subset int\ dom f$ is defined as the necessarily unique vector $\overleftarrow{proj}_C^f(x) \in C$, which satisfies

$$D_f(\overleftarrow{proj}_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Similarly to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex and Gâteaux differentiable functions has a variational characterization ([6], Corollary 4.4, p. 23]).

Lemma 2.3. *Suppose that f is Gâteaux differentiable and totally convex on int dom f. Let $x \in$ int dom$f$ and $C \subset int\ dom f$ be a nonempty, closed, and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*

(i) *The vector $\hat{x} \in C$ is the Bregman projection of x onto C with respect to f.*

(ii) *The vector $\hat{x} \in C$ is the unique solution of the variational inequality*

$$\langle z - y, \nabla f(x) - \nabla f(z) \rangle \geq 0, \qquad \forall y \in C.$$

(iii) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \qquad \forall y \in C.$$

Lemma 2.4. *[11] Let C be a nonempty, closed and convex subset of a real Banach space E, and let $F : C \to E^*$ be pseudo-monotone and continuous. Then x^* belongs to $Sol(C, F)$ if and only if*

$$\langle F(x), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

We need the following lemma:

Lemma 2.5. *[14] Let $\{a_k\}_{k=0}^\infty$, $\{\gamma_k\}_{k=0}^\infty$ and $\{\delta_k\}_{k=0}^\infty$ be sequences of nonnegative real numbers such that*

$$\forall k > 0, \quad a_{k+1} \leq (1 - \gamma_k)a_k + \gamma_k a_{k-1} + \delta_k,$$

where, $\{\gamma_k\}_{k=0}^\infty \subset [0, \frac{1}{2}]$ and $\sum_{k=1}^{\infty} \delta_k < \infty$. Then the sequence $\{a_k\}_{k=0}^\infty$ converges.

Definition 1. *Let C be a nonempty subset of E, the mapping $F: C \to E^*$ is said to be*

(i) *pseudo-monotone on C if for any $x, y \in C$ the following implication holds:*

$$\langle F(x), x - y \rangle \geq 0 \Longrightarrow \langle F(y), y - x \rangle \geq 0,$$

(ii) *L-Lipschitz continuous on C if there exists a scalar $L > 0$ satisfying*

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in C,$$

(iii) *weakly sequentially continuous if for each sequence $\{x_n\}$we have, $\{x_n\}$ converges weakly to x implies $\{F(x_n)\}$ converges weakly to $F(x)$.*

3 Main results and weak convergence

In this section, we assume that $f : X \to \mathbb{R}$ is a Legendre, super coercive and totally convex function on bounded subsets of E such that ∇f^* is bounded on bounded subsets of int $\operatorname{dom} f^*$ and f is σ-strongly convex if

$$D_f(x, y) \geqslant \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in C.$$

. We introduce our algorithms for finding a solution of a variational inequality problem involving pseudo-monotone and Lipschitz continuous operator.

Table 1: Algorithm 1

Initialization: Let $\epsilon > 2$ and $z_0, z_1 \in E$ are arbitrary, $\beta \in [0, 1]$, $0 \leqslant \theta \leqslant \frac{1}{1+\epsilon}$, $0 \leqslant \alpha \leqslant \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$ and $0 < \mu < \min\{1, \sigma^2\}$. Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} \alpha_n < \infty$.

Step 1. Given the current iterates z_n, z_{n-1} compute

$$\begin{aligned}
w_n &= \nabla f^*\Big(\nabla f(z_n) + \alpha(\nabla f(z_n) - \nabla f(z_{n-1}))\Big),\\
u_n &= \nabla f^*\Big(\nabla f(z_n) + \beta(\nabla f(z_n) - \nabla f(z_{n-1}))\Big),\\
y_n &= \overleftarrow{proj}_C^f\Big(\nabla f^*(\nabla f(u_n) - \tau_n F u_n)\Big),\\
z_{n+1} &= \nabla f^*\Big((1-\theta)\nabla f(w_n) + \theta(\nabla f(y_n) - \tau_n(F y_n - F u_n))\Big)
\end{aligned}$$

where

$$\tau_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu\|u_n - y_n\|}{\|F u_n - F y_n\|}, \tau_n + \alpha_n\Big\}, & F y_n \neq F u_n,\\ \tau_n + \alpha_n, & \textit{otherwise.} \end{cases} \tag{3.1}$$

If $z_n{=}w_n$ then stop and z_n is a solution of variational inequality. Otherwise, set $n{=}:n{+}1$ and go back Step 1.

Remark 3.1. We define $x_n := \nabla f^*(\nabla f(y_n) - \tau_n(F y_n - F u_n))$.

It is easy to proof the following Lemma.

Lemma 3.2. *Assume that F is L-Lipschitz continuous on E. Let $\{\tau_n\}$ be the sequence generated by (3.1). Then*

$$\lim_{n\to\infty} \tau_n = \tau \in \Big[\min\{\frac{\mu}{L}, \tau_1\}, \tau_1 + \alpha\Big],$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$.

Lemma 3.3. *Under the conditions of Algorithem 2.4, we have*

$$D_f(x^*, x_n) = D_f(x^*, u_n) - \|y_n - u_n\|^2\Big[\frac{\sigma}{2} - \frac{\tau^2 L^2}{\sigma}\Big].$$

Proof. By definition of x_n, (2.3) and (2.4), we get

$$\begin{aligned} D_f(x^*, x_n) =& D_f(x^*, \nabla f^*(\nabla f(y_n) - \tau_n(Fy_n - Fu_n))) \\ =& V_f(x^*, \nabla f(y_n) - \tau_n(Fy_n - Fu_n)) \\ \leqslant& V_f(x^*, \nabla f(y_n) - \tau_n(Fy_n - Fu_n) + \tau_n(Fy_n - Fu_n)) \\ & - \langle \nabla f^*(\nabla f(y_n) - \tau_n(Fy_n - Fu_n)) - x^*, \tau_n(Fy_n - Fu_n)\rangle \\ =& V_f(x^*, \nabla f(y_n)) - \tau_n\langle x_n - x^*, Fy_n - Fu_n\rangle \\ =& D_f(x^*, y_n) - \tau_n\langle x_n - u_n, Fy_n - Fu_n\rangle - \tau_n\langle u_n - x^*, Fy_n - Fu_n\rangle \end{aligned} \tag{3.2}$$

By definition of y_n and the pseudo-monotonicity of F, we obtain

$$\langle y_n - x^*, \nabla f(u_n) - \tau Fu_n - \nabla f(y_n) + \tau_n Fy_n\rangle \geqslant 0,$$

then

$$\begin{aligned} &\langle y_n - x^*, \nabla f(u_n) - \nabla f(y_n)\rangle + \tau_n\langle y_n - x^*, Fy_n - Fu_n\rangle \geqslant 0 \\ &\langle y_n - x^*, \nabla f(u_n) - \nabla f(y_n)\rangle \geqslant \tau_n\langle y_n - x^*, Fu_n - Fy_n\rangle, \end{aligned} \tag{3.3}$$

Therefor we have

$$\begin{aligned} D_f(x^*, x_n) \leqslant& D_f(x^*, y_n) - \tau_n\langle x_n - u_n, Fy_n - Fu_n\rangle - \tau_n\langle u_n - x^*, Fy_n - Fu_n\rangle \\ =& D_f(x^*, y_n) - \tau_n\langle x_n - u_n, Fy_n - Fu_n\rangle \\ & - \tau_n\langle u_n - y_n, Fy_n - Fu_n\rangle - \tau_n\langle y_n - x^*, Fy_n - Fu_n\rangle \\ \leqslant& D_f(x^*, y_n) - \tau_n\langle x_n - y_n, Fy_n - Fu_n\rangle + \langle y_n - x^*, \nabla f(u_n) - \nabla f(y_n)\rangle, \end{aligned}$$

by the three point identity , we obtain that

$$\begin{aligned} D_f(x^*, x_n) =& D_f(x^*, u_n) - D_f(y_n, u_n) - \tau_n\langle x_n - y_n, Fy_n - Fu_n\rangle \\ \leqslant& D_f(x^*, u_n) - D_f(y_n, u_n) + \tau\|y_n - x_n\|.\|Fy_n - Fu_n\|. \end{aligned} \tag{3.4}$$

This completes the proof. □

The following lemmas can be easily proven.

Lemma 3.4. *Under the conditions of Algorithem 2.4, $\lim_{n\to\infty} D_f(x^*, z_n)$ exists.*

Lemma 3.5. *Any sequential weak cluster point of the sequence $\{z_n\}$ belongs to $Sol(C, F)$.*

Now, we proof a weak convergence theorem for Algorithm2.4.

Theorem 3.6. *Under the conditions of Algorithem 2.4, the sequence $\{z_n\}$ generated by Algorithm 2.4 converges weakly to an element $x^* \in Sol(C, F)$.*

Proof. We claim that the sequence $\{z_n\}$ converges weakly to some point in $Sol(C, F)$. Indeed, we have already shown that, for every $x^* \in Sol(C, F)$, the limit $\lim_{n\to\infty} D_f(x^*, z_n)$ exists and that each sequential weak cluster point of the sequence $\{z_n\}$ belongs to $Sol(C, F)$. Invoking Lemma 3.4, we see that the sequence $\{z_n\}$ converges weakly to an element in $Sol(C, F)$, as asserted. This completes the proof. □

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Extragradient method for solving pseudomonotone equilibrium problems using Bregman distance

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Abstract

We study the new method for solving equilibrium problem involving strongly pseudomonotone mapping in reflexive Banach spaces. The algorithm is constructed around the extragradient method and theorem of strong convergence is established under mild conditions.
Keywords: Extragradient method, Pseudomonotone mapping, Bregman distance.

1 Introduction

Let X be a reflexive real Banach space and C be a nonempty closed and convex subset of X. Throughout this paper, we shall denote the dual space of X by X^*. The norm and the duality pairing between X and X^* are respectively denoted by $\|.\|$ and $\langle ., . \rangle$. We introduce the equilibrium problem (EP) of find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction. The set of solutions of (1.1) is denoted by $EP(f)$. It unifies many important mathematical problems, such as optimization problems, complementarity, fixed point, Nash equilibria, optimization, saddle point, and variational inequality problems can be reformulated as equilibrium problems (cf. [5, 4]). In recent decades, many methods have been proposed and analyzed for approximating solution of equilibrium problems (cf. [11, 12, 9, 18]).
Recently, iterative methods for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of operators in Hilbert spaces have been developed further by several authors [2, 1, 19]. The extragradient method has been extended to equilibrium problems for monotone (more general, pseudomonotone) and Lipschitz-type continuous bifunctions and studied both theoretically and algorithmically [3, 8, 16, 21, 22]. Inspired by the aforementioned results, the main purpose of this paper is to extend the method in [13] to equilibrium problems (1.1) with the Bregman distance.
The paper is organized as follows: In section 2, we present some definitions and preliminaries needed in the paper. We introduce our algorithm and prove the main result in the Section 3. Sections 4 include applications of algorithm.

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2 Preliminaries

In this section, we recall some definition and results that will be used in the sequel.
Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function. The set of minimizers of f is denoted by $Argmin\ f$. If $Argmin\ f$ is a singleton, its unique element is denoted by $argmin_{x\in X}\ f(x)$. Also we denote by dom f, the domain of f, that is the set $\{x \in X : f(x) < \infty\}$. Let $x \in int\ domf$. Then subdifferential of f at x is the convex set defined by:

$$\partial f(x) = \{\xi \in X^* : f(x) + \langle y - x, \xi\rangle \leq f(y),\ \ \forall y \in X\},$$

and the Fenchel conjugate of f is the convex function

$$f^* : X^* \to (-\infty, \infty], \qquad f^*(\xi) = \sup\{\langle x, \xi\rangle - f(x) : x \in X\}.$$

It is well known that $\xi \in \partial f(x)$ is equivalent to

$$f(x) + f^*(\xi) = \langle x, \xi\rangle. \tag{2.1}$$

It is not difficult to check that f^* is proper convex and lower semicontinuous function. The function f is said to be cofinite if dom $f^* = X^*$. For any convex mapping $f : X \to (-\infty, +\infty]$, we denote by $f^\circ(x, y)$ the right-hand derivative of f at $x \in int\ domf$ in the direction y, that is

$$f^\circ(x, y) := \lim_{t\downarrow 0} \frac{f(x + ty) - f(x)}{t}. \tag{2.2}$$

If the limit as $t \to 0$ in (2.2) exists for each y, then the function f is said to be Gâteaux differentiable at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle := f^\circ(x, y)$ for all $y \in X$. When the limit as $t \to 0$ in (2.2) is attained uniformly for any $y \in X$ with $\|y\| = 1$, we say that f is Fréchet differentiable at x. The function f is said to be Legendre if it satisfies the following two conditions:

(L1) $int\ domf \neq \emptyset$ and ∂f is single-valued on its domain,

(L2) $int\ domf^* \neq \emptyset$ and ∂f^* is single-valued on its domain.

Because the space X is assumed to be reflexive, we always have $(\partial f)^{-1} = \partial f^*$. This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f{=}(\nabla f^*)^{-1}, \quad ran\ \nabla f{=}dom\ \nabla f^*{=}int\ dom\ f^*, \quad ran\ \nabla f^*{=}dom\ \nabla f{=}int\ dom\ f.$$

Also, conditions (L1) and (L2), in conjunction with Theorem 5.4 of [4], imply that the functions f and f^* are strictly convex on the interior of their respective domains and f is Legendre if and only if f^* is Legendre. In 1967, Bregman [6] introduced the concept of Bregman distance, and he discovered an elegant and effective technique for the use of the Bregman distance in the process of designing and analyzing feasibility and optimization algorithms.
From now on, we assume that $f : X \to (-\infty, +\infty]$ is also Legendre. The Bregman distance with respect to f, or simply, Bregman distance is the bifunction $D_f : domf \times int\ domf \to [0, +\infty]$, defined by:

$$D_f(y, x) := f(y) - f(x) - \langle y - x, \nabla f(x)\rangle.$$

It should be noted that D_f is not a distance in the usual sense of the term. Clearly, $D_f(x, x) = 0$, but $D_f(y, x) = 0$ may not imply $x = y$. In our case, when f is Legendre this indeed holds (see [4], Theorem 7.3(vi), p. 642). D_f satisfies the three point identity

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle.$$

Recall that the function f is called sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.1. *[7] If dom f contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 2.2. *[17] Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in X$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

In addition if $f : X \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : X^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function (see [15]).

Thus, for all $z \in X$, we have

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.3}$$

where $\{x_i\}_{i=1}^N \subset X$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.3. *[20] Let C be a nonempty convex subset of X and $f : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then f attains its minimum at $x \in C$ if and only if $0 \in \partial f(x) + N_C(x)$, where $N_C(x)$ is the normal cone of C at x, that is*

$$N_C(x) := \{x^* \in X^* : \langle x - z, x^* \rangle \geq 0, \forall z \in C\}.$$

A function $g : C \times C \to (-\infty, +\infty]$, where $C \subset X$ is a closed convex subset, such that $g(x, x) = 0$ for all $x \in C$ is called a bifunction. Throughout this paper we consider bifunctions with the following properties:

(B1) g is monotone, i.e., for all $x, y \in C$,

$$g(x, y) + g(y, x) \leq 0.$$

(B2) g is pseudomonotone, i.e., for all $x, y \in C$,

$$g(x, y) \geq 0 \Rightarrow g(y, x) \leq 0.$$

(B3) g is γ- strongly pseudomonotone if there exists a constant $\gamma \geq 0$ such that for all $x, y \in C$,

$$g(x, y) \geq 0 \Rightarrow g(y, x) \leq -\gamma\Big(D_f(y, x) + D_f(x, y)\Big).$$

(B4) g is Bregman-Lipschitz-type continuous, i.e, there exist two positive constants c_1, c_2 such that

$$g(x, y) + g(y, z) \geq g(x, z) - c_1 D_f(y, x) - c_2 D_f(z, y), \quad \forall x, y, z \in C,$$

where $f : X \to (-\infty, +\infty]$ is a Legendre function. The constants c_1, c_2 are called Bregman-Lipschitz coefficients with respect to f.

Here, we assume that the bifunction g satisfis the following conditions:

(A1) g is pseudomonotone on C and $g(x, x) = 0$ for all $x \in X$.

(A2) g is Bregman-Lipschitz-type continuous on X.

(A3) $g(x,.)$ is convex, lower semicontinuous and subdifferentiable on X for every fixed $x \in X$.

Remark 2.4. If g satisfis $(A1)-(A3)$, then the set of solutions $EP(g)$ of $EP(1)$ is closed and convex.

Let $f : X \to (-\infty, +\infty]$ be a Gâteaux differentiable function, recall that the proximal mapping of a proper convex and lower semicontinuous function $g : C \to (-\infty, +\infty]$ with respect to f is defined by:

$$prox_g^f(x) := argmin\{g(y) + D_f(y,x) : y \in C\}, x \in X. \tag{2.4}$$

Lemma 2.5. *[8] Let $f : X \to (-\infty, +\infty]$ be a super coercive and Legendre function. Let $x \in intdomf, C \subset intdomf$ and $g : C \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then the following inequality holds:*

$$g(y) - g(prox_g^f(x)) + \langle prox_g^f(x) - y, \nabla f(x) - \nabla f(prox_g^f(x))\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.6. *[14] Let $\{a_k\}_{k=0}^{\infty}$, $\{\gamma_k\}_{k=0}^{\infty}$ and $\{\delta_k\}_{k=0}^{\infty}$ be sequences of nonnegative real numbers such that*

$$\forall k > 0, \quad a_{k+1} \leq (1-\gamma_k)a_k + \gamma_k a_{k-1} + \delta_k,$$

where, $\{\gamma_k\}_{k=0}^{\infty} \subset [0, \frac{1}{2}]$ and $\sum_{k=1}^{\infty} \delta_k < \infty$. Then the sequence $\{a_k\}_{k=0}^{\infty}$ converges.

3 Main results

In this section, we assume that $f : X \to \mathbb{R}$ is a Legendre, super coercive and totally convex function on bounded subsets of X such that ∇f^* is bounded on bounded subsets of int domf^* and the bifunction $g : X \times X \to \mathbb{R}$ satisfies $(A1)-(A3)$. We introduce a new algorithm for solving equilibrium problems.

Remark 3.1. By Lemma 2.5 and definition of v_n, it is easy to show that if v_n=w_n, then y_n is the solution of problem $EP(g)$.

Lemma 3.2. *Under conditions $(A1)-(A3)$, for $v^* \in EP(g) \neq \emptyset$ we have*

$$\begin{aligned} D_f(v^*, z_n) \leq & D_f(v^*, w_n) - (1 - c_1\lambda_n)D_f(v_n, w_n) - (1 - c_2\lambda_n)D_f(v_n, z_n) \\ & - \gamma\lambda_n\Big(D_f(v^*, v_n) + D_f(v_n, v^*)\Big). \end{aligned}$$

Proof. By definition of z_n and using Lemma 2.3, we get

$$0 \in \lambda_n \partial_2 g(v_n, z_n) + \nabla_1 D_f(z_n, w_n) + N_{H_n}(z_n),$$

therefore, there exists $w \in \partial_2 g(v_n, z_n)$ and $\overline{w} \in N_{H_n}(z_n)$ such that

$$\lambda_n w + \nabla f(z_n) - \nabla f(w_n) + \overline{w} = 0.$$

Note that, $\overline{w} \in N_{H_n}(z_n)$ implies that $\langle v - z_n, \overline{w}\rangle \leq 0$ for all $v \in H_n$. Thus, we have

$$\begin{aligned} \langle v - z_n, \nabla f(w_n) - \nabla f(z_n)\rangle &= \langle v - z_n, \lambda_n w + \overline{w}\rangle \\ &= \lambda_n\langle v - z_n, w\rangle + \langle v - z_n, \overline{w}\rangle \\ &\leq \lambda_n\langle v - z_n, w\rangle, \quad \forall v \in H_n. \end{aligned} \tag{3.1}$$

Table 1: Algorithm 1

Choose initial points $x_0, x_{-1} \in X$, and three sequences $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ such that

(H1) $\{\lambda_n\} \subset [a,b] \subset \Big(0, \min(\frac{1}{c_1}, \frac{1}{c_2})\Big)$, $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = +\infty$,

(H2) $\{\alpha_n\}$ is nondecreasing and $0 \leq \alpha_n \leq \alpha < \frac{1}{2}$,

(H3) $\{\beta_n\} \subset (0,1]$.

Set $n = 1$ and go to Step 1.

Step 1. Given the current iterates u_{n-1}, u_n compute

$$w_n = \nabla^* f\Big((1-\alpha_n)\nabla f(u_n) + \alpha_n \nabla f(u_{n-1})\Big),$$
$$v_n = prox^f_{\lambda_n g(w_n,.)}(w_n).$$

If v_n=w_n then stop and v_n is a solution of problem $EP(g)$. Otherwise,

Step 2. Construct a half-spaces

$$H_n = \{z \in X | \langle z - v_n, \nabla f(w_n) - \lambda_n t_n - \nabla f(v_n)\rangle \leqslant 0\},$$

where $t_n \in \partial_2 g(w_n, v_n)$.

Step 3. Compute

$$z_n = prox^f_{\lambda_n g(v_n,.)}(w_n),$$
$$u_{n+1} = \nabla^* f\Big((1-\beta_n)\nabla f(w_n) + \beta_n \nabla f(u_{n-1} z_n)\Big).$$

Set $n =: n+1$ and go back Step 1.

On the other hand, $w \in \partial_2 g(v_n, z_n)$ implies that

$$g(v_n, v) - g(v_n, z_n) \geq \langle v - z_n, w\rangle, \quad \forall v \in X. \tag{3.2}$$

Combining (3.1) with relation (3.2), we obtain

$$\lambda_n g(v_n, v) - \lambda_n g(v_n, z_n) \geq \langle v - z_n, \nabla f(w_n) - \nabla f(z_n)\rangle, \quad \forall v \in H_n. \tag{3.3}$$

Assume that $v^* \in EP(g)$ and substituting $v = v^*$ in the last inequality, we obtain

$$\lambda_n g(v_n, v^*) - \lambda_n g(v_n, z_n) \geq \langle v^* - z_n, \nabla f(w_n) - \nabla f(z_n)\rangle, \quad \forall v \in H_n. \tag{3.4}$$

As $v^* \in EP(g)$ and $v_n \in C$, we obtain that $g(v^*, v_n) \geq 0$. Thus $g(v_n, v^*) \leq -\gamma\Big(D_f(v_n, v^*) + D_f(v^*, v_n)\Big)$ because of the γ- strongly pseudomonotonicity of g. From the expression (3.4), we obtain

$$-\gamma\Big(D_f(v_n, v^*) + D_f(v^*, v_n)\Big) - \lambda_n g(v_n, z_n) \geq \langle v^* - z_n, \nabla f(w_n) - \nabla f(z_n)\rangle. \tag{3.5}$$

By relations (3.5) and the Lipschitz-type condition of g , impliy that

$$g(w_n, z_n) \leq g(w_n, v_n) + g(v_n, z_n) + c_1 D_f(v_n, w_n) + c_2 D_f(z_n, v_n). \tag{3.6}$$

Combining (3.5) with relation (3.6), we obtain

$$\langle z_n - v^*, \nabla f(w_n) - \nabla f(z_n)\rangle \geq \lambda_n \Big(g(w_n, z_n) - g(w_n, v_n)\Big) - c_1\lambda_n D_f(v_n, w_n) - c_2\lambda_n D_f(z_n, v_n) + \gamma\lambda_n \Big(D_f(v_n, v^*) + D_f(v^*, v_n)\Big). \tag{3.7}$$

Definition of z_n implies that $z_n \in H_n$, so

$$\langle z_n - v_n, \nabla f(w_n) - \nabla f(v_n)\rangle \leq \lambda_n \langle z_n - v_n, t_n\rangle. \tag{3.8}$$

Since $t_n \in \partial_2 g(w_n, v_n)$, we have

$$g(w_n, z_n) - g(w_n, v_n) \geq \langle t_n, v - v_n\rangle, \quad \forall v \in X.$$

Substituting $v = z_n$ into the above inequality, we obtain that

$$g(w_n, z_n) - g(w_n, v_n) \geq \langle z_n - v_n, t_n\rangle. \tag{3.9}$$

Expressions (3.8) and (3.9) imply that

$$\lambda_n \Big(g(w_n, z_n) - g(w_n, v_n)\Big) \geq \langle z_n - v_n, \nabla f(w_n) - \nabla f(v_n)\rangle. \tag{3.10}$$

From (3.7) and (3.10), we obtain

$$\langle z_n - v^*, \nabla f(w_n) - \nabla f(z_n)\rangle \geq \langle z_n - v_n, \nabla f(w_n) - \nabla f(v_n)\rangle - c_1\lambda_n D_f(v_n, w_n) - c_2\lambda_n D_f(z_n, v_n) + \gamma\lambda_n \Big(D_f(v_n, v^*) + D_f(v^*, v_n)\Big). \tag{3.11}$$

By the three point identity, we obtain that

$$D_f(z_n, v_n) + D_f(v_n, w_n) - D_f(z_n, w_n) = \langle z_n - v_n, \nabla f(w_n) - \nabla f(v_n)\rangle,$$

and semilary, we have

$$D_f(v^*, z_n) + D_f(z_n, w_n) - D_f(v^*, w_n) = \langle v^* - z_n, \nabla f(w_n) - \nabla f(z_n)\rangle.$$

Using inequality (3.11) by taking into account two above equalities, we obtain

$$D_f(v^*, w_n) - D_f(v^*, z_n) \geq D_f(z_n, v_n) + D_f(v_n, w_n) - c_1\lambda_n D_f(v_n, w_n) - c_2\lambda_n D_f(z_n, v_n) + \gamma\lambda_n \Big(D_f(v_n, v^*) + D_f(v^*, v_n)\Big).$$

This completes the proof. □

Now, we proof a strong convergence theorem for Algorithm2.4.

Theorem 3.3. *The sequences $\{w_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ generated by Algorithm2.4 are bounded and strongly converges to a solution $v^* \in EP(g)$.*

Proof. By definition of u_{n+1}, we deduce that

$$\begin{aligned} D_f(v^*, u_{n+1}) &= D_f\Big(v^*, \nabla^* f((1-\beta_n)\nabla f(w_n)+\beta_n \nabla f(u_{n-1}z_n))\Big) \\ &\leq (1-\beta_n)D_f(v^*, w_n)+\beta_n D_f(v^*, z_n), \end{aligned} \tag{3.12}$$

Using Lemma 3.2, we have

$$\begin{aligned} D_f(v^*, z_n) \leq & D_f(v^*, w_n) - (1-c_1\lambda_n)D_f(v_n, w_n) - (1-c_2\lambda_n)D_f(v_n, z_n) \\ & - \gamma\lambda_n\Big(D_f(v^*, v_n) + D_f(v_n, v^*)\Big). \end{aligned} \tag{3.13}$$

Combining (3.12) with relation (3.13), we obtain

$$\begin{aligned} D_f(v^*, u_{n+1}) \leq & D_f(v^*, w_n) - \beta_n(1-c_1\lambda_n)D_f(v_n, w_n) - \beta_n(1-c_2\lambda_n)D_f(v_n, z_n) \\ & - \beta_n\gamma\lambda_n\Big(D_f(v^*, v_n)+D_f(v_n, v^*)\Big) \\ \leq & D_f(v^*, w_n) - \beta_n(1-c_1\lambda_n)D_f(v_n, w_n) - \beta_n(1-c_2\lambda_n)D_f(v_n, z_n) \\ \leq & D_f(v^*, w_n) - \beta_n(1-b\lambda_n)\Big(D_f(v_n, w_n)+D_f(v_n, z_n)\Big), \end{aligned} \tag{3.14}$$

where $b = \max(c_1, c_2)$. From hypothesis $(H1)$, we obtain

$$D_f(v^*, u_{n+1} \leq D_f(v^*, w_n). \tag{3.15}$$

By definitions of u_{n+1} and Bregman distance, we have

$$D_(w_n, u_{n+1}) \leq \beta_n D_f(w_n, z_n). \tag{3.16}$$

Semilarly, by definition of w_n we have the following relations

$$D_f(v^*, w_n) \leq (1-\alpha_n)D_f(v^*, u_n) + \alpha_n D_f(v^*, u_{n-1}), \tag{3.17}$$

and

$$D_f(u_{n+1}, w_n) \leq (1-\alpha_n)D_f(u_{n+1}, u_n) + \alpha_n D_f(u_{n+1}, u_{n-1}). \tag{3.18}$$

By combining relations (3.15) and (3.17), we get

$$D_f(v^*, u_{n+1}) \leq D_f(v^*, w_n) \leq (1-\alpha_n)D_f(v^*, u_n) + \alpha_n D_f(v^*, u_{n-1}). \tag{3.19}$$

It follows from hypothesis $(H2)$ and Lemma 2.6, that the sequence $\{D_f(v^*, u_{n+1})\}$ converges. Hence, there exists $\lim_{n\to\infty} D_f(v^*, u_{n+1})$ and the sequence $\{D_f(v^*, u_{n+1})\}$ is bounded. Therefore, by Lemma 2.2 the sequence $\{u_n\}$ is bounded. From relations (3.17) and (3.13), clearly $\{w_n\}$ and $\{z_n\}$ are bounded. On the other hand, from (3.13) we derive that

$$0 \leq \gamma\lambda_n D_f(v^*, v_n) \leq D_f(v^*, w_n) - D_f(v^*, z_n), \tag{3.20}$$

this implies that, the sequence $\{D_f(v^*, v_n)\}$ is bounded. Therefore, by Lemma 2.2 the sequence $\{v_n\}$ is bounded.

Now, we are ready to prove that $v_n \to v^*$. Considering the limit in the relation (3.19) as $n \to \infty$, using relations (3.12) and (3.13) these imply that, there exists $\lim_{n\to\infty} D_f(v^*, w_n)$ and $\lim_{n\to\infty} D_f(v^*, z_n)$ and also they

are equail to $\lim_{n\to\infty} D_f(v^*, u_n)$. By equation (3.20), we obtain $\lim_{n\to\infty} D_f(v^*, v_n) = 0$, it follows from Lemma 6, that $\lim_{n\to\infty} \|v^* - v_n\| = 0$. Thuse, $v_n \to v^*$. By the use of triangular inequality, we obtain

$$\lim_{n\to\infty} \|v_n - w_n\| = 0.$$

By the three point identity, we obtain that

$$D_f(v^*, v_n) + D_f(v_n, w_n) - D_f(v^*, w_n) = \langle v^* - v_n, \nabla f(w_n) - \nabla f(v_n)\rangle, \tag{3.21}$$

passing to the limit in this relation, since ∇f is uniformly continuous on bounded subset of X, the secuences $\{w_n\}$ and $\{v_n\}$ are bounded and these imply that $\lim_{n\to\infty} D_f(v^*, w_n) = 0$, it follows from Lemma 6, that $\lim_{n\to\infty} \|v^* - w_n\| = 0$. Therefore, $w_n \to v^*$. So, we obtain

$$\lim_{n\to\infty} \|v^* - w_n\| = \lim_{n\to\infty} \|v^* - v_n\| = \lim_{n\to\infty} \|v^* - z_n\| = \lim_{n\to\infty} \|v^* - u_n\| = 0.$$

This completes the proof. □

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Almost S-Atrinian modules

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Abstract

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. In this paper we introduse the concept of almost S-Artinian modules. Let M be an R-module and S be a multiplicatively closed subset of R. We recall that a decsending chain $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ of submodules of M is S-stationary if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sM_k \subseteq M_n$ for each $n \geq k$. We call a decsending chain $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ of submodules of M almost S-stationary if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sM_n \subseteq M_{n+1}$ for each $n \geq k$.We say that M is an almost S-Artinian module whenever every descending chain of submodules of M is almost S-stationary. A ring R is called almost S-Artinian if it is an almost S-Artinian as an R-module.
Keywords: Artinian module, weakly S-Artinian module, Almost S-Artinian module.

1 Introduction

Throughout this paper, all rings are assumed to becommutative with nonzero identity and all modules are nonzero unital. let R denote such a ring and M denote such an R-module. Chain conditions on rings and modules have been widely studied in commutative algebra by this time, [7, 2] and [4]. Let R be a ring. A nonempty subset S of R is said to be a multiplicatively closed subset if $1 \in S$ and $ab \in S$ for each $a, b \in S$. We denote the set of all zero divisors, regular elements and units in R by $zd(R)$, $reg(R)$ and $u(R)$, respectively.Also, we denote the set of all minimal prime, prime and maximal ideals of R by $\min(R)$, $Spec(R)$ and $\max(R)$, respectively. Let R denote such a ring and M denote such an R-module. Let N be a submodule of M, the residual of N by M, $(N :_R M)$, denotes the ideal $\{r \in R : rM \subseteq N\}$. In special case in which $N = 0$, the ideal $(0 :_R M)$ is called the annihilator of M and it is denoted by $ann_R(M)$. An R-module M is said to be a faithful module if the $ann_R(M)$ is the zero ideal. M is called a multiplication module if every submodule N of M has the form $N = (N :_R M)M$. In [23], Khaninasab and Hamed introdused the concept of weakly S-Artinian modules. An R-module M is called a weakly S-Artinian module if for each descending chain of submodules $\{M_n\}_{n\in\mathbb{N}}$ of M, there exist $s_n \in S$ and $k \in \mathbb{N}$ such that $s_n M_k \subseteq M_n$ for all $n \geq k$. They investigate conditions under which weakly S-Artinian implies Artinian. In this paper we introduse the concept of almost S-Artinian modules. Let M be an R-module and S be a multiplicatively closed subset of R. We recall that a decsending chain $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ of submodules of M is S-stationary if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sM_k \subseteq M_n$ for each $n \geq k$. We call a decsending chain $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ of submodules of M almost S-stationary if there exist $s \in S$ and $k \in \mathbb{N}$

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such that $sM_n \subseteq M_{n+1}$ for each $n \geq k$. We say that M is an almost S-Artinian module whenever every descending chain of submodules of M is almost S-stationary. A ring R is called almost S-Artinian if it is an almost S-Artinian as an R-module. We briefly study this modules and trivially observe that Artinian rings are almost S-Artinian, see Remark 2.4. Moreover each almost S-Artinian module is weakly S-Artinain, but the convers is not true in general, see Remark 2.4 and Lemma 2.3.

We investigate some results of almost S-artinian modules. Let R be a commutative ring and S a multiplicatively subset of R. We recall that S is a strongly-multiplicatively set if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of elements of S we have $(\cap_{\alpha\in\Lambda} s_\alpha R)\cap S \neq \varnothing$. For example every finite multiplicatively subset of a commutative ring R is a strongly multiplicatively set. Finally, we show that if M is an R-module and S a strongly multiplicatively closed subset of R. Then M is an S-Artinian module if and only if $S^{-1}M$ is an Artinian $S^{-1}R$-module, see Proposition 2.6.We observe that if M an R-module and S a strongly multiplicatively closed subset of R, then M is an almost S-Artinian module if and only if it is a weakly S-Artinian module, see Lemma 2.11. We also show that if M is an almost S-Artinian module, where $ann(M)\cap S = \varnothing$, then M contains an essential submodule which is a direct sum of S-minimal, see corollray 2.13.

2 Main results

Example 2.1. Let S be a multiplicatively closed subset of R. Suppose that M is an R-module such that $S \cap ann_R(M) \neq \varnothing$. Then M is an almost S-Artinian R-module.

Example 2.2. Every Artinian R-module M is an almost S-Artinian module, where S is a multiplicatively closed subset of R.

Lemma 2.3. *Every almost S-Artinian module is weakly S-Artinian.*

Proof. Let M be an almost S-Artinian R-module and let $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ be a descending chain of submodules of M. Since M is an almost S-Artinian R-module, so there exist $s \in S$ and $k \in \mathbb{N}$ such that $sM_n \subseteq M_{n+1}$ for each $n \geq k$. Thus $s^2M_n \leq sM_{n+1} \leq M_{n+2}$. And also we have $s^3M_n \leq s^2M_{n+1} \leq sM_{n+2} \leq M_{n+3}$. Hence $s^iM_n \leq M_{n+i}$. So there exist $k \in \mathbb{N}$ such that $s^{n-k}M_k \subseteq M_n$ for each $n \geq k$. Therefor M is a weakly S-Artinian module. □

Remark 2.4. Notice every Artinian module is an almost S-Artinian module but the coverse is not true in general. For example if $R = \mathbb{Z}\times\mathbb{Q}$ and $S = \{(m, \frac{a}{b})|m \in \mathbb{Z}, \frac{a}{b} \in \mathbb{Q}\}$ be a multiplicatively closed subset of R then R_R is an almost S-Artinian module which it is not Artinian. Morover by Lemma 2.3, every almost S-Artinian module is weakly S-Artinian but the coverse of this fact is not true in general. Let $S = \mathbb{Z} - \{0\}$, the following chain of submodule $p_1\mathbb{Z} \supseteq p_1p_2\mathbb{Z} \supseteq \cdots$ of $\mathbb{Z}_\mathbb{Z}$ where each p_i is a prim number, show $\mathbb{Z}_\mathbb{Z}$ is not an almost S-Artinian module, but it is a weakly S-Artinian module. See [23, example (2.6)].

We recall for each multiplicatively closed subset $S \subseteq R, S^* = \{r \in R : \frac{r}{1}$ is a unit of $S^{-1}R\}$ denotes the saturation of S. Note that, S^* is a multiplicatively closed subset containing S. A multiplicatively closed subset S of R is called saturated if $S = S^*$. It is clear that S^* is always a saturated multiplicatively closed subset of R.

Lemma 2.5. *Let M be an R-module and S a multiplicatively closed subset of R. Then the following statements are satisfied:*

1. *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M is an almost S_1-Artinian module, then M is an almost S_2-Artinian module.*
2. *Suppose that S^* is the saturation of S. Then M is an almost S-Artinian module if and only if M is an almost S^*-Artinian module.*

3. *Let M be a multiplication R-module and $S \subseteq U_M(R)$ be a multiplicatively closed subset of R. Then M is an Artinian module if and only if M is an almost S-Artinian module.*

4. *Let M be an almost S-Artinian module with $ann_R(M) \cap S = \varnothing$. Then $S^{-1}M$ is an Artinian $S^{-1}R$-module.*

Proof. 1. It is explicit.

2. Let M be an almost S-Artinian module. Since $S \subseteq S^*$ and S^* is a multiplicatively closed subset of R, by (1), M is an almost S^*-Artinian module. For the converse, assume that M is an almost S^*-Artinian Module. Consider the descending chain of submodules $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \ldots$ of M. Since M is an almost S^*-Artinian module, there exist $r \in S^*$ and $k \in \mathbb{N}$ such that $rM_n \subseteq M_{n+1}$ for all $n \geq k$. Since $r \in S^*$, $\frac{r}{1}$ is a unit of $S^{-1}R$ and so $\frac{a}{s}\frac{r}{1} = 1$ for some $s \in S$ and $a \in R$. This implies that $uar = us$ for some $u \in S$. Now, put $s' = us$. Then $s' \in S$ and also $uarM_n = s'M_n \subseteq rM_n \subseteq M_{n+1}$ for all $n \geq k$. Hence, M is an almost S-Artinian module.

3. It is clear.

4. Let M be an almost S-Artinian module. Assume that $S^{-1}M_1 \supseteq S^{-1}M_2 \supseteq \cdots \supseteq S^{-1}M_n \supseteq \ldots$ is a descending chain of submodules of $S^{-1}M$, where M_i is an submodule of M for each $i \in \mathbb{N}$. Consider the natural homomorphism $\pi : M \to S^{-1}M$, defined by $\pi(m) = \frac{m}{1}$ for each $m \in M$. Then, we have descending chain of submodules $\pi^{-1}(S^{-1}M_1) \supseteq \pi^{-1}(S^{-1}M_2) \supseteq \cdots \supseteq \pi^{-1}(S^{-1}M_n) \supseteq \ldots$ of M. Since M is an almost S-Artinian module, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s\pi^{-1}(S^{-1}M_n) \subseteq \pi^{-1}(S^{-1}M_{n+1})$ for all $n \geq k$. This implies that $\frac{s}{1}S^{-1}M_n \subseteq S^{-1}M_{n+1}$ for all $n \geq k$. Since $\frac{s}{1}$ is a unit of $S^{-1}R$, we conclude that $S^{-1}M_n \subseteq S^{-1}M_{n+1}$ for all $n \geq k$ and thus $S^{-1}M$ is an Artinian $S^{-1}R$-module.

□

Proposition 2.6. *Let M be an R-module and S a strongly multiplicatively closed subset of R. Then M is an almost S-Artinian module if and only if $S^{-1}M$ is an Artinian $S^{-1}R$-module.*

Proof. The "if" part is clear and straightforward. Assume that $S^{-1}M$ is an Artinian $S^{-1}R$-module and

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$$

is a descending chain of submodules of M. Then $\{S^{-1}M_i\}_{i\in\mathbb{N}}$ is a family of descending chain of submodules of $S^{-1}M$ such that $S^{-1}M_k = S^{-1}M_n$ for each $n \geq k$. Let $m \in M_n$. Then $\frac{m}{1} \in S^{-1}M_n = S^{-1}M_{n+1}$ and so $s_im \in M_n$ for some $s_i \in S$. Since S is a strongly multiplicatively closed subset of R, thus $(\cap_{s_i\in S}s_iR)\cap S \neq \varnothing$. Let $s \in (\cap_{s_i\in S}s_iR) \cap S$. Then $sm \in M_{n+1}$ and so $sM_n \subseteq M_{n+1}$ which completes the proof. □

Theorem 2.7. *Let M be an R-module and $S = \{s_1, s_2, s_3, \ldots s_n\}$ be a finite multiplicatively closed subset of R. Then M is an almost S-Artinian module if and only if $S^{-1}M$ is an Artinian $S^{-1}R$-module.*

In the next proposition we characterize almost S-Artinian modules in terms of almost S-Artinian rings.

Proposition 2.8. *Let M be an R-module and S a multiplicatively closed subset of R, and π is the canonical homomorphism $\pi : R \to \frac{R}{ann_R(M)}$. Then the following statements are satisfied:*

1. *Suppose that M is a multiplication module. If $\frac{R}{ann_R(M)}$ is an almost $\pi(S)$- Artinian ring, then M is an almost S-Artinian module.*

2. *Suppose that M is a finitely generated multiplication module. If M is an almost S-Artinian module, then $\frac{R}{ann_R(M)}$ is an almost $\pi(S)$-Artinian ring.*

Proof. 1. Note that if

$$M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$$

is a descending chain of submodules of M, then the following chain

$$\frac{(M_1:_R M)}{ann_R(M)} \supseteq \frac{(M_2:_R M)}{ann_R(M)} \supseteq \cdots \supseteq \frac{(M_n:_R M)}{ann_R(M)} \supseteq \cdots$$

is a descending chain of ideals of $\frac{R}{ann_R(M)}$.

(2) Assume that M is a finitely generated multiplication module. Let M be an almost S-Artinian module. Take a descending chain of ideals

$$J_1 \supseteq J_2 \supseteq ... \supseteq J_n \supseteq ...$$

of $\frac{R}{ann_R(M)}$. Then there exist some ideals I_i of R containing $ann_R(M)$ such that $\frac{I_i}{ann_R(M)} = J_i$. Then by the above chain, it is clear that $\{I_i\}_{i\in\mathbb{N}}$ is a descending chain of ideals of R. This implies that

$$I_1M \supseteq I_2M \supseteq ... \supseteq I_nM \supseteq ...$$

is a descending chain of submodules of M. As M is an almost S-Artinian module, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sI_nM \subseteq I_{n+1}M$ for all $n \geq k$. Since M is a finitely generated multiplication module, by ([30, Theorem 9]), $sI_n \subseteq I_{n+1} + ann_R(M) = I_{n+1}$ and so $(s + ann_R(M))J_n \subseteq J_{n+1}$ for all $n \geq k$. Hence, $\frac{R}{ann_R(M)}$ is an almost $\pi(S)$-Artinian ring. □

Theorem 2.9. *Let M be a finitely generated multiplication module and S a multiplicatively closed subset of R. Then the following statements are equivalent:*

1. *M is an almost S-Artinian module.*
2. *$\frac{R}{ann_R(M)}$ is an almost $\pi(S)$-Artinian ring.*

Proof. Follows from Proposition 2.8. □

We recall the following definitions from [21].

Definition 1. *Let $\mathcal{F}$ be a nonempty family of submodules of M and S be a multiplicatively closed subset of R. Then $Y \in \mathcal{F}$ is called an S-minimal element of $\mathcal{F}$ if there exists $s \in S$ and whenever $K \subseteq Y$ for some $K \in \mathcal{F}$ then $sY \subseteq K$. In particular, M satisfies $(S-MIN)$ condition if every nonempty family of submodules $\mathcal{F}$ of M has an S-minimal element.*

Definition 2. *Let $\mathcal{F}$ be a nonempty family of submodules of M and S a multiplicatively closed subset of R. Then $\mathcal{F}$ is said to be an S-cosaturated family of submodules if whenever $sY \subseteq K$ for some $s \in S$; $Y \in \mathcal{F}$ and a submodule K of M, then $K \in \mathcal{F}$.*

Theorem 2.10. *Let M be an R-module and S a strongly multiplicatively closed subset of R. Then the following statements are equivalent:*

1. *M is an almost S-Artinian module.*
2. *Every nonempty S-cosaturated family $\mathcal{F}$ of submodules of M has a minimal element.*
3. *M satisfies $(S$-$MIN)$-condition.*

4. Every factor module $\frac{X}{Y}$ is a finitely S-cogenerated R-module.

Proof. (1 $\Rightarrow$ 2). Suppose that M is an almost S-Artinian module and $\mathcal{F}$ is an arbitrary nonempty S-cosaturated family of submodules of M. Assume that for any $Y, K \in \mathcal{F}$, $Y \preceq K$ if and only if $K \subseteq Y$. Then $(\mathcal{F}, \preceq)$ is clearly a partially ordered set. Take any chain $Y_1 \preceq Y_2 \preceq \cdots \preceq Y_n \preceq \ldots$ of $\mathcal{F}$. Then it is clear that the supremum of the chain $\{Y_i\}_{i\in\mathbb{N}}$ is $Y = \vee_{i\in\mathbb{N}} Y_i = \cap_{i\in\mathbb{N}} Y_i$. Now, we will show that $Y \in \mathcal{F}$. By the above chain, we have the descending chain of submodules $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n \supseteq \cdots$ of M. Since M is an almost S-Artinian module, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sY_n \subseteq Y_{n+1}$ for each $n \geq k$. Then we can easily get $s^i Y_k \subseteq Y_{k+i}$ for every i. Since $(s^i)_{i\in\mathbb{N}} \subseteq S$ and S is a strongly multiplicatively closed subset of R. Then $((\cap_{i\in\mathbb{N}} s^i R) \cap S) \neq \varnothing$. Let $s' \in ((\cap_{i\in\mathbb{N}} s^i R) \cap S)$, so $s'Y_k \subseteq Y_{k+i}$ for every i. Thus $s'Y_k \subseteq \cap_{i\in\mathbb{N}} Y_i = Y$. Since $Y_k \in \mathcal{F}$ and $\mathcal{F}$ is an S-cosaturated family of submodules of M, we have $Y \in \mathcal{F}$. Hence by Zorn's Lemma, $\mathcal{F}$ has a maximal element $K \in \mathcal{F}$ with respect to $\preceq$. It is clear that K is the minimal element of $(\mathcal{F}, \supseteq)$.
2 $\Rightarrow$ 3.Suppose that every nonempty S-cosaturated family of submodules of M has a minimal element. Let $\mathcal{F}$ be an arbitrary nonempty family of submodules of M. Now, we will show that $\mathcal{F}$ has an S-minimal element. Consider the family $\mathcal{F}^* = \{K : K$ is a submodule of M, $\exists s \in S$, $Y^* \in \mathcal{F}$ such that $sY^* \subseteq K\}$. First, we will show that $\mathcal{F}$ is an S-cosaturated family of submodules of M. Take a submodule K of M such that $sY \subseteq K$ for some $s \in S$ and $Y \in \mathcal{F}^*$.Then there exist $s_1 \in S$ and $Y^* \in \mathcal{F}$ such that $s_1 Y^* \subseteq Y$. Now put $s' = ss_1$ and note that $s' \in S$. Also it is clear that $s'Y^* = s(s_1 Y^*) \subseteq sY \subseteq K$ and this implies that $K \in \mathcal{F}^*$. Hence $\mathcal{F}^*$ is an S-cosaturated family of submodules of M. Then by assumption, $\mathcal{F}^*$ has a minimal element $L \in \mathcal{F}^*$. Then there exist $s^* \in S$ and $Y' \in \mathcal{F}$ such that $s^*Y' \subseteq L$. Now, we will show that Y' is an S-minimal element of $\mathcal{F}$. To see this, take an element $K' \in \mathcal{F}$ such that $K' \subseteq Y'$. Then $s^*K' \subseteq s^*Y'$ and so $s^*K' \subseteq L \cap K'$. Then we have $L \cap K' \in \mathcal{F}^*$. By the minimality of L, we have $L \cap K' = L$ and this yields $L \cap K'$ and so $s^*Y' \subseteq L \subseteq K'$. Hence, Y' is an S-minimal element of $\mathcal{F}$ that is, M satisfies (S-MIN) condition.
3 $\Rightarrow$ 4. Let $\cap_{i\in\Delta} \frac{Y_i}{Y} = 0_{\frac{X}{Y}}$ for some family of submodules $\{\frac{Y_i}{Y}\}_{i\in\Delta}$ of $\frac{X}{Y}$. Then we have $\cap_{i\in\Delta} Y_i = Y$. Now, put $\mathcal{F} = \{\cap_{i\in\Delta'} Y_i : \Delta' \subseteq \Delta$ is a finite subset $\}$. Then by (S-MIN) condition, $\mathcal{F}$ has an S-minimal element, say $Y' = \cap_{i=k_1}^{k_n} Y_i$ for some $k_1, k_2, \cdots, k_n \in \Delta$. Take an element $k \in \Delta - \{k_1, k_2, \cdots, k_n\}$ then note that $Y' \cap Y_k \subseteq Y'$. By the S-minimality of Y', there exists $s \in S$ such that $sY' \subseteq Y' \cap Y_k$. This implies that $sY' \subseteq \cap_{i\in Delta} Y_i = Y$, so $s(\cap_{i=k_1}^{k_n} Y_i) \subseteq Y$ hence $s[\cap_{i=k_1}^{k_n} \frac{Y_i}{Y}] = 0_{\frac{X}{Y}}$. So $\frac{X}{Y}$ is finitly S-cogenerated module.
4 $\Rightarrow$ 1. Assume that every factor module $\frac{X}{Y}$ is finitly S-cogenerated R-module. Consider the descending chain of submodules

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

of R-module M. Put $Y = \cap_{i=k_1}^{\infty} Y_i$. By assumption, $\frac{X}{Y}$ is finitly S-cogenerated R-module and $\cap_{i=k_1}^{\infty} \frac{Y_i}{Y} = 0_{\frac{X}{Y}}$, then there exist $s \in S$ and $k_1 \leq k_2 \leq \cdots \leq k_n$ such that $s(\cap_{i=k_1}^{k_n} \frac{Y_i}{Y}) = 0_{\frac{X}{Y}}$ hence $s(\cap_{i=k_1}^{k_n} \frac{Y_i}{Y}) = s(\frac{Y_{k_n}}{Y}) = 0_{\frac{X}{Y}}$. This implies that $sY_{k_n} \subseteq Y = \cap_{i=1}^{\infty} Y_i \subseteq Y_n$ for all $n \geq k_n$, so $sY_{k_n} \subseteq Y_{k_{n+1}}$ for all $n \geq k_n$, hence M is an almost S-Artinian module. □

Lemma 2.11. *Let M be an R-module and S a strongly multiplicatively closed subset of R. Then M is an almost S-Artinian module if and only if it is a weakly S-Artinian module.*

Proof. $\Rightarrow$. By Lemma 2.3, it is clear.
conversly, let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chian of submodules of M, since M is a weakly S-Artinian module, then there exist $s_n \in S$ and $k \in \mathbb{N}$ such that $s_n M_k \subseteq M_n$ for all $n \geq k$. Let $m \in M_k$, since S is a strongly multiplicatively closed subset of R, then $(\cap_{s_n\in S} s_n R) \cap S \neq \varnothing$. Consider $s \in (\cap_{s_n\in S} s_n R) \cap S$, so $sm \in M_k$ and $sm \in M_n$, thus $sM_k \subseteq M_n$ for all $n \geq k$. Hence $sM_k \subseteq M_{k+1}$ for all $n \geq k$, so M is an almost S-Artinian module. □

Now, we have the following theorem.

Theorem 2.12. *Let M be an R-module and S a strongly multiplicatively closed subset of R. Then the following statements are equivalent:*

1. *M is a weakly S-Artinian module.*
2. *Every nonempty S-cosaturated family $\mathcal{F}$ of submodules of M has a minimal element.*
3. *M satisfies (S-MIN)-condition.*
4. *Every factor module $\frac{X}{Y}$ is a finitely S-cogenerated R-module.*

Proof. This proof by Theorem 2.10 and Lemma 2.11, is clear. □

Theorem 2.13. *Let M be a left R-module and S a strongly multiplicative subset of R such that $ann(M)\cap S = \varnothing$. If M is an almost S-Artinian module, then M contains an essential submodule which is a direct sum of S-minimal submodules.*

Proof. Let $M \neq 0$. By Theorem 2.10, M contains an S-minimal submodule. Let $\mathcal{F}$ be the set of all independent set of S-minimal submodules of M with $\subseteq$ as an order relation. Clearly $\mathcal{F} \neq \varnothing$ and by Zorn's Lemma, we can find a maximal independent family of an S-minimal submodules of M, say $\{M_\alpha\}_{\alpha\in\Lambda}$. Consider the submodule $\bigoplus_{\alpha\in\Lambda} M_\alpha$. Let N be a nonzero submodule of M. Then by Theorem 2.10, N contains an S-minimal submodule, say K. By maximality of $\{M_\alpha\}_{\alpha\in\Lambda}$, $\{M_\alpha\}_{\alpha\in\Lambda}\cup\{K\}$ is not independent, so $K \cap (\bigoplus_{\alpha\in\Lambda} M_\alpha) \neq 0$. Hence $N \cap (\bigoplus_{\alpha\in\Lambda} M_\alpha) \neq 0$. This shows that $\oplus_{\alpha\in\Lambda} M_\alpha$ is essential in M. □

Proposition 2.14. *Let M and M' be left R-modules, S a multiplicative subset of R and $f : M \to M'$ a surjective module homomorphism. Suppose that $ann(M') \cap S = \varnothing$. If M is an almost S-Artinian module, then M' is an almost S-Artinian module.*

Proof. Since $ann(M') \cap S = \varnothing$ and f is an epimorphism, then $ann(M) \cap S = \varnothing$. Consider a descending chain of submodules $M'_1 \supseteq M'_2 \supseteq ... \supseteq M'_n \supseteq ...$ of M' , then $f^{-1}(M'_1) \supseteq f^{-1}(M'_2) \supseteq ... \supseteq f^{-1}(M'_n) \supseteq ...$ is a descending chain of submodules of M. Since M is an almost S-Artinian module, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $sf^{-1}(M'_n) \subseteq f^{-1}(M'_{n+1})$.
Now, since f is an epimorphism, $f(sf^{-1}(M'_n)) = s(f(f^{-1}(M'_n))) = sM'_n$. Hence $sM'_n \subseteq M'_{n+1}$ for all $n \geq k$, therefore M' is an almost S-Artinian module. □

Theorem 2.15. *Let M be a left R-module, N a proper submodule of M and S a multiplicative subset of R. Suppose that $S \cap (N :_R M) = \varnothing$ and $S \cap ann(N) = \varnothing$. Then the following assertions are equivalent:*

1. *M is an almost S-Artinian module.*
2. *N and $\frac{M}{N}$ are both almost S-Artinian modules.*

Proof. (1) $\Rightarrow$ (2). Let M be an almost S-Artinian R-module. Every descending chain of submodules of N and $\frac{M}{N}$ is a descending chain of submodules of M, so N and $\frac{M}{N}$ are almost S-Artinian R-modules.
(2) $\Rightarrow$ (1) let N and $\frac{M}{N}$ are almost S-Artinian R-modules. We show that M is an almost S-Artinian R-module. Take descending chain of submodules

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$$

of M, then $M_0\cap N \supseteq M_1\cap N \supseteq M_2\cap N \supseteq ... \supseteq M_n\cap N \supseteq ...$ and $\frac{M_0+N}{N} \supseteq \frac{M_1+N}{N} \supseteq \frac{M_2+N}{N} \supseteq ... \supseteq \frac{M_n+N}{N} \supseteq ...$ are descending chain of submodules of N and $\frac{M}{N}$. Since N and $\frac{M}{N}$ are almost S-Artinian R-modules, there exist $s_1, s_2 \in S$, $k_1, k_2 \in \mathbb{N}$ such that $s_1(M_n \cap N) \subseteq (M_{n+1} \cap N)$ and $s_2(\frac{M_n+N}{N}) \subseteq (\frac{M_{n+1}+N}{N})$. Now put $s = s_1s_2$ and $k = \max\{k_1, k_2\}$. Thus $s(M_n \cap N) \subseteq (M_{n+1} \cap N)$ and $s(\frac{M_n+N}{N}) \subseteq (\frac{M_{n+1}+N}{N})$ for all $n \geq k$.

Let $s^2m_n \in s^2M_n$ for every $m_n \in M_n$. Since $sm_n \in sM_n \subseteq s(M_n + N) \subseteq (M_{n+1} + N)$, we infer that $sm_n \in M_{n+1} + N$ and so $sm_n = m_{n+1} + x$ for some $m_{n+1} \in M_{n+1}$ and $x \in N$. Since $M_{n+1} \subseteq M_n$ for all $n \geq k$, we have $m_{n+1} \in M_n$, hence $x \in M_n \cap N$ and $s^2m_n - sm_{n+1} = sx$. This yields that $sx \in s(M_n \cap N) \subseteq (M_{n+1} \cap N)$ and $s^2m_n = sm_{n+1} + sx \in M_{n+1}$. Thus, we have $s^2M_n \subseteq M_{n+1}$ for each $n \geq k$ where $s^2 \in S$. Hence M is an almost S-Atrinian module. □

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Some fixed point results for multivalued mappings on orthogonal b-metric space

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Abstract

In this paper, we consider the concept of multivalued rational type F_b-contraction mappings and prove fixed point theorems for this type mappings in complete orthogonal b-metric space.
Keywords: fixed point, multivalued mappings, orthogonal b-metric space.

1 Introduction

The theory of fixed point for multivalued mappings has been developed after the famous paper of Nadler [9]. Let (X, d, b) be a complete metric space. Respectively, let $CB(X)$ and $K(X)$ be the class of nonempty closed bounded and nonempty compact subsets of the nonempty set X. For $A, B \in CB(X)$, the map $H : CB(X) \times CB(X) \to [0, \infty)$ denote by

$$H(A, B) = \max\{\sup_{a\in A} \Delta(a, B), \sup_{b\in B} \Delta(b, A)\}$$

where $\Delta(a, B) = \inf_{b\in B} d(a, b)$, is a metric on $CB(X)$, and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. For $A, B \in CB(X)$, it is obvious that

$$d(A, B) \leq H(A, B).$$

A well-known generalization of metric space is b-metric spaces introduced by Czerwik in [3]. Then many researchers extended fixed point theorems to b-metric spaces, see [7]

Definition 1. *[3] Let X be a non-empty set, $b \geq 1$ and let the function $d : X \times X \to [0, \infty)$ satisfy the following conditions:*

i) $d(s, t) \geq 0$ *for all* $s, t \in X$ *and* $d(s, t) = 0$ *if and only if* $s = t$;

ii) $d(s, t) = d(t, s)$ *for all* $s, t \in X$;

iii) $d(s, p) \leq b[d(s, t) + d(t, p)]$ *for all* $s, t, p \in X$.

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Then, the pair (X, d, b) is called a b-metric space.
Clearly, for $b = 1$, b-metric space is a metric space. Then, a metric space is also a b-metric space, but the converse is not always true.

Example 1.1. Let $X = [0, 1]$ and $d : X \times X \to (0, \infty)$ be defined by $d(s, t) = |s - t|^2$ for all $s, t \in X$. Clearly, $(X, d, 2)$ is a b-metric space that is not a metric space.

The notion of F-contraction was introduced by Wardowski in [8]. It was shown that it possesses a unique fixed point in a complete metric space and that each sequence of successive approximations converges to the fixed point. Banach contraction is an F-contraction for a function $F(x) = ln(x), x \in (0, \infty)$ and, in that way, the main theorem of [4] extends the well-known Banach fixed point theorem. Many papers have been published on this topic by observing the new setting (b-matric space, cone metric space, partial metric space, fuzzy metric space, etc.) or modifying the contractive condition, see [2, 6]
Inspired by Wardowski [8], Cosentino et al. [4] gave the following definition. In [4], the author presents a modification of the concept of F-contraction by adding condition F_4 to it to prove fixed point theorems for multivalued mapping in complete b-metric spaces.

Definition 2. *[4] Let $b \geq 1$ and $\mathcal{F}_b$ be the set of all function $F : (0, \infty) \to \mathbb{R}$ such that*

F_1) *F is strictly increasing;*

F_2) *For each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers then $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$;*

F_3) *there exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$;*

F_4) *For each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\theta + F(bt_n) \leq F(t_{n-1})$ for all $n \in \mathbb{N}$ and some $\theta \in (0, \infty)$, then $\theta + F(b^n t_n) \leq F(b^{n-1} t_{n-1})$ for all $n \in \mathbb{N}$.*

Example 1.2. Let $F : (0, \infty) \to (-\infty, \infty)$ be defined by $F(t) = t + lnt$. Clearly, F satisfies $(F_1) - (F_4)$. Then $F \in \mathcal{F}_b$.

Definition 3. *[4] Let (X, d, b) be a b-metric space. A multivalued mapping $T : X \to CB(X)$ is called an F-contraction of Nadler type if there exist $F \in \mathcal{F}_b$ and $\tau > 0$ such that*

$$2\tau + F(bH(Ts, Tt)) \leq F(d(s, t)),$$

for all $s, t \in X$ with $Ts \neq Tt$.

Theorem 1.3. *[4] Let (X, d, b) be a complete b-metric space and let $T : X \to CB(X)$. Assume that there exists a continuous from the right function $F \in \mathcal{F}_b$ and $\tau > 0$ such that*

$$2\tau + F(bH(Ts, Tt)) \leq F(d(s, t)),$$

for all $s, t \in X$ with $Ts \neq Tt$. Then T has a fixed point.

Now, we recall some concepts of orthogonality, which will be needed in the sequel.

Definition 4. *[5] Let X be a non-empty set and $\perp \subseteq X \times X$ to be a binary relation. Then $(X, \perp)$ is called an orthogonal set if the following condition holds*

$$\exists s_0 \in X : (\forall s \in X, s \perp s_0) \quad or \quad (\forall s \in X, s_0 \perp s).$$

We denote this orthogonal set or O-set by $(X, \perp)$.

Example 1.4. Let $X = [0, \infty)$ and define $s \perp t$ if $st \in \{s, t\}$. Then, by setting $s_0 = 1$ or $s_0 = 0$, $(X, \perp)$ is an O-set.

Definition 5. *Let $(X, \perp)$ be an O-set and d be a b-metric space on X. Then $(S, \perp, d)$ is named an orthogonal b-metric space (briefly, O-b-metric space).*

Gordji et al. [5] presented the definition of an orthogonal sequence in 2017 as follows:

Definition 6. *Let $(X, \perp, d)$ be an O-b-metric space.*
i) A sequence $\{t_n\}$ is named an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, t_n \perp t_{n+1}) \quad or \quad (\forall n \in \mathbb{N}, t_{n+1} \perp t_n).$$

ii) A Cauchy sequence $\{t_n\}$ is named an orthogonal Cauchy sequence (briefly, Cauchy O-sequence) if

$$(\forall n \in \mathbb{N}, t_n \perp t_{n+1}) \quad or \quad (\forall n \in \mathbb{N}, t_{n+1} \perp t_n).$$

iii) X is said to be a complete O-b-metric space if every Cauchy O-sequence is converges in X.

The concept of orthogonal continuity was defined by Gordji et al. [5] as follows.

Definition 7. *Let $(X, \perp, d)$ be an O-b-metric space. A function $f : X \to X$ is said to be orthogonal continuous (or $\perp$-continuous) at $t \in X$ if for O-sequence $\{t_n\} \subseteq X$ such that $t_n \to t$, we have $f(t_n) \to f(t)$ as $n \to \infty$. Further f is $\perp$-continuous on X if f is $\perp$-continuous at any $t \in X$.*

Definition 8. *Let $(X, \perp)$ be an O-set, A and B are nonempty subsets of X. The set A is orthogonal to set B is denoted by $\perp'$ and defined as:*

$$A \perp' B, \ if \ \forall a \in A \quad and \quad \forall b \in B, a \perp b.$$

Inspired by the multivalued rational type F_b-contraction by Akar [1], we implement a new orthogonally multivalued rational type F_b-contraction and present some fixed point results in an orthogonal complete b-metric space for this contraction map.

2 Main Results

First, we introduce the concept of an orthogonally multivalued rational type F_b-contraction.

Definition 9. *Let $(X, \perp, d)$ be an O-b-metric space with $b \geq 1$ and $T : X \to CB(X)$ be a mapping. Then T is said to be an orthogonally multivalued rational type F_b-contraction if $F \in \mathcal{F}_b$ and there exists $\tau > 0$, such that $s, t \in X$ with $s \perp t$,*

$$H(Ts, Tt) > 0 \Rightarrow \tau + F(bH(Ts, Tt)) \leq F(M(s, t)),$$

where

$$M(s,t) = \max\{d(s,t), \Delta(s,Ts), \Delta(t,Tt), \frac{\Delta(s,Ts)\Delta(t,Tt)}{1+d(s,t)}, \frac{\Delta(s,Ts)\Delta(t,Tt)}{1+H(Ts,Tt)}\}.$$

Theorem 2.1. *Let $(X, \perp, d)$ be a complete O-b-metric space. Let $T : X \to K(X)$ satisfy the following conditions:*

i) There exists $t_0 \in X$ such that $\{t_0\} \perp' Tt_0$ or $Tt_0 \perp' \{t_0\}$,

ii) For all $s,t \in X$, $s \perp t$ implies $Ts \perp' Tt$,

iii) T is an orthogonally multivalued rational type F_b-contraction,

iv) If $\{t_n\}$ is an O-sequence in X such that $t_n \to t$ then $t_n \perp t$ or $t \perp t_n$ for all $n \in \mathbb{N}$.

Therefore, T has at least a fixed point such that $t \in Tt$ in X.

Below, we add a condition to the members of the set $\mathcal{F}_b$ so that in Theorem 2.1, we consider $CB(X)$ instead of $K(X)$.

Definition 10. *Let $b \geq 1$ and $\mathcal{F}_{b'}$ be the set of all functions $F : (0,\infty) \to \mathbb{R}$ such that*

F_1) F is strictly increasing;

F_2) For each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers then $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$;

F_3) there exists $k \in (0,1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$;

F_4) For each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\theta + F(bt_n) \leq F(t_{n-1})$ for all $n \in \mathbb{N}$ and some $\theta \in (0,\infty)$, then $\theta + F(b^n t_n) \leq F(b^{n-1} t_{n-1})$ for all $n \in \mathbb{N}$.

F_5) $F(inf A) = inf F(A)$ for all $A \subset (0,\infty)$ with $inf A > 0$.

Definition 11. *T is said to be an orthogonally generalized multivalued rational type $\mathcal{F}_{b'}$-contraction if, in Definition 9, $F \in \mathcal{F}_{b'}$.*

Theorem 2.2. *Let $(X, \perp, d)$ be a complete O-b-metric space. Let $T : X \to K(X)$ satisfy the following conditions:*

i) There exists $t_0 \in X$ such that $\{t_0\} \perp' Tt_0$ or $Tt_0 \perp' \{t_0\}$,

ii) For all $s,t \in X$, $s \perp t$ implies $Ts \perp' Tt$,

iii) T is an orthogonally generalized multivalued rational type $\mathcal{F}_{b'}$-contraction,

iv) If $\{t_n\}$ is an O-sequence in X such that $t_n \to t$ then $t_n \perp t$ or $t \perp t_n$ for all $n \in \mathbb{N}$.

Therefore, T has at least a fixed point such that $t \in Tt$ in X.

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On (C,D)-Controlled Continuous Frames

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Abstract

In this paper, we define controlled continuous frames with two not necessarily positive controller operators and give some results about them. Finally, we state a sufficient condition that the sum of two (C,D)-controlled continuous frames is also a (C,D)-controlled continuous frame.

Keywords: continuous frame, controlled continuous frame, dual, sum of controlled continuous frames.

1 Introduction

In 1952, Duffin and Schaeffer [4] introduced the concept of discrete frames in Hilbert spaces. Weighted and controlled frames have been introduced in [2]. They investigated relations of weighted frames and controlled frames. Also, they showed that controlled frames are equivalent to standard frames, and the concept of controlled frame gives us a generalized way to check the frame condition, while offering a numerical advantage in the sense of preconditioning.

Throughout this paper, (Ω, μ) is a measure space with a positive measure μ and $\mathcal{H}$ is a complex Hilbert space, the set of all bounded linear operators on $\mathcal{H}$ is represented by $B(\mathcal{H})$, the set of all bounded linear operators on $\mathcal{H}$ which have bounded invers is denoted by $GL(\mathcal{H})$, and the set of all positive elements of $GL(\mathcal{H})$ will be showed by $GL^+(\mathcal{H})$.

Controlled and weighted continuous frames introduced in [3] and here we recall the definition of continuous frame and controlled continuous frame.

Definition 1. *A continuous frame is a map $F : \Omega \to \mathcal{H}$ such that*
(i) F is weakly-measurable, i.e, for any $f \in \mathcal{H}$, the mapping $\omega \longmapsto \langle f, F(\omega)\rangle$ is measurable on Ω,
(ii) There exist constants $0 < A < B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1.1}$$

A linear opertor T_F is called the *Synthesis operator* and weakly defined by

$$T_F : L^2(\Omega, \mu) \to \mathcal{H}, \qquad \langle T_F\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle d\mu(\omega),$$

for all $f \in \mathcal{H}$, $\varphi \in L^2(\Omega, \mu)$ and $\omega \in \Omega$.

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The adjoint opertor T_F^* is called the *Analysis operator* and weakly defined by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \qquad (T_F^* f)(\omega) = \langle f, F(\omega)\rangle,$$

for all $f \in \mathcal{H}$ and $\omega \in \Omega$.

The *continuous frame operator* S_F of F is weakly defined by

$$S_F : \mathcal{H} \to \mathcal{H}, \qquad \langle S_F f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), g\rangle d\mu(\omega),$$

for all $f, g \in \mathcal{H}$.

Definition 2. *Let $C \in GL(\mathcal{H})$. A C-controlled continuous frame is a map $F : \Omega \to \mathcal{H}$ such that*
(i) F is weakly-measurable, i.e, for any $f \in \mathcal{H}$, the mapping $\omega \longmapsto \langle f, F(\omega)\rangle$ is measurable on Ω,
(ii) There exist constants $0 < A < B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega \langle f, F(\omega)\rangle\langle CF(\omega), f\rangle d\mu(\omega) \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1.2}$$

2 Main results

As stated, controlled frames were introduced with one positive and invertible operator by Balazs [2] and then they were studied by using two controller operators by many authors [5] . Now, we define controlled frames with two not necessarily positive controller operators.

Definition 3. *Let $C, D \in GL(\mathcal{H})$. A (C,D)-controlled continuous frame is a map $F : \Omega \to \mathcal{H}$ such that*
(i) F is weakly-measurable, i.e, for any $f \in \mathcal{H}$, the mapping $\omega \longmapsto \langle f, F(\omega)\rangle$ is measurable on Ω,
(ii) There exist constants $0 < A_{CD} < B_{CD} < \infty$ such that

$$A_{CD}\|f\|^2 \leq \int_\Omega \langle f, F(\omega)\rangle\langle CF(\omega), f\rangle d\mu(\omega) \leq B_{CD}\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

The constants A_{CD}, B_{CD} are called *lower* and *upper* frame bounds, respectively. The mapping F is called *(C,D)-controlled continuous Bessel mapping* if the right inequality in (1.4) holds and is called *tight* if $A = B$.

If μ is counting measure and $\Omega = \mathbb{N}$, then F is called *(C,D)-controlled discrete frame.*

Suppose that $P \in B(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ is a continuous Bessel mapping with the synthesis operator T_F and Bound B_F. We introduce a bounded linear operator $T_{PF} : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by

$$\langle T_{PF}\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle PF(\omega), f\rangle d\mu(\omega), \quad (f \in H). \tag{2.2}$$

It is easy to see that $T_{PF} = PT_F$ and $\|T_{PF}\| \leq \sqrt{B_F}\|P\|$. This implies that unlike continuous frames, (C,D)-controlled continuous frames have two synthesis operators as following:

Definition 4. *Let $F : \Omega \to \mathcal{H}$ be a (C,D)-controlled continuous Bessel mappig. Then*
(i) Two linear opertors T_{CF} and T_{DF} are called the Synthesis operators and weakly defined by

$$T_{CF} : L^2(\Omega, \mu) \to \mathcal{H}, \qquad \langle T_{CF}\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle CF(\omega), f\rangle d\mu(\omega),$$

$$T_{DF} : L^2(\Omega, \mu) \to \mathcal{H}, \qquad \langle T_{DF}\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle DF(\omega), f\rangle d\mu(\omega).$$

for all $f \in \mathcal{H}$, $\varphi \in L^2(\Omega, \mu)$ *and* $\omega \in \Omega$.

(ii) The adjoint opertors T^*_{CF} *and* T^*_{DF} *are called the Analysis operators and weakly defined by*

$$T^*_{CF} : \mathcal{H} \to L^2(\Omega, \mu), \qquad (T^*_{CF} f)(\omega) = \langle f, CF(\omega)\rangle,$$

$$T^*_{DF} : \mathcal{H} \to L^2(\Omega, \mu), \qquad (T^*_{DF} f)(\omega) = \langle f, DF(\omega)\rangle.$$

for all $f \in \mathcal{H}$ *and* $\omega \in \Omega$.

Definition 5. *Let* $F : \Omega \to \mathcal{H}$ *be a (C,D)-controlled continuous frame for* $\mathcal{H}$. *Then the controlled continuous frame operator* S_{CDF} *of* F *is weakly defined by*

$$S_{CDF} : \mathcal{H} \to \mathcal{H}, \qquad \langle S_{CDF} f, g\rangle = \int_\Omega \langle f, CF(\omega)\rangle\langle DF(\omega), g\rangle d\mu(\omega),$$

for all $f, g \in \mathcal{H}$.

It is easy to see that S_{CDF} is a positive, self-adjoint and invertible operator in $B(\mathcal{H})$. Indeed, $A_{CD} I_{\mathcal{H}} \leq S_{CDF} \leq B_{CD} I_{\mathcal{H}}$ and $S_{CDF} = DS_F C^*$. Also,

$$f = \int_\Omega \langle f, CF(\omega)\rangle S^{-1}_{CDF} DF(\omega) d\mu(\omega) = \int_\Omega \langle f, S^{-1}_{CDF} CF(\omega)\rangle DF(\omega) d\mu(\omega),$$

for all $f \in \mathcal{H}$. Now we state the relationship between controlled continuous frames and continuous frames.

Proposition 2.1. *Let* $F : \Omega \to \mathcal{H}$ *be a weakly measurable mapping and* $C, D \in GL(\mathcal{H})$. *Then the following assertions are hold:*
(i) If F *is a (C,D)-controlled continuous frame for* $\mathcal{H}$, *then* F *is a continuous frame for* $\mathcal{H}$.
(ii) If F *is a continuous frame for* $\mathcal{H}$ *and* $DS_F C^*$ *is a positive operator, then* F *is a (C,D)-controlled continuous frame for* $\mathcal{H}$.

According to [5], we can charactrize (C,D)-controlled continuous frames.

Theorem 2.2. *Let* $F : \Omega \to \mathcal{H}$ *be a continuous Bessel mapping and* $C, D \in GL(\mathcal{H})$. *Then the following are equivalent:*
(i) F *is a (C,D)-controlled continuous frame for* $\mathcal{H}$,
(ii) T_{CF} *and* T_{DF} *are bounded and onto operators and* $T_{DF} T^*_{CF}$ *is bounded below,*
(iii) CF *and* DF *are continuous frames and* $T_{DF} : R(T^*_{CF}) \to \mathcal{H}$ *is injective.*

In the following we define the concept of *(C,D)-controlled dual continuous frame operator.*

Definition 6. *Let* $C, D \in GL(\mathcal{H})$ *and* $CF, DG : \Omega \to \mathcal{H}$ *be continuous Bessel mappings for* $\mathcal{H}$. *A (C,D)-controlled dual continuous frame operator* $R_{CFDG} : \mathcal{H} \to \mathcal{H}$ *defines as follows:*

$$R_{CFDG} f = T_{DG} T^*_{CF} f = \int_\Omega \langle f, CF(\omega)\rangle DG(\omega) d\mu(\omega),$$

for all $f \in \mathcal{H}$.

If $R_{CFDG} = I_{\mathcal{H}}$, then G is called *(C,D)-controlled dual* of F.

Proposition 2.3. *Let* $C, D \in GL(\mathcal{H})$ *and* $CF, DG : \Omega \to \mathcal{H}$ *be continuous Bessel mappings for* $\mathcal{H}$. *Then*
(i) R_{CFDG} *is a well-defined and bounded operator.*
(ii) If R_{CFDG} *is bounded below, then* CF *and* DG *are continuous frames.*

Next, we state a sufficient condition that the sum of two (C,D)-controlled continuous frames is also a (C,D)-controlled continuous frame.

Theorem 2.4. *Let $C, D \in GL(\mathcal{H})$ and $F, G : \Omega \to \mathcal{H}$ be two (C,D)-controlled continuous frames. If $D(T_F T_G^* + T_G T_F^*)C^*$ is positive, then $F + G$ is also a (C,D)-controlled continuous frame.*

Corollary 2.5. *Let $C, D \in GL(\mathcal{H})$ and $F, G : \Omega \to \mathcal{H}$ be two (C,D)-controlled continuous frames which G is (C,D)-controlled dual of F. Then $F + G$ is also a (C,D)-controlled continuous frame.*

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Separable Programming Technique to Solve Multi objective Non-Linear Fractional Programming Problem

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Abstract

This article deals with multi objective non-linear fractional programming problems in separable form .i.e both the numerator and denominator of objective functions are separable and attempt to propose a method to solve such problems. We linearize Firstly numerators and denominators of the objective functions to form a multi objective linear fractional problem on using piecewise linear approximation for linearization. The converted linear program can be solved by any classical method. We use a parametric algorithm which uses linear programming to obtain efficient solutions. Finally, a numerical example is provided to illustrate the effectiveness and efficiency of the proposed approach.

Keywords: Multi objective programming; Fractional programming; Piecewise linear approximation; Separable functions.

1 Introduction

The MOFP problem is called a MOLFP problem when numerators and denominators of the objective functions are linear. Solving MOFP problems is usually difficult and sometimes finding even one efficient solution is of importance issue.

Costa [33] proposed an iterative algorithm for solving sum of linear ratios problems and finding efficient solutions. Dinkelbach [2] used a parametric approach to solve linear fractional problems.Valipour and Yagoobi [3] presented an iterative algorithm that extends Dinkelbach, ... approach [2] to solve MOLFP problems. They considered a parametric linear objective function and inserts new constraints that the convergency of the algorithm is guaranteed. Antczak [4] used a modified objective function method for solving multi objective nonlinear fractional programming problem. In this article we are concerned on multi objective problems whose each objective functions are similar to the problems considered by Gaur and Arora [5]. Firstly by linearizing numerators and denominator of the objective functions, the multi objective non-linear fractional problem is reduced to a MOLFP problem.

2 Preliminaries

The real valued function f which has n variables, is said to be separable if it can be expressed as the sum of n functions which have one variable. This function is defined as:

$$f(x_1, x_2, ..., x_n) = f(x_1) + f(x_2) + ... + f(x_n)$$

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Let us consider the multi objective fractional programming given by

$$\begin{aligned} \max\ & Z(x) = (z_1(x), ..., z_p(x))^t \\ & s.t\ \ x \in X \end{aligned} \tag{2.1}$$

Where X is compact, $z_i(x) = \dfrac{N_i(x)}{D_i(x)} = \dfrac{\sum_{j=1}^{n} N_{ij}(x_j) + \alpha_0}{\sum_{j=1}^{n}(x_j) + \beta_0}$ for $i = 1, ..., p$,
$N_i(x)$ and $D_i(x)$ are convex and separable nonlinear functions on feasible region

$$X = \{x \in R^n :\ Ax \leqslant b,\ x \geqslant 0\},\ A = [a_{rj}]_{m\times n},\ b \in R^m \& D_i(x) > 0,\ \forall\ x \in X.$$

Definition 1. *A point $\bar{x} \in X$ is an efficient solution of problem (2.1) if there is no other $x \in X$ such that $z_i(x) \geqslant z_i(\bar{x})$ for all $i = 1, ..., p$ and $z_j(x) > z_j(\bar{x})$ j for at least one j.*

3 On solving multi objective non-linear fractional programming problem

Antczak [4] used a modified objective function method for solving multi objective non-linear fractional programming problem.

Definition 2. *A differentiable function $f : X \longrightarrow R^p$ is called V-invex with respect to η at $\bar{x} \in X$ on X if, there exist functions $X \times \times X \longrightarrow R^n$ and $\alpha_i X \times X \longrightarrow R + \backslash\{0\}$, for $i = 1, ..., p$ such that for any $x \in X$ the following relation holds.*

$$f_i(x) - f_i(\bar{x}) \geqslant \alpha_i(x, \bar{x}) \bigtriangledown f_i(\bar{x})\eta(x, \bar{x})$$

Antczak considered the vector optimization program $(FP_\eta(\bar{x}))$ as follows:

$$\begin{aligned} \max\ & (z_i(\bar{x} + \bigtriangledown z_1(\bar{x})\eta(x, \bar{x}), ..., z_p(\bar{x}) + \bigtriangledown z_p(\bar{x})\eta(x, \bar{x})) \\ & s.t\ \ x \in X \end{aligned}$$

4 How to convert the problem to MOLFP

Suppose that the non-linear terms of the objectives with respect to the variable x_j are replaced by their piecewise linear approximations via the break-points x_{jv}, for $v = 1, .., k_j$ (x_{jv} and λ_{jv} are the v th breakpoint value and the weight of the v th break-point of x_j, respectively).
Each of variable x_j in linear constraints must be replaced by

$$x_j = \sum_{v=1}^{k_j} \lambda_{jv} x_{jv} \tag{4.1}$$

Let $L = \{j : N_{ij}$ and D_{ij} are linear for $i = 1, ..., p\}$. The resulting problem can be formulated as shown in [5].

$$\max\ z_i^{'}(y) = \frac{N_i^{'}(y)}{D_i^{'}(y)} = \frac{\sum_{j\in L} N_{ij}(x_j) + \sum_{j\notin L}\sum_{v=1}^{k_j} \lambda_{jv} N_{ij}(x_{jv}) + \alpha_0}{\sum_{j\in L} D_{ij}(x_j) + \sum_{j\notin L}\sum_{v=1}^{k_j} \lambda_{jv} D_{ij}(x_{jv}) + \beta_0} \quad \text{for } i = 1, ..., p \tag{4.2}$$

$$\begin{aligned} s.t\ & \sum_{j\in L} a_{rj}x_j + \sum_{j\notin L}\sum_{v=1}^{k_j} \lambda_{jv} a_{rj} x_{jv} \leqslant b_r \\ & \lambda_{jv} \geqslant 0 \quad v = 1, ..., k_j;\ j \notin L \\ & \sum_{v=1}^{k_j} \lambda_{jv} = 1 \ \text{ for } j \notin L \\ & x_j \geqslant 0 \ \text{ for } \ j \in L \end{aligned}$$

5 Iterative approach to solve MOL FP problem

Valipour and yaghoobi [3] presented an iterative algorithm which uses the parametric approach of Dinkelbach [2] to find efficient solutions of MOLFP problems.
At each iteration of the algorithm, the following weighted linear program is solved.

$$\max\ \mathcal{V}^{k+1}(y) = \sum_{i=1}^{p} w_i (N^{'}(y) - t_i^k D_i^{'}(y)) \tag{5.1}$$

$$\begin{aligned} s.t\ & z_i^{'}(y) \geqslant t_i^k \\ & \sum_{j\in L} a_{rj}x_j + \sum_{j\notin L}\sum_{v=1}^{k_j} \lambda_{jv} a_{rj} x_{jv} \leqslant b_r \\ & \lambda_{jv} \geqslant 0 \quad \text{for } v = 1, ..., k_j;\ \ j \notin L \\ & \sum_{v=1}^{k_j} \lambda_{jv} = 1 \ \text{ for } j \notin L \\ & x_j \geqslant 0 \ \text{ for } \ j \in L \end{aligned}$$

Where $t^k = (z_1^{'}(y^k), ..., z_p^{'}(y^k))$ In the first iteration, set $k = 0$ and $t^0 = (z_1^{'}(y^\circ), ..., z_p^{'}(y^\circ))$ where y° be an arbitrary feasible solution of problem (4.1). The problem ..... can be written as follows:

$$\begin{aligned}
\max \ \mathcal{V}^{k+1}(y) = &\sum_{i=1}^{p} w_i \big(\big(\sum_{j \in L} N_{ij}(x_j) + \sum_{j \notin L} \sum_{v=1}^{k_j} \lambda_{jv} N_{ij}(x_{jv}) + \alpha_0 \big) \\
&- t_i^k \big(\sum_{j \in L} D_{ij}(x_j) + \sum_{j \notin L} \sum_{v=1}^{k_j} \lambda_{jv} D_{ij}(x_{jv}) + \beta_0 \big) \\
s.t \ \ & z_i^{'}(y) \geqslant t_i^k \quad \text{for } i = 1, ..., p \\
& \sum_{j \in L} a_{rj} x_j + \sum_{j \notin L} \sum_{v=1}^{k_j} \lambda_{jv} a_{rj} x_{jv} \leqslant b_r \ \text{ for } r = 1, ..., m \\
& \lambda_{jv} \geqslant 0 \ \text{ for } \ v = 1, ..., k_j; \ \ j \notin L \\
& x_j \geqslant 0 \ \text{ for } j \in L
\end{aligned}$$

Let y^{k+1} and $\mathcal{V}^{k+1}$ be optimal solution and optimal value of the problem (5.1), respectively and assume that $w = \dfrac{1}{\min_{i=1,...,p} w_i}$.
If $wv^{k+1} < \varepsilon$, then y^{k+1} is an efficient solution of problem (4.2).Otherwise, considering $t^{k+1} = (z_1^{'}(y^{k+1}), ..., z_p^{'}(y^{k+1}))$ in the next iteration, the problem (4.2) is solved again.
The algorithm proposed above computed the efficient solution of MOLFP of problem (4.2). Therefore, on using relation (4.1), the efficient solution of problem (2.1) can easily be obtained.

6 Numerical example

Consider the following multi objective separable fractional programming problem.

$$\begin{aligned}
\max \ z_1 &= \frac{6x_1^2 + 3x_2^2}{5x_1^2 + 2x_2^2 + 1} \\
\max \ z_2 &= \frac{15x_1^2 + 10x_2^2}{x_1^2 + x_2^2 + 1} \\
s.t \ \ z_2 &= 3x_1 + 5x_2 \leqslant 15 \\
& 5x_1 + 2x_2 \leqslant 10 \\
& x_1, x_2 \geqslant 0
\end{aligned} \tag{6.1}$$

By considering the break-points for x_2 as $0, 1, 2, 3$, and for x_1 as $0, 1, 2$, these variables are then can be defined as:

$$x_1 = 0\lambda_{11} + 1\lambda_{12} + 2\lambda_{13} \tag{6.2}$$
$$x_2 = 0\lambda_{21} + 1\lambda_{22} + 2\lambda_{23} + 3\lambda_{24} \tag{6.3}$$

Table 1: Feasible solution x^* of problem (6.1)

(x^*)	Efficient point(x^*)	Iterations(k)
$(0.68, 2)$	$(1.0528, 2.3682)$	4
$(0.5, 2.7)$	$(0.3494, 2.7903)$	4
$(0.87, 1.2)$	$(1.3272, 1.6818)$	3
$(1.05, 2.36)$	$(0.9842, 2.4096)$	2
$(0.3, 2.82)$	$(0.3026, 2.8185)$	1

Therefore, the nonlinear terms of objective functions replaced with piecewise linear approximations and problem (6.1) reduces to the following MOLFP problem:

$$\begin{aligned}
\max\ z_1^{'} &= \frac{6\lambda_{12} + 24\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} + 27\lambda_{24}}{5\lambda_{12} + 20\lambda_{13} + 2\lambda_{22} + 8\lambda_{23} + 18\lambda_{24} + 1} \\
\max\ z_2^{'} &= \frac{15\lambda_{12} + 60\lambda_{13} + 10\lambda_{22} + 40\lambda_{23} + 90\lambda_{24}}{\lambda_{12} + 4\lambda_{13} + \lambda_{22} + 4\lambda_{23} + 9\lambda_{24} + 1} \\
s.t\ \ & 3\lambda_{12} + 6\lambda_{13} + 5\lambda_{22} + 19\lambda_{23} + 15\lambda_{24} \leqslant 15 \\
& 5\lambda_{12} + 10\lambda_{13} + 2\lambda_{22} + 4\lambda_{23} + 6\lambda_{24} \leqslant 10 \\
& \lambda_{11} + \lambda_{12} + \lambda_{13} = 1 \\
& \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1 \\
& \lambda_{1v} \geqslant 0 \ \text{ for } v = 1, 2, 3 \\
& \lambda\lambda_{2v} \geqslant 0 \ \text{ for } \ v = 1, 2, 3, 4
\end{aligned} \tag{6.4}$$

Now, we consider an admissible tolerance $\varepsilon = 0.1$ and a weight vector $w = (1/3, 2/3)$. By applying the proposed algorithm of the article to MOLFP problem , then efficient solutions of the problem (6.1) are obtained.
For instance, suppose $x^{\circ} = (0.68, 2)$ be an arbitrary feasible solution of problem (6.1). We attempt to solve following system of equations:

$$\begin{aligned}
\lambda_{11} + \lambda_{12} + \lambda_{13} &= 1 \\
\lambda_{12} + 2\lambda_{13} &= 0.68 \\
\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} &= 1 \\
\lambda_{22} + 2\lambda_{23} + 3\lambda_{24} &= 2
\end{aligned} \tag{6.5}$$

On solving we get $\lambda^* = (0.4736, 0, 0.5264, 0.2106, 0, 0, 0.7894)$ and $x^* = (1.0528, 2.3682)$
Table 1.1 presents five feasible solution x^* of problem (6.1), the resulting efficient solutions and the number of iterations.

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Inverse Eigenvalue Problem for Constructing a Specific Pentadiagonal Matrix

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Abstract

In the inverse eigenvalue problem for matrices with specific structures, it is possible to reconstruct the matrix using partial spectral information. In this paper, we aim to construct the original matrix using the minimal and maximal eigenvalues of its principal submatrices along with a pair of eigenvalues of the full matrix. The target matrix is an nonsymmetric pentadiagonal matrix with a special structure.
Keywords: inverse eigenvalue problem; symmetric pentadiagonal matrices; nonsymmetric pentadiagonal matrices; leading principal submatrices.

1 Introduction

The inverse eigenvalue problem (IEP) involves constructing a matrix from given spectral data, such as eigenvalues, eigenvectors, or partial spectral information. Chu and Golub in [1] gave a perfect characterization of IEPs. IEPs usually come with a practical background and its applications have been studied in various scientific fields such as control, image processing, graph theory, finite element method. In this article, we investigate the inverse eigenvalue problem (IEP) for reconstructing a specific pentadiagonal matrix using the minimal eigenvalues of each of its leading principal submatrices the minimal and maximal eigenvalues of its principal submatrices and the eigenvector corresponding to the maximal eigenvalue of the given matrix. Several papers discuss this problem for different structured matrices and some applications [2],[3]. In most cases, the reconstructed matrices are symmetric, and as known, these matrices have real eigenvalues.

In this paper the pentadiagonal matrix has the following form:

$$B_{2m} = \begin{pmatrix} a_1 & b_1 & c_1 & 0 & \dots & \dots & 0 & 0 & 0 \\ f_1 & a_2 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ t_1 & 0 & a_3 & b_3 & c_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & f_3 & a_4 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & t_3 & 0 & a_5 & b_5 & c_5 & \vdots & 0 \\ 0 & 0 & 0 & 0 & f_5 & a_6 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & t_{2m-3} & 0 & a_{2m-1} & b_{2m-1} \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 & f_{2m-1} & a_{2m} \end{pmatrix} \tag{1.1}$$

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where $b_j f_j > 0$ for all $j = 1, 2, \ldots, m$ and $c_k t_k > 0$ for $k = 1, 2, \ldots, m-1$.

The matrix B_{2m} of the form (1.1) is diagonally similar to the following symmetric pentadiagonal matrix $DB_{2m}D^{-1}$

$$DB_{2m}D^{-1} = \begin{pmatrix} a_1 & \sqrt{b_1 f_1} & \sqrt{c_1 t_1} & 0 & \ldots & 0 & 0 & 0 \\ \sqrt{b_1 f_1} & a_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \sqrt{c_1 t_1} & 0 & a_3 & \sqrt{b_3 f_3} & \sqrt{c_3 t_3} & \ldots & 0 & 0 \\ 0 & 0 & \sqrt{b_3 f_3} & a_4 & 0 & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & a_{2m-1} & \sqrt{b_{2m-1} f_{2m-1}} \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & \sqrt{b_{2m-1} f_{2m-1}} & a_{2m} \end{pmatrix} \tag{1.2}$$

$D = diag(d_1, d_2, \ldots, d_{2m})$ such that

$$d_i = \begin{cases} \frac{b_{i/2}}{f_{i/2}} \prod_{j=1}^{\frac{i}{2}-1} \frac{c_j}{t_j}, & i = 2, 4, \ldots, 2m \\ \prod_{j=1}^{\frac{i}{2}-1} \frac{c_j}{t_j}, & i = 3, 5, \ldots, 2m-1 \end{cases} \tag{1.3}$$

which implies that, under this transformation, the eigenvalues of the leading principal submatrices $B_j, j = 1, 2, \ldots, 2m$ of the matrix B_{2m} remain invariant.

2 Main results

In this paper, we aim to solve the following problem.

Problem 1: Given a set of real $\lambda_1^{(j)}$, $j = 1, 2, \ldots, 2m$ and $\lambda_2^{(j)}$, $j = 2, \ldots, 2m$, a positive integer m and nonzero vector $X = (x_1, x_2, \ldots . x_{2m})$. Construct a nonsymmetric pentadiagonal matrix of form 1.1 such that $\lambda_1^{(j)}$ and $\lambda_2^{(j)}$ are, respectively, the minimal and maximal eigenvalues of the principal submatrices of matrix B_{2m}, and X is an eigenpair of the matrix.

In the following, we investigate the relation between successive leading principal minors of $\lambda I_{2m} - B_{2m}$.

Lemma 2.1. *The sequence* $\{\phi_j(\lambda) = \det(\lambda I_j - B_j)\}_{j=1}^{2m}$ *of characteristic polynomials of* B_j *satisfies the following recurrence relations:*

i. $\phi_1(\lambda) = (\lambda - a_1)$

ii. $\phi_{2j}(\lambda) = (\lambda - a_{2j})\phi_{2j-1}(\lambda) - b_{2j-1} f_{2j-1} \phi_{2j-2}(\lambda) \qquad j = 1, 2, \ldots, m$

iii. $\phi_{2j+1}(\lambda) = (\lambda - a_{2j+1})\phi_{2j}(\lambda) - c_{2j-1} t_{2j-1} (\lambda - a_{2j})\phi_{2j-2}(\lambda)$
$j = 1, 2, \ldots, m-1$

where $\phi_0(\lambda) = 1$.

Proof. It is immediate by expanding the determinant.

In the following lemma, we examine the relationship between the components of the eigenvector corresponding to the maximal eigenvalue of matrix 1.1, and it can be observed that its elements are proportional to the first component.

Lemma 2.2. *Let $X_{2m} = (x_1, x_2, \ldots, x_{2m})^T$ be an eigenvector of B_{2m} corresponding to eigenvalue $\lambda_2^{(2m)}$ then $x_1 \neq 0$, and the components of this eigenvector are given by*

$$x_{2j+1} = \frac{(-1)^j \phi_{2j}(\lambda_2^{(2)}) x_1}{\prod_{i=1}^{j}(a_{2i} - \lambda_2^{(2m)}) c_{2i-1}}, \qquad j = 1, 2, \ldots, m-1, \tag{2.1}$$

$$x_{2j} = \frac{(-1)^j f_{2j-1}\phi_{2j-2}(\lambda_2^{(2m)}) x_1}{(a_{2j} - \lambda_2^{(2m)})\prod_{i=1}^{j-1}(a_{2i} - \lambda_2^{(2m)}) c_{2i-1}}, \qquad j = 1, 2, \ldots, m. \tag{2.2}$$

Hereafter, we will adopt the following notations

$$D_{2j} = \phi_{2j-1}(\lambda_1^{(2j)})\phi_{2j-2}(\lambda_2^{(2j)}) - \phi_{2j-1}(\lambda_2^{(2j)})\phi_{2j-2}(\lambda_1^{(2j)}).$$

$$M_j = \lambda_1^{(2j)}\phi_{2j-1}(\lambda_1^{(2j)})\phi_{2j-2}(\lambda_2^{(2j)}),$$

$$N_j = \lambda_2^{(2j)}\phi_{2j-1}(\lambda_2^{(2j)})\phi_{2j-2}(\lambda_1^{(2j)}),$$

$$P_j = (\lambda_2^{(2j)} - \lambda_1^{(2j)})\phi_{2j-1}(\lambda_1^{(2j)})\phi_{2j-1}(\lambda_2^{(2j)}).$$

$$D_{2j+1} = \phi_{2j}(\lambda_1^{(2j+1)})\phi_{2j-2}(\lambda_2^{(2j+1)})(\lambda_2^{(2j+1)} - a_{2j}) - \phi_{2j}(\lambda_2^{(2j+1)})\phi_{2j-2}(\lambda_1^{(2j+1)})(\lambda_1^{(2j+1)} - a_{2j}),$$

$$S_j = \lambda_1^{(2j+1)}\phi_{2j}(\lambda_1^{(2j+1)})\phi_{2j-2}(\lambda_2^{(2j+1)})(\lambda_2^{(2j+1)} - a_{2j}),$$

$$T_j = \lambda_2^{(2j+1)}\phi_{2j}(\lambda_2^{(2j+1)})\phi_{2j-2}(\lambda_1^{(2j+1))})(\lambda_1^{(2j+1)} - a_{2j}).$$

In the following Theorem, by applying the interlacing condition of the eigenvalues, we determine the unknown elements of the matrix and construct the matrix.

Theorem 2.3. *If the list of real numbers $\lambda_1^{(j)}, \lambda_2^{(j)}$ satisfying*

$$\lambda_1^{(2m)} < \lambda_1^{(2m-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_2^{(2m)} \tag{2.3}$$

and $x_j \neq 0$ for $j = 1, 2, \ldots, 2m$, then problem 1 has a unique solution and we obtain:

$$a_{2j} = \frac{M_j - N_j}{D_{2j}},$$

$$b_{2j-1}f_{2j-1} = \frac{P_j}{D_{2j}}.$$

$$f_{2j-1} = (\lambda_2^{(2m)} - a_{2j})\frac{x_{2j}}{x_{2j-1}} \quad j = 1, 2, \ldots, m.$$

$$b_{2j-1} = \frac{(\lambda_2^{(2j)} - a_{2j})\phi_{2j-1}(\lambda_2^{(2j)})}{f_{2j-1}\phi_{2j-2}(\lambda_2^{(2j)})}.$$

$$a_{2j+1} = \frac{S_j - T_j}{D_{2j+1}},$$

$$c_{2j-1}t_{2j-1} = \frac{(\lambda_2^{(2j+1)} - \lambda_1^{(2j+1)})\phi_{2j}(\lambda_1^{(2j+1)})\phi_{2j}(\lambda_2^{(2j+1)})}{D_{2j+1}},$$

$$c_{2j-1} = \frac{(-1)^j \phi_{2j}(\lambda_2^{(2m)}) x_1}{x_{2j+1}(a_{2j} - \lambda_2^{(2m)}) \prod_{i=1}^{j-1}(a_{2i} - \lambda_2^{(2m)}) c_{2i-1}}.$$

$$t_{2j-1} = \frac{(\lambda_2^2 - a_{2j+1})\phi_{2j}(\lambda_2^{2m})}{c_{2j-1}(\lambda_2^{2m} - a_{2j})\phi_{2j-2}(\lambda_2^{2m})}.$$

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Solving mixed integer convex geometric programming problems

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Abstract

The investigation on geometric programming (GP) with integer variables is an interesting subject in many optimization areas. This study proposes an optimization approach for solving GP. Our approach is first to convert all non-convex terms into convex by negative power transformation method. Then, the convex terms are further treated with the proposed piece-wise linearization technique. Finally, we integrate the problem by using the non-linear Branch and Bound algorithm. Our proposed method is to use the linear Branch and Bound method after performing piece-wise linear approximation. Computational results show that the proposed method is much more efficient and faster than the conventional one. A numerical example is illustated in the end cto ensure the effectiveness and efficiency of our approach.

Keywords: geometric programming, mixed integer non- linear programming, Branch-and-Bound algorithm, negative power transformation, piece-wise linearization method.

1 Introduction

Consider the following geometric program (GP)

$$\begin{aligned} &Minimize\ f_0(x) \\ &f_i(x) \leqslant 1 \quad i = 1, ..., m \\ &g_i(x) = 1 \quad j = 1, ..., p \end{aligned} \tag{1.1}$$

Where $g_i(x)$, $j = 1, ..., p$ are monomials and $f_i(x)$ for $i = 1, ..., m$ are posynomials. Many optimization issues in science such as engineering and in general purpose applications, including discrete decision-making and dynamic non-linear systems, exist in the literature that affects the final design quality. Therefore, considering integer variables and non-linear systems that solve the problem are called optimization problems in discrete space. Geometric programming is a class of non-linear optimization with many useful theoretical computational properties. Geometric programming is an essential broadens in the scope of

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linear programming applications and is originally suited to model some important non-linear programming problems in science and engineering. Geometric programming had been applied in many fields of applications, including analog/digital circuit design, chemical engineering, mechanical engineering, power control and communication network systems. To solve mixed integer geometric programming problems, many papers have been proposed. In 1960, many applications were being found in mechanical, civil, chemical engineering, statistics, finance and economics, and many more applications in GP. [22, 23, 24]. Therefore, considering integer variables and non-linear systems that solve the problems become very important in the literature of optimization problems in discrete space. Today, many of important issues of optimizing including binary and integer variables as well as non-linear systems can directly affect the quality of solving these problems. Some techniques for solving mixed integer non-linear programming problems were proposed by Leyffer et al. (2009) [3]. For the typical non-linear programming problems Melo et al. 2014 [4] proposed a new hybrid algorithm for convex mixed integer non-linear programming problem, which is an improved version for branch-and-bound technique. For mixed integer polynomial programming Dua [5] suggested an algorithm that reformulate the mixed integer polynomial programming to multi-parametric programming. A global optimization method was proposed by Lundle et al. [6]. For solving non-convex mixed integer non-linear programming Shen et al. [7] described global optimization for signomial geometric programming using linear relaxation. Numerous useful results have been obtained in solving various MOGPPs. [2, 3, 4].

In recent years, many papers have been represented to optimize geometrical problems based on special ordered sets. [8, 9, 10]. In a research work in 2009, Vielma proposed a method that is superior to the SOS approach without binary variables.

Here in this paper we developed an optimization approach for solving GP. The paper is organized as follows. (i) Convert all nonconvex terms into convex ones. (ii) The convex terms are further treated with the proposed piecewise linearization method where only $\lceil \log 2(m-1) \rceil$ binary variables are used.

Section 2 of article is devoted to introduce the negative power transformation methods and linear approximations. Section 3 introduces the non-linear branch -and - bound algorithm to solve the geometric programming problems. Section 4 introduces the proposed method and provides some numerical examples to illustrate the modeling idea and the usefulness of the proposed method. Section 5 is devoted to conclusion.

2 Convex Approximation

Convex approximation of non- convex function is a basic part in approximate global optimization methods. The best convex approximation should be contained a minimal number of variables and constraint. For the negative power transportation, we have the following proposition:

Lemma 2.1. *Let A be a nonzero $n \times n$ symmetric matrix.*

(a) If A is positive semidefinite, then $\triangle_k \geqslant 0$, $k = 1, 2, ..., n$ and not all $\triangle_k = 0$.

(b) A is positive definite if and only if $\triangle_k > 0$, for all $k = 1, 2, ..., n$

Lemma 2.2. *Let A be a nonzero symmetric matrix. Then, the following will hold.*

(a) A is positive semidefinite if and only if all of its eigenvalues are nonnegative.

(b) A is positive definite if and only if all of its eigenvalues are positive.

Let $f_1 : R^n_{++} \longrightarrow R$ defined as $f_1(x) = c_1 \prod_{i=1}^{n} x_i^{\alpha_i}$, where $c_1 > 0$ and $\alpha_i \leqslant 0$ for all $i = 1, 2, ..., n$. Then f_1 is a convex function.

Proof. Since $c_1 > 0$, it is enough to show that $\tilde{f}(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ is convex. Let $g(x) = \ln \tilde{f}_1(x) = \sum_{i=1}^{n} \ln x_i^{\alpha_i} = \sum_{i=1}^{n} \alpha_i \ln x_i$ Then, we have

$$\nabla g(x) = \Big[\frac{\alpha_1}{x_1}\frac{\alpha_2}{x_2}...\frac{\alpha_n}{x_n}\Big]^T$$

$$\nabla^2 g(x) = \Big[\frac{-\alpha_1}{x_1^2}0....00\frac{-\alpha_2}{x_2^2}\cdot\ddots 00\ \ 0...\frac{-\alpha_n}{x_n^2}\Big]$$

Because $\alpha_i \leqslant 0$ for all $i = 1, 2, ..., n$ we know that all eigenvalues of $\nabla^2 g(x)$ are nonnegative, which implies (by Lemma 2.1(a)) that $\nabla^2 g(x)$ is positive semidefinite. Thus, $(x) = \ln(x)$ is a convex function which yields that $\tilde{f}_1(x)$ is a convex function. □

Proposition 2.3. *Let $f_2 : R^n_{++} \longrightarrow R$ defined as $f_2(x) = c_2 \prod_{i=1}^{n} x_i^{\alpha_i}$ where $\alpha_i > 0$. $c_2 < 0$ for all $i = 1, 2, ..., n$ with $1 - \sum_{i=1}^{n} \alpha_i \geqslant 0$. Then f_2 is a convex function.*

Proof. It is not hard to compute that $= c_2\alpha_i x_i^{\alpha_i - 1} \prod_{\substack{j=1 \\ j\neq i}}^{n} x_j^{\alpha_j} [\nabla f_2(x)]_i$. In other words,

$$\nabla f_2(x) = [c_2\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}...x_n^{\alpha_n} c_2\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha\alpha_2}...x_n^{\alpha_n}...c_2\alpha_n x_1^{\alpha_1-1}x_2^{\alpha_2}...x_n^{\alpha_n}]$$

In addition, it can be verified that

$$[\nabla^2 f_2(x)]_{i,j} = \frac{\partial^2 f_2(x)}{\partial x_i \partial x_j} = \Big\{ \frac{\alpha_i\alpha_j}{\partial x_i \partial x_j} f_2(x) \quad i \neq j \quad \frac{\alpha_i(\alpha_i - 1)}{\partial x_i^2} f_2 \ \ i = j$$

Moreover, the determinant of $\nabla^2 f_2(x)$ can be computed and be shown by induction as

$$\det[\nabla^2 f_2(x)] = (-c_2)^n \Big(\prod_{i=1}^{n} \alpha_i x_i^{n\alpha_i - 2})(1 - \sum_{i=1}^{n} \alpha_i \Big) \tag{2.1}$$

We will complete the proof by discussing the following two cases. Case i. If $1 - \sum_{i=1}^{n} \alpha_i = 0$,then it is not hard to verify that $\nabla^2 f_2(x)x$ is a zero vector for any $x \in R^n$ Hence, $x^T \nabla^2 f_2(x) = 0$ which says that $\nabla^2 f_2(x)$ is a positive semidefinite matrix by definition. Therefore, $f_2(x)$ is convex under this case.
Case ii. If $1 - \sum_{i=1}^{n} \alpha_i > 0$ then we know from (2.1) that $\triangle_i = (-c_1)^i (\prod_{j=1}^{i} \alpha_j x_j^{n\alpha_j - 2})(1 - \sum_{j=1}^{i} \alpha_j)$ where $\triangle_i$ denotes the i th principal minor of the Hessian matrix of $f_2(x)$. Note that $c_2 < 0.\alpha_i > 0$ for all $i = 1, 2, ..., n$ and $1 - \sum_{i=1}^{n} \alpha_i > 0$. Therefore, it can be seen that $\triangle_i > 0$ for all $i = 1, 2, ..., n$ which implies (by Lemma 2.1(b)) that $\nabla^2 f_2(x)$ is a positive definite matrix. This says that $f_2(x)$ is strictly convex under this case. From all the previous, the desired result follows. □

Proposition 2.4 ((Lu et al. 2010))**.** *Suppose $\underline{x_i}$ and $\underline{x_i}$ be a upper and lower bound of $\underline{x_i}$, respectively , and let ε be the accuracy of the computer. The lowest possible value of α_j such that there exist a computationally distinguishable distance between the inverse transformation function and the piece wise linear understimater can be selected by following statements:*

i) if $(\underline{x}_j \cdot \underline{x}_i) <= 0$ then $\alpha_j >= \sqrt{\varepsilon}/(\ln(\underline{x}_j/\underline{x}_j))$.

ii) if $(\underline{x}_j \cdot \underline{x}_i) > 0$ then $\alpha_j >= (4 \ln \underline{x}_j/\underline{x}_j - \sqrt{G}/(2 \ln(\underline{x}_j/\underline{x}_j) \cdot (\underline{x}_j \cdot \underline{x}_j))$.

Where $G = 16(\underline{x}_j/\underline{x}_j) \cdot (\underline{x}_j \cdot \underline{x}_j) - 2\sqrt{\varepsilon}(\underline{x}_j \cdot \underline{x}_j)$ and $\varepsilon <= (\ln(\underline{x}_j/\underline{x}_j)/\ln((\underline{x}_j \cdot \underline{x}_j))^2$. After finding the lowest value of α_j, we should approximate the inverse transformation function by piecewise linear approximation. An injective function for modelling piecewise linear function is proposed as following remark:

Remark 2.5. : (Vielma And Nemhauser 2011): An injective function $B : \{1, ..., m\} \longrightarrow \{0.1\}^{\theta},\ \theta = [m]$ where the vectors B and $B(p+1)$ differ in at most one component for all $p \in \{1, 2, ..., m+1\}$ can be constracted.wherer $B(p) = (u_1....u_\theta) \forall uk \in \{0, 1\}, k = 1, ..., \theta$ and $B(0) = B(1)$.
Some terms are introduced below: (FA TSAI 2011) $S^+(k) = \{p|\ \forall B(p)$ and $B(p+1),\ uk = 1, p = 1, ..., m-1\} \cup \{p|\ \forall B(p), uk = 1, p \in \{0, ..., m\}\}$ $S^-(k) = \{p|\ \forall B(p)$ and $B(p+1),\ uk = 0, p = 1, ..., m-1\} \cup \{p|\ \forall B(p), uk = 0, p \in \{0, ..., m\}\}$

To approximate a univariate function with the proposed method by Vielma and Nemhauser, we have the following theorem: Given a univariate function $f(x)$, $a_0 \leqslant x \leqslant a_m$ denote $L(f(x))$ as a piece wise linear function of $f(x)$, where $a_0 < a_1 < ... < a_m$ are $m+1$ break point of $L(f(x)) \cdot L(f(x))$ can be denote as:

$$L(f(x)) = \sum_{p=0}^{m} f(ap)\lambda p \quad x = \sum_{p=0}^{m} ap\lambda p \quad \sum_{p=0}^{m} \lambda p = 1 \tag{2.2}$$

$$\sum_{p \in s+(k)} \lambda p \leqslant u_k \qquad \sum_{p \in s-(k)} \lambda p \leqslant 1 - u_k \quad \forall\ \lambda_p \in R_+,\ \forall\ u_k \in \{0, 1\}$$

Above theorem goes beyond other piecewise linearization formulation, including the SOS2 model without binary variables.
3.non – linear branch-and-bound algorithm:
A node in the branch-and-bound tree is uniquely defined by a set of bounds, (l, u), on the integer variables and corresponds to the $NLP(l, u)$.

$$\begin{aligned} &\min f(x) \\ &s.t\ g(x) \leqslant 0 \\ &\quad x \in X,\ l_i \leqslant x_i \leqslant u_i\ \ \forall\, i \in I\ \ I \subseteq \{1, 2, ..., n\} \end{aligned}$$

branching and pruning rules for branch-and-bound.
Branching. If the solution $x^{'}$ of $(NLP(l, u))$ is feasible but not integer, then we branch on any non-integer variable, say $x_i^{'}$. Branching introduces two new NLP nodes, also referred to as child nodes of $(NLP(l, u))$. In particular, we initialize bounds for two new problems as $(l6-, u^-) := (l, u)$ and then modify the bound corresponding to the branching variable

$$u_i^- := \lfloor x_i^{'} \rfloor \quad l_i^+ := \lceil x_i^{'} \rceil$$

The two new NLP problems are then defined as $NLP(l^-, u^-)$ and $NLP(l^+, u^+)$ In practice, the new problems are stored on a heap H, which is updated with these two new problems.
Pruning rules: The pruning rules for NLP branch-and-bound are based on optimality and feasibility of NLP sub problems. We let U be an upper bound on the optimal value of non-linear programming problem. (initialized as $U = \infty$).

- **Infeasible nodes**. If any node, $(NLP(l, u))$ is infeasible, then any problem in the subtree rooted at this node is also infeasible. Thus, we can prune infeasible nodes.
- Integer feasible nodes. If the solution, $x^{(l,u)}$ of $(NLP(l, u))$ is integral, then we obtain a new incumbent solution if $f(x^{l,u}) < U$, and we set $x^* = x^{l,u}$ and $U = f(x^{l,u})$. Otherwise, we prune the node because its solution is dominated by the upper bound.
- Upper bounds on NLP nodes. If the optimal value of $(NLP(l, u),\ f(x^{l,u}))$ (or in fact a lower bound on the optimal value) is dominated by the upper bound, that is, if $f(x^{(l,u)})$, then we can prune this node because there cannot be any better integer solution in the subtree rooted at $(NLP(l, u))$.

Algorithm 1: Branch-and-bound for MINLP.

Choose a tolerance $\varepsilon > 0$, set $U = \infty$, and initialize the heap of open problems $H = \phi$;
Add $(NLP(-\infty, +\infty))$ to the heap: $H = H \cup NLP(-\infty, +\infty)$.
while $H \neq \phi$ do
Remove a problem $(NLP(l; u))$ from the heap: $H = H - (NLP(l; u))$
Solve $(NLP(l; u))$ and let its solution be $x^{(l,u)}$
if $(NLP(l; u))$ is infeasible then
Node can be pruned because it is infeasible.
else if $f(x^{(l,u)}) > U$ then
Node can be pruned, because it is dominated by upper bound.
else if $x_I^{(l,u)}$ integral then
Update incumbent solution: $x^* = x^{(l,u)}$, $u = f(x^{(l,u)})$
else
Branch On Variable $x_i^{(l,u),l,u,H}$, see **Algorithm 2** .

The complete nonlinear branch-and-bound algorithm is described in Algorithm 1.

The following Proposition establishes the convergence of algorithm:

Proposition 2.6. *Consider solving nonlinear branch-and-bound. Assume that the problem functions f and g are convex and twice continuously differentiable and that X is a bounded polyhedral set. Then it follows that branch-and-bound terminates at an optimal solution after searching a finite number of nodes or with an indication that the non-linear problem has no solution.*

We note that for the convergence analysis of nonlinear branch-and-bound, it requires only to ensure that every node that is pruned is solved to global optimality. The convexity assumptions are convenient sufficient condition that ensures global optimality, but clearly not the only one!

Algorithm 2: Branch on a fractional variable.

Subroutine: $S \longleftarrow$ Branch On Variable $(x_i^{(l,u)}, l, u, H)$
Branch on a fractional $(x_i^{(l,u)}, i \in I$
Set $u_i^- = \lfloor x_i^{(l,u)} \rfloor, l^- = l, l_i^+ = \lfloor x_i^{(l,u)} \rfloor, u^+ = u$
Add $NLP(l^-, u^-)$ and $NLP(l^+, u^+)$ to the heap $H = H\{NLP(l^-, u^-), NLP(l^+, u^+)\}$

3 proposed method

Linear branch and bound method for integrating geometric problems:
Step 1:
Solved the convex geometric problem without considering integer variables. There are two cases:
Case 1: If the variables get the integer value, the problem is solved.
Case 2: If the variables not integer, we go to the second step.
Step 2:
2. let $Z_I = \infty$
3. Branching: consider the variable x_I that not integer and put $x^* = x_I$. We have
$[x_I] < x_I < [x_I + 1]$. The problem is divided into two sub-problems $p_1 \cdot p_2$.
Problem p_1: The problem p_2 is the original problem that added the condition $x_I + t = [x_I]$ to the problem and solved the problem again.

Table 1: The solutions of the problem without consider the integer variables by lingo

Variable	Value	Reduced Cost
U1	1.000000	2.800357
U2	1.000000	5.524081
U3	1.000000	23.75966
U4	0.000000	-5.504592
U5	1.000000	78.77829
U6	0.000000	-26.25066
X1	1.844603	0.000000
X2	1.204137	0.000000
X3	0.4748030	0.000000
Y2	0.9606432	0.000000
Y3	1.274361	0.000000
Y1	0.8603850	0.000000
LANDA0	0.000000	0.000000
LANDA1	0.3107943	0.000000
LANDA2	0.6892057	0.000000
LANDA3	0.000000	0.000000
LANDA4	0.000000	1.052905
LANDA5	0.5917260	0.000000
LANDA6	0.4082740	0.000000
LANDA7	0.000000	0.000000
LANDA8	0.000000	0.000000
LANDA9	0.000000	0.000000
LANDA1	0 0.4503939	0.000000
LANDA11	0.5496061	0.000000
LANDA12	0.000000	0.000000
LANDA13	0.000000	20.50109
LANDA14	0.000000	0.000000

Local optimal solution found

Objective value	91.00418
Objective bound	91.00418
Infeasibilities	0.2220446E-15
Extended solver steps	48
Total solver iterations	1393

Problem p_2: The problem p_2 is the original problem that added the condition $x_I = t+[x_I+1]$ to the problem and solved the problem again.

4. We solve the problem p_1 and p_2 and replace Z_I with whichever one had a better answer.
5. Sub-branches have one of the following three states:

i. All variables have the integer value.

ii. The sub-problem is infeasible.

iii. The value of the function becomes worse than Z_I

Table 2: The integer solution to the problem with the nonlinear branch and bound algorithm

Variable	Value	Reduced Cost
U1	1.000000	2.921953
U2	1.000000	5.763946
U3	1.000000	0.000000
U4	0.000000	0.000000
U5	1.000000	80.50255
U6	0.000000	-26.82522
X1	2.000000	1.251907
X2	1.000000	-23.13104
X3	0.4949785	0.000000
Y2	1.000000	0.000000
Y3	1.258137	0.000000
Y1	0.8408964	0.000000
LANDA0	0.000000	0.000000
LANDA1	0.000000	0.000000
LANDA2	1.000000	0.000000
LANDA3	0.000000	0.000000
LANDA4	0.000000	1.098623
LANDA5	1.000000	0.000000
LANDA6	0.000000	1.690446
LANDA7	0.000000	0.000000
LANDA8	0.000000	0.000000
LANDA9	0.000000	0.000000
LANDA10	0.4100429	0.000000
LANDA11	0.5899571	0.000000
LANDA12	0.000000	0.000000
LANDA13	0.000000	0.000000
LANDA14	0.000000	0.000000

Local optimal solution found

Objective value:	92.33431
Objective bound:	92.33431
Infeasibilities:	0.1110223E-15
Extended solver steps:	53
Total solver iterations:	2647

To present the proposed method, we consider the problem of designing a gravel box. We want to find a box with the integer dimensions that has the lowest cost of transportation. The problem is as follows:

$$\min 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_2x_3 + 20x_1x_3 + 10x_1x_2$$

$$\begin{aligned} Min\ & 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_2x_3 + 20x_1x_3 + 10x_1x_2 \\ & 1 \leqslant x_1 \leqslant 3.1 \leqslant x_2 \leqslant 3.0.2 \leqslant x_3 \leqslant 2 \\ & x_1 \cdot x_2 \in Z \end{aligned}$$

After solving the following problem by dual method we have:

Table 3: The GRAVEL box design problem

Variable	Value	Reduced Cost
U1	0.000000	-2.921953
U2	1.000000	8.685900
U3	1.000000	0.000000
U4	0.000000	0.000000
U5	1.000000	80.50255
U6	0.000000	-26.82522
X1	2.000000	0.000000
X2	1.000000	-18.53904
X3	0.4949785	0.000000
Y2	1.000000	0.000000
Y3	1.258137	0.000000
Y1	0.8408964	0.000000
LANDA0	0.000000	0.000000
LANDA1	0.000000	0.000000
LANDA2	1.000000	0.000000
LANDA3	0.000000	0.000000
LANDA4	0.000000	6.942530
LANDA5	1.000000	0.000000
LANDA6	0.000000	1.690446
LANDA7	0.000000	0.000000
LANDA8	0.000000	0.000000
LANDA9	0.000000	0.000000
LANDA10	0.4100429	0.000000
LANDA11	0.5899571	0.000000
LANDA12	0.000000	0.000000
LANDA13	0.000000	0.000000
LANDA14	0.000000	0.000000
T1	0.000000	0.000000
T2	0.000000	0.000000

Local optimal solution found

Objective value:	92.33431
Objective bound:	92.33431
Infeasibilities:	0.1110223E-15
Extended solver steps:	18
Total solver iterations:	1491

$$x_1^* = 2m \;\; x_2^* = 1m \;\; x_3^* = 0.5m$$

The volume of the box is $1m^3$, the number of $trips = 400$, and the minimum total $cost = \$\ 100$.
Consider the objective function of the gravel box design problem, we want to solve this problem again so that the dimensions of the box are integers. This function is non-convex. Using the NPT method, we make the problem convex and then piecewise linear approximation. The convexity of the following problem with

Table 4: Comparing the solutions with the proposed method

Total solver iterations	Extended solver steps	Objective function	
		100	Dual solution
592	5	100	The solution without convexification
1393	48	91.00418	The solution of the problem after convexification and piecewise linear approximation before integration
2647	53	92.32431	The solution to the problem after integration
1491	18	92.32431	Proposed method

the assumption $\beta 1{=}\beta 2{=}\beta 3{=}0.25$ and the four break points of convexity of the following problem is:

$$40x_1^{-1}x_2^{-1}x_3^{-1} + 40y_2^{-\frac{1}{0.25}}y_3^{\frac{1}{0.25}} + 20y_1^{-\frac{1}{0.25}}y_3^{-\frac{1}{0.25}} + 10y_1^{-\frac{1}{0.25}}y_2^{-\frac{1}{0.25}}$$

$$s.t \;\; y_i = l(x_i^{-\beta_i}) = \sum_{p=0}^{m} f(a_p)\lambda_p \;\; i = 1 \cdot 2 \cdot 3$$

$$x_i = \sum_{p=0}^{m}(a_p)\lambda_p \qquad i = 1 \cdot 2 \cdot 3$$

$$\sum_{p=0}^{m}\lambda_p = 1 \qquad \sum_{p\in s^+(k)}\lambda_p \leqslant u_k \qquad \sum_{p\in s^-(k)}\lambda_p \leqslant 1 - u_k$$

The solutions of the problem without consider the integer variables by lingo is as Table 1.

Now, after considering the integer variables and finding the integer solution to the problem with the nonlinear branch and bound algorithm, we have the results as given in Table 2.

By implying the linear branch and bound algorithm after convexification and piecewise linear approximation, we have the GRAVEL box design problem provided in Table 3:

By comparing the solutions with the proposed method, we have the results as given in Table 4:

4 Conclusion

By using Linear branch and bound method for integrating geometric problems, after convexification and piecewise linear approximation, as well as using the linear branch and bound algorithm for integrating, we came to the conclusion that with the proposed method, we reached the minimum cost for the design of the GRAVEL box, which is less than using the non- linear branch and bound algorithm. This result guarantees the efficiency of our approach.

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A New Distorted Distribution Family based power of the logistic function

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Abstract

In this paper, we introduce a flexible and general family of probability distributions by employing the power of the logistic function as a generating variable within the distortion function method framework. The theoretical properties of the proposed family are thoroughly investigated, including the cumulative distribution and density functions, hazard rate function, reliability function, and the total time on test (TTT) transform. Closed-form expressions for moments, mean, variance, and measures of dispersion are derived, and the conditions for monotonicity and unimodality of the hazard rate function are analytically studied. Parameter estimation is carried out via the maximum likelihood method (MLE). Furthermore, by selecting different baseline distributions, several unit-generated special cases of the proposed family are constructed. Their performance is evaluated and compared using goodness-of-fit indices and information criteria (AIC, BIC, etc.). The results demonstrate that the proposed framework offers high flexibility for modeling diverse reliability data and serves as a powerful tool for developing new distributional classes and enhancing reliability analysis.

Keywords:

1 Introduction

The development of new probability distributions plays a pivotal role in enhancing the modeling of lifetime data and reliability analysis across diverse fields such as engineering, medicine, and risk management. Classical distributions, while foundational, often lack the flexibility required to capture complex patterns observed in practice, particularly in the shape and behavior of hazard rate functions. This has motivated researchers to construct broader classes of distributions using methods such as compounding, parameter induction, and transformation techniques. Among these approaches, transformation-based frameworks such as the T-X family have gained prominence for their ability to generalize baseline distributions into more flexible models with richer structures (Alzaatreh et al., 2013). Recent advances include the Pi-Power Logistic-G family, which demonstrates remarkable adaptability in generating monotonic, J-shaped, or reverse-J hazard rate functions (Afify et al., 2024), as well as the Alpha Power Transformed Logistic distribution, which further illustrates the potential of logistic-based power transformations (Mahmoudi et al., 2021). Additionally, Arslan (2023) introduced a new family of unit-distributions, highlighting the growing importance of unit-based transformations for constructing versatile distributional forms and providing practical applications in reliability modeling. Within this growing body of work, the present study introduces

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a novel family of distributions by employing the power of the logistic function as a generating variable within the transformation method framework. The key properties of this new family—including the cumulative distribution function, probability density function, reliability and hazard rate functions, and the total time on test (TTT) transform are analytically investigated. Closed-form expressions for moments are derived, and parameters are estimated via the maximum likelihood method (MLE). Furthermore, by considering different baseline distributions, several unit-generated special cases are constructed and compared using standard information criteria such as AIC and BIC. The remainder of this paper is organized as follows. In the next section, we introduce the proposed transformed model and study its main statistical properties. In the final section, we analyze a real dataset to demonstrate that the proposed model not only possesses the necessary reliability in comparison with some related models but also exhibits superior performance.

2 The Proposed Model and Its Properties

Let X be a random variable with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Considering the conversion $Z = [1 + \exp(-X)]^{-\frac{1}{\alpha}}$ we have

$$F_Z(z) = F_X(-\ln(z^{-\alpha} - 1)), \tag{2.1}$$

$$f_Z(z) = \alpha[z(1 - z^{\alpha})]^{-1} f_X(-\ln(z^{-\alpha} - 1)), \quad z \in (0, 1), \quad \alpha > 0. \tag{2.2}$$

Note that power of the logistic function is used for the Z-conversion. In the following, a new family of distribution called distorted distribution is introduced. For this purpose, suppose $g(t) = e^{-\theta(1-t)} - e^{-\theta}(1-t)$ be a distortion function. Therefore, according to the distortion transform theorem, we have the $F_D(x) = g(K(x)) = e^{-\theta\bar{K}(x)} - e^{-\theta}\bar{K}(x)$ be a family of distribution functios that named distorted distribution function, where in $K(x)$ is a baseline distribution function. Esfahani et al (2021) have studied the properties and application of $g(.)$ and $F_D(.)$ in detail.

Now for $K(x) = F_Z(x)$, we have a new family of distribution that named family of distorted distribution and denoted by $F_D(x) = g(F_Z(x)) = e^{-\theta\bar{F}_Z(x)} - e^{-\theta}\bar{F}_Z(x)$. So it can be said if X be a random variable with distribution function $F(x)$, then considering Z-conversion we have a new distorted distribution by form

$$F_D(x) = e^{-\theta\bar{F}_X(-\ln(z^{-\alpha}-1))} - e^{-\theta}\bar{F}_X(-\ln(z^{-\alpha} - 1)). \tag{2.3}$$

By substituting different baseline distributions into the above formulation, new transformed distributions can be introduced and their statistical properties and applications can be studied. For this purpose, we consider the exponential baseline distribution with parameter $\lambda > 0$ and propose a new distorted distribution and denoted by $F_{ED}(.)$.

$$F_{ED}(x) = e^{-\theta(z^{-\alpha}-1)^{-\lambda}} - e^{-\theta}(z^{-\alpha} - 1)^{-\lambda}. \tag{2.4}$$

It is clear that

$$f_{ED}(x) = \lambda(z^{-\alpha} - 1)^{-\lambda-1}(-\alpha z^{-\alpha-1})[\theta e^{-\theta(z^{-\alpha}-1)^{-\lambda}} - e^{-\theta}]. \tag{2.5}$$

The hazard rate function of $F_{ED}(.)$ is

$$h_{ED}(x) = \frac{\lambda(z^{-\alpha} - 1)^{-\lambda-1}(-\alpha z^{-\alpha-1})[\theta e^{-\theta(z^{-\alpha}-1)^{-\lambda}} - e^{-\theta}]}{1 - e^{-\theta(z^{-\alpha}-1)^{-\lambda}} + e^{-\theta}(z^{-\alpha} - 1)^{-\lambda}}. \tag{2.6}$$

Examining the elasticity of the hazard rate function:

For $(\alpha, \lambda, \theta) = (0.5, 1, 0.5)$ The function is gently increasing.

For $(\alpha, \lambda, \theta) = (1, 1, 1)$ The hazard rate increases rapidly at first and then remains almost constant (almost flat).

For $(\alpha, \lambda, \theta) = (1.5, 0.5, 2)$ The hazard rate has a U-shape, first decreasing and then increasing.

For $(\alpha, \lambda, \theta) = (2, 2, 1.5)$ The function is continuously increasing and unimodal.

Overall conclusion: The hazard rate function is very flexible and can exhibit ascending, descending, bathtub-shaped, and unimodal behaviors depending on the choice of parameters. This flexibility indicates that the model can model well different reliability and survival time data with diverse hazard rate behaviors.

2.1 Maximum likelihood estimator

Next, we briefly examine the maximum likelihood method for estimating the parameters of the model.

The likelihood function is a function of the parameter θ and represents the probability of observing the given data:

$$L(\Theta) = \Pi_{i=1}^{n} f(x_i, \Theta).$$

For convenience in calculations, we often use the log-likelihood function:

$$\ell(\Theta) = \log L(\Theta) = \sum_{i=1}^{n} \log f(x_i, \Theta),$$

where in $\Theta = (\theta_1, ..., \theta_k)$.

To find $\hat{\theta}$ (MLE of Θ), we take the derivative of the log-likelihood with respect to Θ and set it to zero:

$$\frac{\partial \ell(\Theta)}{\partial \theta_i} = 0$$

Solving this equation gives the parameter value that maximizes the likelihood function. Hence MLE of parameters of $F_{ED}(x)$ from solving the following equations we get

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = 0, \qquad \frac{\partial \ell(\Theta)}{\partial \lambda} = 0, \qquad \frac{\partial \ell(\Theta)}{\partial \theta} = 0.$$

3 Application

To demonstrate the practical utility of the proposed distribution, we analyze the well-known carbon fiber stress rupture dataset reported by Padgett and Spurrier (1990). This dataset, which has frequently been employed in reliability and lifetime studies, provides an appropriate benchmark for assessing the performance of competing models. In this study, the proposed transformed exponential distribution is fitted to the data and compared with classical alternatives such as the exponential and Weibull distributions. Parameter estimation is carried out via the maximum likelihood method, and model comparisons are based on standard goodness-of-fit measures. The empirical findings reveal that the proposed model achieves a superior fit, thereby highlighting its flexibility and applicability in modeling lifetime data.

Tables 1 and 2 summarize the main empirical findings of the analysis. Specifically, Table 1 reports the maximum likelihood estimates of the model parameters for the exponential, Weibull, and the proposed transformed exponential distributions. Table 2 presents the corresponding model selection criteria, including the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Consistent Akaike Information Criterion (CAIC), and the Hannan–Quinn Information Criterion (HQIC). These results provide a comprehensive basis for evaluating and comparing the goodness-of-fit of the competing models.

The comparative analysis of the exponential, Weibull, and the proposed transformed ED model clearly demonstrates the advantages of the ED model in terms of both parameter estimation and overall model fit. As shown in Table 1, the standard errors of the estimated parameters under the exponential and Weibull

Table 1: Estandard Error of Estimator

Model	$SE(\hat{\alpha})$	$SE(\hat{\theta})$	$SE(\hat{\lambda})$
$Exponential$	$---$	$---$	112.53
$Weibull$	104.15	$---$	107.97
ED	105.86	102.31	109.79

Table 2: Model evaluation criteria

Model	AIC	$CAIC$	BIC	$HQIC$
$Exponential$	105.47	107.12	110.24	106.62
$Weibull$	104.15	104.41	107.97	105.61
ED	98.26	99.45	102.17	101.04

models are relatively large, reflecting instability and higher variability in parameter estimation. In contrast, the ED model provides smaller and more balanced standard errors across all estimated parameters, which suggests that the estimators derived from the ED model are more efficient and reliable. The stability of these estimates is particularly important in reliability and survival analysis, where precise inference about the underlying lifetime distribution directly affects decision-making and system evaluation. Moreover, the evaluation criteria reported in Table 2 (AIC, CAIC, BIC, and HQIC) further confirm the superiority of the ED model. It is well established that lower values of these information criteria indicate a better trade-off between goodness-of-fit and model complexity. The ED model consistently achieves the minimum values across all four criteria, whereas both the exponential and Weibull models yield higher values. This consistent improvement across multiple criteria highlights the robustness of the ED model and its capacity to capture the underlying structure of the data more accurately. Taken together, these findings provide strong evidence that the proposed ED model offers significant improvements over traditional models such as the exponential and Weibull distributions. By yielding more precise parameter estimates and achieving better performance in terms of information criteria, the ED model can be considered a more suitable and reliable choice for modeling lifetime data. Consequently, it has the potential to serve as a valuable alternative in practical applications of reliability analysis, survival studies, and related fields where accurate modeling of lifetime behavior is essential.

4 Conclusion

In this study, we introduced a new transformed ED model and compared its performance with the exponential and Weibull models. The results of the analysis showed that the ED model provides smaller and more balanced standard errors for parameter estimates, indicating greater stability and precision. Moreover, all information criteria achieved their minimum values under the ED model, confirming its superior goodness-of-fit. Therefore, the proposed ED model can be regarded as a more reliable and appropriate choice for lifetime data analysis, offering a promising alternative to classical models in reliability and survival studies.

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Functional Linear Regression for Investigating the Influence of Social Media Usage on Mental Health Trends

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Abstract

This study employs functional linear regression analysis to explore the relationship between social media usage and mental health trends. We aim to highlight how varying levels of social media engagement correlate with different mental health outcomes, particularly focusing on anxiety and depression among various demographics. The findings reveal significant associations, suggesting that social media can serve as both a tool for support and a source of distress, especially during critical life events.

Keywords: functional linear regression. social media usage, mental health trends, longitudinal data analysis

1 Introduction

In the digital age, social media platforms have become integral to everyday life, fundamentally altering the landscape of interpersonal communication. While these platforms foster connection and community, substantial evidence suggests that they can negatively impact mental health. This paper aims to investigate the multifaceted relationship between social media usage and mental health trends, specifically examining anxiety and depression through the lens of functional linear regression.

The relevance of this study is underscored by rising concerns regarding the mental health crisis among young adults, particularly as they navigate both social media and traditional stressors such as academic pressures. In this context, this research seeks to clarify how prolonged engagement with social media may influence mental health outcomes, contributing to a growing body of literature on the subject.

Existing research on the relationship between social media usage and mental health has yielded mixed results. Some studies suggest that high levels of social media engagement correlate with increased feelings of loneliness and depression. For instance, [2] found a significant association between excessive social media use and heightened anxiety symptoms among adolescents. They report that engagement with negative content and peer comparison drives these outcomes.

Conversely, other studies highlight the potential for social media to positively influence mental health by providing platforms for social support and information sharing during crises. [2] argue that, particularly during the COVID-19 pandemic, social media served as a vital resource for emotional support, enhancing resilience among users.

Functional linear regression has emerged as a suitable method for elucidating complex relationships, allowing researchers to analyze how social media usage impacts mental health over time. It provides flexibility in modeling time-varying effects and offers insights into individual trajectories [2, 2].

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Table 1: Case Study Results: Average Anxiety Scores Over Time

Week	Average Social Media Usage (Hours/Day)	Average GAD-7 Score	Change from Baseline
0	3	10.2	0
8	5	12.8	+2.6
16	7	15.4	+5.2

2 Methodology

To conduct this study, we utilized a dataset collected through an online survey distributed to participants aged 18-30. The survey assessed social media usage (measured in hours per day) and mental health indicators, including standardized scales for anxiety (GAD-7) and depression (PHQ-9), over a one-year period. The final sample consisted of 500 respondents, balanced across different demographics to ensure representativeness.

This analysis uses functional linear regression to model the relationship between social media usage and mental health outcomes. The model can be described mathematically as follows:

$$Y(t) = \beta_0(t) + \int_0^T \beta(s,t)X(s)ds + \epsilon(t)$$

where:
- $Y(t)$ represents the mental health outcomes at time t,
- $X(t)$ represents the function for social media usage over time,
- $\beta_0(t)$ is the intercept function,
- $\beta(s,t)$ is the coefficient function that captures the impact of social media usage at different time points,
- $\epsilon(t)$ is the error term encompassing unobserved influences on mental health.

The model allows for the integration of continuous social media usage patterns, providing a nuanced understanding of its effects.

Data analysis was performed using R, leveraging packages `fda` for functional data analysis and `lm` for regression modelling. The analysis involved fitting the functional linear regression model to the dataset, followed by diagnostic checks for model assumptions.

3 Case Study

To illustrate the practical application of our findings, we present a case study involving a cohort of 100 college students participating in a semester-long well-being program. Participants reported their daily social media usage and were assessed for anxiety and depression bi-weekly using the GAD-7 and PHQ-9 scales.

Throughout the semester, average daily social media usage was tracked, yielding the following statistics:
- **Baseline Anxiety (Week 0)**: Average GAD-7 score of 10.2 (SD = 4.3).
- **Mid-Semester Anxiety (Week 8)**: Average GAD-7 score of 12.8 (SD = 3.9).
- **Final Anxiety (Week 16)**: Average GAD-7 score of 15.4 (SD = 4.7).

Table 1 summarizes the average anxiety scores across the semester, categorized by social media usage. The results indicate a progressive increase in anxiety scores correlating with an increase in social media usage throughout the semester. The change from baseline indicates an increase of 5.2 points by the final assessment, confirming that heightened social media engagement may exacerbate anxiety levels among college students.

Qualitative feedback from participants suggested that while many used social media for networking and support, the pressure to maintain a curated online presence contributed significantly to their anxiety. The

fitted functional linear regression model is:

$$Y(t) = 10.2 + \int_0^T 0.065\, X(s)\, ds + \epsilon(t),$$

where, in this case study, $T = 16$ weeks.

4 Discussion

The findings from this study highlight the critical role of social media in shaping mental health outcomes, particularly among young adults. The functional linear regression model demonstrated significant associations between social media usage and mental health trends, emphasizing the need for nuanced policy and intervention strategies. These results are particularly relevant for mental health professionals, educators, and policymakers. Efforts should be directed toward promoting healthy social media use and developing tools to help individuals navigate online environments more effectively.

Future research should consider additional factors such as the type of social media platform used, the nature of interactions (positive vs. negative), and individual differences in coping strategies that may influence these relationships over time.

5 Conclusion

This study provides valuable insights into the intricate relationship between social media usage and mental health trends. By employing functional linear regression, we have elucidated how varying levels of engagement impact mental health, particularly concerning anxiety and depression among young adults. The dual nature of social media—acting as both a support system and a source of distress-necessitates a comprehensive understanding of its effects.

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Analyzing the Impact of Temperature on Energy Consumption in Heating and Cooling Systems: A Functional Linear Regression Model

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Abstract

This study investigates the relationship between temperature and energy consumption in heating and cooling systems using a functional linear regression model. By analyzing data over specified time periods, we aim to understand how variations in temperature affect energy use. Preliminary results indicate a significant correlation between temperature fluctuations and energy consumption patterns, highlighting the implications for energy management strategies.

Keywords: functional linear regression, heating and cooling systems, energy management strategies.

1 Introduction

In an era of climate change and rising energy costs, understanding the dynamics of energy consumption is critical for optimizing heating and cooling systems. The demand for energy in these systems is highly dependent on external temperature conditions. This study aims to model the relationship between temperature as a functional variable and energy consumption as a response variable.

Recent studies highlight the correlation between temperature variations and energy consumption in residential and commercial buildings. For example, [1] emphasized the importance of temperature degree days in estimating heating and cooling needs. Previous work by [4] outlined how temperature patterns influence energy demand, showing an increase in heating requirements with lower temperatures. Similarly, [2] found significant seasonal patterns in energy consumption tied to temperature changes, arguing for the necessity of advanced modeling techniques.

Moreover, [3] proposed the use of functional regression models to account for temperature's continuous nature and variability over time. Their findings suggest that traditional linear regression methods overlook crucial information regarding the functional relationship between temperature and energy use.

2 Methodology

The primary objective of this study is to analyze energy consumption $Y(t)$ in relation to temperature $X(t)$ over time, using the following functional linear regression model:

$$Y(t) = \beta_0(t) + \int_0^T \beta(s,t)X(s)ds + \epsilon(t),$$

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where:

- $Y(t)$ is the energy consumption at time t,
- $X(t)$ is the temperature as a function of time,
- $\beta_0(t)$ is the intercept function of the model,
- $\beta(s,t)$ represents the coefficients varying with temperature,
- $\epsilon(t)$ is the error term.

Data were collected from smart meters installed in various residential buildings over a one-year period. Temperature data were obtained from local weather stations, recording hourly temperatures.

3 Real-Life Example

A case study was conducted in a residential complex located in Shiraz. The table below represents energy consumption data for heating during winter months recorded over several weeks, along with the average daily temperature.

Week	**Average Temperature (°C)**	**Energy Consumption (kWh)**
1	5	150
2	3	180
3	0	220
4	-2	250
5	1	200

Table 1: Energy consumption versus average temperature data from a residential complex during winter months.

The consumption data illustrates that as the average daily temperature decreases, the energy consumption for heating increases significantly. For instance, when the average temperature fell to 0 °C, energy consumption peaked at 220 kWh, while at 5 °C, the consumption was only 150 kWh.

From the collected data, we can observe that every degree drop in temperature correlates with an increase in energy consumption for heating purposes. This presents a critical insight for energy management, indicating that maintaining indoor comfort levels during colder weeks can significantly increase energy costs.

The fitted functional linear regression model is:

$$Y(t) = 220 - 14 \int_0^T X(s)\, ds + \epsilon(t),$$

where $\epsilon(t)$ denotes the residual error. In this fitted model, T specifically refers to the 5-week observation window of the case study. This formulation shows that energy consumption rises as temperature decreases, with approximately 14 kWh additional consumption for each degree drop in average temperature. At 5°C (Week 1), the model predicts 150 kWh, while at 0°C (Week 3), it predicts 220 kWh, consistent with the observed data.

4 Discussion

The functional linear regression model showed that energy consumption responds strongly to temperature fluctuations, validating its use in predicting energy usage based on varying temperature profiles.

According to previous studies, the increase in energy consumption is consistent with findings by [4], who reported similar patterns in their research on heating requirements. Moreover, [1] found that each degree day correlates positively with energy consumption, supporting the results found in this study.

The findings emphasize the importance of designing energy-efficient systems that can adapt to varying temperature conditions. As highlighted by [3], a robust understanding of this relationship can lead to improved energy forecasts and potentially lower energy costs for consumers.

5 Conclusion

This research demonstrates the significant impact of temperature on energy consumption in heating and cooling systems. The functional linear regression model provides a robust framework for analyzing this relationship, allowing for a more detailed understanding of how temperature fluctuations affect energy use over time.

The findings underscore the need for building managers and policymakers to consider temperature patterns when designing energy-efficient systems and strategies. Future research should focus on integrating additional variables such as humidity, occupancy rates, and building characteristics to enhance the predictive power of the model. Moreover, implementing smart metering technology could yield more precise data, improving the accuracy of energy consumption forecasts in relation to temperature variations.

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Global Stability Analysis of Viral Infection Model with Cellular and Humoral Immunity

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Abstract

This paper develops a model to study the dynamics of biological systems. The main result of the stability analysis of the equilibrium points is obtained using the Lyapunov function and LaSalle's invariance principle, which is carried out by determining the equilibrium points and the basic reproduction number. Additionally, a numerical simulation of the system will be presented in the final Section.
Keywords: Cellular Immunity, Humoral Immunity, Global Stability.

1 Introduction

Mathematical models are used to investigate how the immune system controls viral infections. The immune system consists of two main components: cellular immunity (involving Cytotoxic T lymphocytes or CTLs) and humoral immunity (involving the production of antibodies by B cells). Both types of immunity play important roles in eliminating or controlling viruses. Some viruses are more affected by cellular immunity, while others are more influenced by humoral immunity. There are also studies and models that examine both types of immune responses at the same time. In [1, 2, 3, 4], some viral infection model have been studied.

2 Model Formulation

In this work, we consider the following system:

$$\begin{aligned} \dot{x}(t) &= \Lambda - \alpha xv - \beta xy + \delta y - \mu_0 x, \\ \dot{y}(t) &= \alpha xv + \beta xy - \delta y - \mu_1 x - ayz, \\ \dot{v}(t) &= \gamma y - \mu_2 v - bvA, \\ \dot{z}(t) &= pyz - \mu_3 z, \\ \dot{A}(t) &= qvA - \mu_4 A, \end{aligned} \tag{2.1}$$

where $x(t)$, $y(t)$, $v(t)$, $z(t)$, $A(t)$ It shows the density of susceptible cells, infected cells, virus, Cytotoxic T-cells, and B lymphocyte cells, respectively. The system parameters, rate of recruitment (Λ), rate of infection

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transmission (α, β), recovery rate of infected cells (δ), rate of apoptosis $(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$, Z-cell induced death rate of infected cells (a), rate of virus production by infected cells (γ), A-cell-induced death rate of virus cells (b), Z-cell activation rate (p), A-cell activation rate (q).
The functions $f(x, v) = xv$ and $g(x, y) = xy$ describing the virus to cell and the cell to cell transmission, respectively. Furthermore, the function $f(x, v)$ and $g(x, y)$ are assumed to satisfy the following properties:

(I1) $f(0, v) = 0$, for all $v \geq 0$ and $g(0, y) = 0$, for all $y \geq 0$,
(I2) $\frac{\partial f}{\partial x}(x, v) > 0$ and $\frac{\partial g}{\partial x}(x, y) > 0$, for all $x > 0$, $y \geq 0$, $v \geq 0$,
(I3) $\frac{\partial f}{\partial v}(x, v) \leq 0$ and $\frac{\partial g}{\partial y}(x, y) \leq 0$, for all $x \geq 0$, $y \geq 0$, $v \geq 0$.

This paper is organized as follows. In Section 3, the definitions of five threshold numbers and the form of the equilibrium points of the system are presented; the global stability of the equilibrium points is also analyzed in this Section. Finally, a numerical simulation of the system is given in Section 4.

Proposition 2.1. *All solutions of system (1.1) with non-negative initial conditions exist for all $t > 0$ and remain bounded and non-negative.*

3 Global Stability

In this Section, the global stability analysis of model (1.1) will be shown. System (1.1) has five equilibrium points in the following form:

1. $\mathcal{E}_0 = (x_0, 0, 0, 0, 0)$ (Infection-free equilibrium),
2. $\mathcal{E}_1 = (x_1, y_1, v_1, 0, 0)$ (Immune-free equilibrium),
3. $\mathcal{E}_2 = (x_2, y_2, v_2, 0, A_2)$ (CTL immune-free equilibrium),
4. $\mathcal{E}_3 = (x_3, y_3, v_3, z_3, 0)$ (Humoral immune-free equilibrium),
5. $\mathcal{E}_4 = (x_4, y_4, v_4, z_4, A_4)$ (Chronic equilibrium).

As in [1], the basic reproduction number $(\mathcal{R}_0)$ of system (1.1) is as follows:

$$\mathcal{R}_0 = \frac{\alpha x_0 \gamma}{\mu_2(\delta + \mu_1)} + \frac{\beta x_0}{(\delta + \mu_1)}.$$

In addition to $\mathcal{R}_0$, we need the another immune response reproduction rates. As in [1], these threshold numbers have the following form:

$$\mathcal{R}_{CTL} = \frac{py_1}{\mu_3}, \quad \mathcal{R}_{Hum} = \frac{qv_1}{\mu_4}, \quad \mathcal{R}_{CTLC} = \frac{py_3}{\mu_3}, \quad \mathcal{R}_{HumC} = \frac{qv_2}{\mu_4}.$$

Theorem 3.1. *If $\mathcal{R}_0 \leq 1$, then Infection-free equilibrium $\mathcal{E}_0$ is globally asymptotically stable.*

Theorem 3.2. *If $\mathcal{R}_0 > 1$, $\mathcal{R}_{Hum} \leq 1$ and $\mathcal{R}_{CTL} \leq 1$, then the Immune-free equilibrium $\mathcal{E}_1$ is globally asymptotically stable.*

Theorem 3.3. *If $\mathcal{R}_{Hum} > 1$, $\mathcal{R}_{CTLC} \leq 1$ and $\mathcal{R}_{HumC} > 1$, then CTL-immune-free equilibrium $\mathcal{E}_2$ is globally asymptotically stable.*

Theorem 3.4. *If $\mathcal{R}_{CTL} > 1$, $\mathcal{R}_{HumC} \leq 1$ and $\mathcal{R}_{CTLC} > 1$, then Humoral immune-free equilibrium $\mathcal{E}_3$ is globally asymptotically stable.*

Theorem 3.5. *If $\mathcal{R}_{HumC} > 1$ and $\mathcal{R}_{CTLC} > 1$, then Chronic equilibrium $\mathcal{E}_4$ is globally asymptotically stable.*

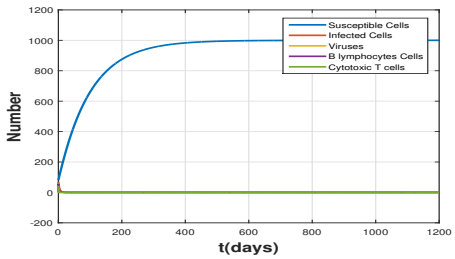


Figure 1: Global stability of Infection-free equilibrium $\mathcal{E}_0$ with $\alpha = 0.0002$, $\beta = 0.0003$, $p = 0.04$ and $q = 0.03$ ($\mathcal{R}_0 = 0.49$).

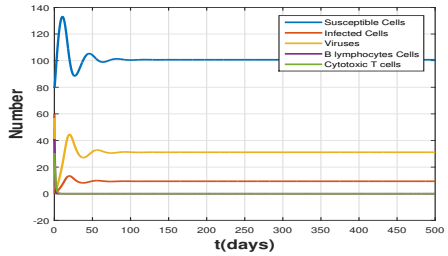


Figure 2: Global Stability of Immune-free equilibrium $\mathcal{E}_1$ with $\alpha = 0.002$, $\beta = 0.003$, $p = 0.04$ and $q = 0.03$ ($\mathcal{R}_0 = 4.94$, $\mathcal{R}_{CTL} = 0.37$ and $\mathcal{R}_{Hum} = 0.62$).

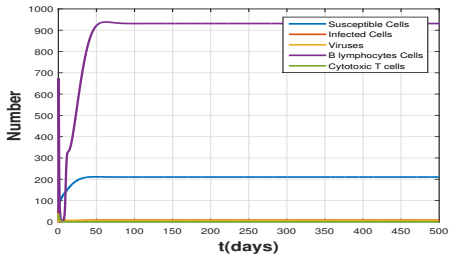


Figure 3: Global stability of CTL immune-free equilibrium $\mathcal{E}_2$ with $\alpha = 0.002$, $\beta = 0.003$, $p = 0.3$ and $q = 0.08$ ($\mathcal{R}_0 = 4.94$, $\mathcal{R}_{CTLC} = 0.86$, $\mathcal{R}_{HumC} = 1.88$ and $\mathcal{R}_{Hum} = 4.67$).

4 Numerical Simulation

In this Section, to illustrate the theoretical results, applying MATLAB with ODE45 method, some numerical simulation of system (1.1) will be presented. For simulation, we consider a set of parameters: $\Lambda = 10$, $\mu_0 = 0.01$, $\mu_1 = 0.8$, $\mu_2 = 3$, $\mu_3 = 2$, $\mu_4 = 0.755$, $\delta = 0.01$, $b = 0.01$, $a = 0.9$, $\gamma = 10$.

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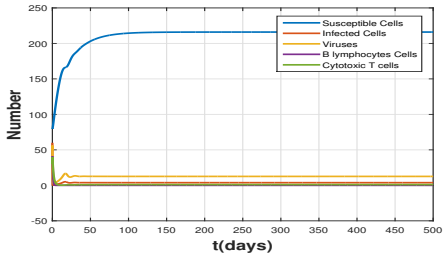


Figure 4: Global stability of Humoral immune-free equilibrium $\mathcal{E}_3$ with $\alpha = 0.002$, $\beta = 0.003$, $p = 0.04$ and $q = 0.02$ ($\mathcal{R}_0 = 4.94$, $\mathcal{R}_{CTL} = 2.47$, $\mathcal{R}_{CTLC} = 2.17$ and $\mathcal{R}_{HumC} = 0.33$).

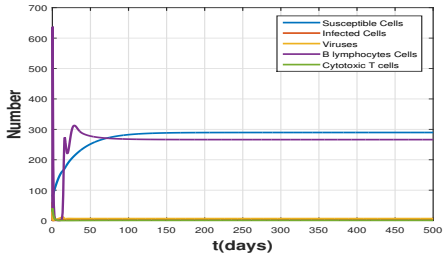


Figure 5: Global stability of Chronic equilibrium $\mathcal{E}_4$ with $\alpha = 0.002$, $\beta = 0.003$, $p = 0.3$ and $q = 0.2$ ($\mathcal{R}_0 = 4.94$, $\mathcal{R}_{CTLC} = 2.17$ and $\mathcal{R}_{HumC} = 1.88$).

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A new algorithm for multiple-sets split equilibrium problem

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Abstract

In this paper, we present parallel linesearch algorithm for finding the solution of the multiple-sets split equilibrium problem in p-uniformly convex and uniformly smooth Banach spaces. To prove the strong convergence of the generated iterates by our algorithms, we do not need any auxiliary equilibrium problem.To prove our method and compare it with other methos, we present some numerical example and solve it using the FMINNCON optimization toolbox in MATLAB R2015a.
Keywords: Bregman projection, Multiple-sets, equilibrium problem, Parallel linesearch

1 Introduction

Suppose that E_1 and E_2 are real Banach spaces and $C_i, (i = 1, 2, ..., k)$ and $Q_j, (j = 1, 2, ..., s)$ are nonempty, convex and closed subsets of E_1 and E_2 respectively, which $C = \bigcap_{i=1}^{k} C_i$ and $Q = \bigcap_{j=1}^{s} Q_j$, also assume that $A : E_1 \to E_2$ is a linear bounded operator and $f_i : C_i \times C_i \to \mathbb{R}, (i = 1, 2, ..., k)$ and $g_j : Q_j \times Q_j \to \mathbb{R}, (j = 1, 2..., s)$ are equilibrium bifunctions. Then the multiple-sets split equilibrium problem MSSEP [5] is formulated as

$$\text{“find} \quad x^* \in C = \bigcap_{i=1}^{k} C_i \quad \text{s.t.} \quad f_i(x^*, y) \geq 0, \quad \forall\, y \in C_i, \quad \forall\, i = 1, 2, ..., k, \text{”}$$

$$\text{“}Ax^* \in Q = \bigcap_{j=1}^{s} Q_j \quad \text{s.t.} \quad g_j(Ax^*, w) \geq 0, \quad \forall\, w \in Q_j, \quad \forall\, j = 1, 2, ..., s.\text{”}$$

The solution set of MSSEP is indicated by $\texttt{ME}$. In the case of $k = s = 1$, then MSSEP reduces to split equilibrium problem SEP, where the solution set of SEP is indicated by $\texttt{SE}$ [1]. If $g_1 \equiv 0$, then SEP reduces to equilibrium problem EP.

In 1980, Cohen introduced a valuable tool called the auxiliary problem principle for solving optimization problems [2]. Recently, Mastroeni expanded the auxiliary problem principle to equilibrium problems, assuming that “the equilibrium bifunction f satisfies a Lipschitz-type condition.” To address these difficulties,

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the linesearch technique in a Hilbert space is utilized to develop convergent algorithms for solving equilibrium problems (EP)[4, 5].

In this paper, new parallel linesearch algorithms—algorithms employing a two-step linesearch—are introduced for finding elements of ME utilizing Bregman projections. Using these algorithms, we establish several strong convergence theorems under suitable conditions. Additionally, we provide numerical examples related to our algorithms and compare their results with some existence results in other articles.

2 Preliminaries

Let E be a real Banach space,E^* its dual and $p > 1$ is a real number. Then $J_{p_E} : E \longrightarrow 2^{E^*}$ defined by $J_{p_E}x = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|^{p-1}\} \quad \forall\, x \in E$, is termed the generalized duality mapping. If E is p-uniformly convex and uniformly smooth, then J_{p_E} is monotone, one-to-one, onto and $J_{p_E}^{-1} = J_{q_E}^*$ [1].

Suppose that E is a p-uniformly convex and uniformly smooth. The Bregman distance $\Delta_p^E : E \times E \to \mathbb{R}$ is defined as $\Delta_p^E(x,y) = \frac{1}{q}\|x\|^p + \frac{1}{p}\|y\|^p - \langle y, J_{p_E}x\rangle, \quad \forall x, y \in E$. One can easily observe that

$$\Delta_p^E(x,y) = \Delta_p^E(x,z) + \Delta_p^E(z,y) + \langle z - y, J_{p_E}x - J_{p_E}z\rangle, \quad \forall\, x, y, z \in E. \tag{2.1}$$

$$\Delta_p^E(x,y) + \Delta_p^E(y,x) = \langle x - y, J_{p_E}x - J_{p_E}y\rangle, \quad \forall\, x, y \in E. \tag{2.2}$$

Moreover, there is always a fixed number $\tau > 0$ such that [4]

$$\tau\|x - y\|^p \le \Delta_p^E(x,y) \le \langle x - y, J_{p_E}x - J_p y\rangle. \tag{2.3}$$

Lemma 2.1. *[3] Let C be a closed nonempty subset of a uniformly smooth and p-uniformly convex space E. Suppose $(x,z) \in E \times C$, and let Π_C denote the Bregman projection from E onto C. Then, the following propositions hold:*

(i) $z = \Pi_C x \iff \langle y - z, J_{p_E}x - J_{p_E}z\rangle \le 0, \quad \forall\, y \in C,$

(ii) $\Delta_p^E(\Pi_C x, z) + \Delta_p^E(x, \Pi_C x) \le \Delta_p^E(x,z),$

3 Main Results

Throughout this paper, let $C_i, (i = 1,2,...,k)$ and $Q_j, (j = 1,2,...,s)$ be nonempty, convex and closed subsets of p-uniformly convex, uniformly smooth and real Banach spaces E_1 and E_2 respectively, where $C = \bigcap_{i=1}^k C_i$ and $Q = \bigcap_{j=1}^s Q_j$. Let $M_i : E_1 \to E_1^*$, $(\forall\, i = 1,2,...,k)$ and $N_j : E_2 \to E_2^*$, $(\forall\, j = 1,2,...,s)$ be maximal monotone operators. We consider single valued mappings $J_{\lambda_n^i}^{M_i} : E_1 \to D(M_i)$ by $J_{\lambda_n^i}^{M_i} = (J_{p_{E_1}} + \lambda_n^i M_i)^{-1} J_{P_{E_1}}$ and $J_{\mu_n^j}^{N_j} : E_2 \to D(N_j)$ by $J_{\mu_n^j}^{N_j} = (J_{p_{E_2}} + \mu_n^j N_j)^{-1} J_{P_{E_2}}$.

Algorithm 1

Step 0 For all $i = 1,2,...,k$ and all $j = 1,2,...,s$, choose $\xi^j, \rho^j \in (0,1)$ and the sequences $\{\eta_n^i\}$ and $\{\gamma_n^j\} \subseteq (a,1)$ such that $0 < a$. Also, $\{\alpha_n^i\}, \{\beta_n^j\} \subseteq [e,d]$ such that $\sum_{i=1}^k \alpha_n^i = \sum_{j=1}^s \beta_n^j = 1$, $\eta_{n,k}^{\cdot} = \frac{1}{k}\sum_{i=1}^k \eta_n^i$, $\gamma_{n,s}^{\cdot} = \frac{1}{s}\sum_{j=1}^s \gamma_n^j$.

Step 1 Calculate y_n^i such that

$$y_n^i \in C_i \quad s.t \quad f_i(y_n^i, x) + \frac{1}{r_n^i}\langle x - y_n^i, J_{p_{E_1}} y_n^i - J_{p_{E_1}} x_n\rangle \geq 0, \qquad \forall\, x \in C_i, \tag{3.1}$$

Step 2 Put $y_n = J^*_{q_{E_1}}(\eta^{\cdot}_{n,k} J_{p_{E_1}} x_n + (1-\eta^{\cdot}_{n,k})\sum_{i=1}^k \alpha_n^i J_{p_{E_1}} J^{M_i}_{\lambda_n^i} y_n^i)$

Step 3 Compute v_n^j such that $v_n^j \in \arg\min_{w\in Q_j}\left\{\gamma_n^j h_j(t_n, w) + \Delta_p^{E_2}(t_n, w)\right\}$,

Step 4 If $v_n^j = t_n$, then $z_n^j = t_n$. Otherwise Find the smallest non-negative integer m such that $h_j(z_{n,m}^j, t_n) - h_j(z_{n,m}^j, v_n^j) \geq \dfrac{\xi^j}{\gamma_n^j}\Delta_{p,j}^{E_2}(t_n, v_n^j)$.

Step 5 Calculate $g_n^j \in \partial_2 h_j(z_n^j, t_n)$ and

$$w_n = J^*_{q_{E_2}}(\gamma^{\cdot}_{n,s} J_{p_{E_2}} t_n + (1-\gamma^{\cdot}_{n,s})\sum_{j=1}^s \beta_n^j J_{p_{E_2}} J^{N_j}_{\mu_n^j} w_n^j)$$

$$D_n = \{z \in D_{n-1} \,:\, \Delta_p^{E_1}(y_n, z) \leq \Delta_p^{E_1}(x_n, z) \leq 0\}$$

Step 6 Set $n := n+1$ and go to Step 2.

Lemma 3.1. *Algorithm 1 ensures that for every $x^* \in \mathtt{ME}$ and $n \in \mathbb{N}\cup\{0\}$*

(i) $\Delta_p^{E_1}(y_n^i, x^*) \leq \Delta_p^{E_1}(x_n, x^*), \quad \forall\, i = 1, 2, ..., k$

(ii) $\Delta_p^{E_1}(y_n, x^*) \leq \Delta_p^{E_1}(x_n, x^*),$

Lemma 3.2. *Algorithm 1 ensures that for every $j = 1, 2, ..., s$ and $n \in \mathbb{N}\cup\{0\}$*

(i) $\langle w - v_n^j, J_{p_{E_2}} v_n^j - J_{p_{E_2}} t_n\rangle \leq \gamma_n^j h_j(t_n, w) - \gamma_n^j h_j(t_n, v_n^j), \qquad \forall\, w \in Q_j.$

Theorem 3.3. *Assume that $\mathtt{ME} \neq \emptyset$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm1 converges strongly to the some solution $\upsilon \in \mathtt{ME}$, where $\upsilon = \lim_{n\to\infty} \Pi_{\mathtt{ME}} x_n$.*

4 Numerical examples

To illustrate Theorem 3.3, we present a numerical example. Furthermore, we analyze the behavior of the sequence $\{x_n\}$ generated by the parallel linesearch Algorithm 1 and compare it with certain existence results reported in the literature.

Example 4.1. We consider Example 4.1 of [4], with $\gamma_n^1 = \frac{1}{8}$, $\gamma_n^2 = \frac{1}{16}$, $\eta_n^1 = \lambda_n^1$, $\eta_n^2 = \lambda_n^2$,

$$f_1(x, y) = 10\|y\|^2 + \langle y, x\rangle - 11\|x\|^2 \quad \& \quad f_2(x, y) = 7\|y\|^2 + 3\langle y, x\rangle - 10\|x\|^2,$$

Moreover,$r_n^1 = 3 + \frac{n+2}{n+1}$, $r_n^2 = 4 + \frac{3n+1}{n+1}$. we assume that the parameters are given as follows: $c_2 = 1$, $\tau = \frac{1}{2}$, $\xi^1 = \xi^2 = \frac{1}{3}$, $\rho^1 = 0.04$, $\rho^2 = 0.005$ Now, we consider the following algorithms and theorems under the above conditions with initial value $x_0 = (-2, 2)$ and stopping criteria $\|x_n\| < 10^{-5}$ and we then compare their results.

Table 1: Convergence behavior of the sequence $\{x_n\}$, generated by algorithms $(O_1) - (O_2)$, with the initial point $x_0 = (-2, 2)$.

n	(O_1)	(O_2)
0	4.24	4.2426
1	0.88	1.28
2	0.48	1.02
3	0.26	0.86
⋮	⋮	⋮
$STOP$	19	No
$CPU(s)$	$2.24s$	No

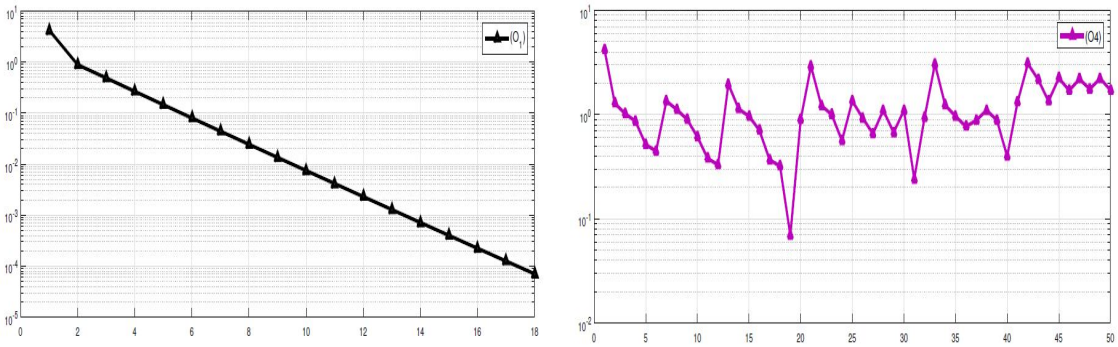


Figure 1: Comparison of the convergence results for the sequence $\{x_n\}$ generated by methods $(O_1) - (O_2)$, with the initial point $x_0 = (-2, 2)$.

(O_1) Our Algorithm 1 and Theorem 3.3 with $J^{M_1}_{\lambda^1_n} = \frac{1}{9}I, J^{M_2}_{\lambda^2_n} = \frac{1}{6}I, J^{N_1}_{\mu^1_n} = \frac{1}{8}I$ and $J^{N_2}_{\lambda^2_n} = \frac{1}{12}I$. In this case, it follows that $\texttt{ME} = \{0\}$.

(O_2) Algorithm 1 and Theorem 3.1 of [5] with parameters $\rho^i_k = \eta^i_n$, $r^j_k = \gamma^j_n$ and $\mu = 0.2$.

These results indicate that $\{\|x_n\|\}$, and consequently $\{x_n\}$, converge strongly to 0. Moreover, our parallel linesearch method for solving MSSEP, namely algorithms (O_1) and (O_2), achieves the solution with fewer iterations and reduced CPU time compared to the other algorithm, highlighting its computational efficiency.

5 Conclusion

In this paper, we introduced new parallel linesearch algorithm for solving MSSEP in p-uniformly convex and uniformly smooth Banach spaces. To check performance, we conducted numerical experiments and compared the results with existing algorithms in the literature. The results show that the convergence of the algorithms is faster than other algorithms.

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On the solutions for a class of biharmonic fractional schrödinger problems with sign-changing functions on $\mathbb{R}^n$

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Abstract

Here we investigate the existence and multiplicity of solutions for the following biharmonic fractional schrödinger problem

$$\Delta u_p^2 + (-\Delta)_p^s u + v(x)|u|^{p-2}u = f(x,u), \tag{0.1}$$

where $f(x,u) = b(x)|u|^{q-2}u + \lambda g(x,u) + h(x)$, $x \in \mathbb{R}^n$, $p < q < p_s^*$ that $p_s^* = \frac{np}{n-ps}$ is the fractional Sobolev exponent, $n > ps$ with $s \in (0,1)$, $v(x)$, $b(x) \in L^\infty(\mathbb{R}^n)$ are sign-changing functions, $\lambda > 0$, $h(x) \in L^\infty(\mathbb{R}^n)$ and

$$(-\Delta)_p^s u(x) := 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)}{|x-y|^{n+ps}} dy.$$

Moreover, using the Mountain Pass Lemma, we establish the existence of at least one solution with positive energy.

Keywords: Critical points, Fractional partial differential equations, Weak solutions, Lack of compactness, Mountain Pass Lemma.

1 Introduction and preliminaries

In this paper, due to the widespread use of fractional differential equations, we will use variational methods to the study of existence and multiplicity of solutions for problem (0.1) and for this purpose, we consider the fractional Sobolev space $W := W^{s,p}(\mathbb{R}^n) \bigcap W^{2,p}(\mathbb{R}^n)$ with the norm

$$\|u\|_W^p := \left(\int_{\mathbb{R}^n} |\Delta u|^p dx \right) + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right). \tag{1.1}$$

One of the main difficulties in problem (0.1), will be the nonlinearity of $g(x,u)$. To overcome this difficulty we need to restrict $g(x,u)$ to the following conditions:

(g1) $g(x,0) = 0$, $\frac{\partial}{\partial t} g(x,tu)|_{t=1} \in C(\mathbb{R}^N \times \mathbb{R}^+)$ and the integral $\int_{\mathbb{R}^N} \frac{\partial}{\partial t} g(x,t|u|)dx$ has the same sign where $u \in L^p(\mathbb{R}^N)$ and $t > 0$.

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(g2) There exists $K_g(x) \in L^\infty(\mathbb{R}^N)$ such that

$$G(x,u) \le \frac{1}{p} g(x,u)u \le \frac{1}{p^2-p} g_u(x,u)u^2 \le K_g(x)u^p,$$

and $\lim_{R\to\infty} \|K_g(x)\|_{L^\infty(\Omega_R)} = 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^+$.

In this paper we consider the problem (0.1) on $\mathbb{R}^N$ and $\mathbb{R}^N$ is an unbounded domain, so the Sobolev embeddings aren't compact. To preserve this compactness in our problems, we need to impose some conditions on $b(x)$ and $h(x)$. Therefore, throughout this paper, we make the following assumptions over $b(x)$ and $h(x)$: (K) For $\rho = b, h$, $\lim_{R\to\infty} \|\rho(x)\|_{L^\infty(\Omega_R)} = 0$, where $\Omega_R = \mathbb{R}^N \backslash B_R(0)$.

In order to study of problem (0.1), we set:

$$L^\alpha(\mathbb{R}^N, \rho) = \{u | \int_{\mathbb{R}^N} \rho(x)|u|^\alpha dx < \infty\}, \ (\alpha > 0),$$

with the norm

$$\|u\|_{L^\alpha(\mathbb{R}^N,\rho)} = \left(\int_{\mathbb{R}^N} |\rho(x)||u|^\alpha dx) \right)^{\frac{1}{\alpha}}.$$

Lemma 1.1. *(see [3]) If $\rho(x) \in L^\infty(\mathbb{R}^N)$ and $\lim_{R\to\infty} \|\rho(x)\|_{L^\infty(\Omega_R)} = 0$, then the embedding $W \hookrightarrow L^q(\mathbb{R}^N, \rho(x))$ for $1 \le q < p^*$ is compact.*

Remark 1.2. Using lemma 2.1, (g1) and (g2), we conclude that the functional $J(u) = \int_{\mathbb{R}^N} G(x,u)dx$ is weakly continuous, i.e., if $u_n \rightharpoonup u$, then $J(u_n) \to J(u)$; moreover, the operators $J'(u)$ and $J''(u)$ are weak to strong continuous, i.e., if $u_n \rightharpoonup u$, then $J'(u_n) \to J'(u)$ and $J''(u_n) \to J''(u)$.

Throughout this paper, we denote by S_q^q the best Sobolev constant for the embeddings of W into $L^q(\mathbb{R}^n)$. Indeed, the sharp constant S_p^p is equal to $\frac{1}{\mu_p}$, where

$$\mu_p := \inf\{\frac{\|u\|_W}{\|u\|_{L^p(\mathbb{R}^n)}} : 0 \ne u \in W\}.$$

The Euler-Lagrange functional associated with this problem is $I : W \to \mathbb{R}$ such that

$$I(u) = \frac{1}{p}\big(M(u) + V(u)\big) - \frac{1}{q}B(u) - \lambda \int_{\mathbb{R}^n} G(x,|u|)dx - H(u), \tag{1.2}$$

where $M(u) := \|u\|_W^p$, $V(u) := \int_{\mathbb{R}^n} v(x)|u|^p dx$, $B(u) := \int_{\mathbb{R}^n} b(x)|u|^q dx$, $G(x,u) := \int_0^u g(x,s)ds$ and $H(u) := \int_{\mathbb{R}^n} h(x)|u|dx$.

Besides, we set

$$v^+ := \operatorname{ess\,sup}\{v(x), x \in \mathbb{R}^n\}, \tag{1.3}$$

and

$$\tilde{V}(u) := M(u) + V(u) = \|u\|_W^p + \int_{\mathbb{R}^n} v(x)|u|^p dx. \tag{1.4}$$

Lemma 1.3. *(see [3]) If $v^+ < \mu_p$, then there exists $\delta_1 > 0$ such that for every $u \in W$, $\tilde{V}(u) \ge \delta_1 \|u\|_W^p$.*

Remark 1.4. According to lemma 1.1 we know that if $v^+ < \mu_p$ then $\tilde{V}(u) \ge \delta_1 \|u\|_W^p$ for $\delta_1 > 0$. On the other hand, using the continouse embedding $W \hookrightarrow L^p(\mathbb{R}^N)$ we have $\tilde{V}(u) \le \|u\|_W^p + \|a\|_\infty S_p^p \|u\|_W^p$. Therefore, in the sequel for $v^+ < \mu_p$, we consider W with the following norm:

$$\|u\|_{\tilde{V}} := \big(\tilde{V}(u)\big)^{\frac{1}{p}} = \left(\|u\|_W^p + \int_{\mathbb{R}^n} v(x)|u|^p dx \right)^{\frac{1}{p}}. \tag{1.5}$$

Also we know that if $I(u)$ denotes the energy functional corresponding to a problem, then all of the critical points of $I(u)$ must lie in the set $N := \{u; \langle I'(u), u\rangle = 0\}$, which is known as the Nehari manifold (see [3]). On the other hand, the fibering map

$$\varphi_u(t) := \frac{t^p}{p}\big(M(u) + V(u)\big) - \frac{t^q}{q}B(u) - \lambda\int_{\mathbb{R}^n} G(x, t|u|)dx - tH(u), \tag{1.6}$$

is closely linked to the Nehari manifold, i.e., $\varphi'_u(1) = 0$ if and only if $u \in N$, (see [2]). So it is reasonable to divide the Nehari manifold into three parts corresponding to local minima, local maxima and inflection points of the critical points of $\varphi_u(t)$. So we define $N^{\pm} := \{u \in N, \varphi''_u(1) \gtrless 0\}$ and $N^0 := \{u \in N, \varphi''_u(1) = 0\}$.

2 Main results

The following lemma shows that minimizers for $\tilde{I}$ on N are usually critical points for $\tilde{I}$, as proved by Brown and Zhang in [2]

Lemma 2.1. *Let u_0 be a local minimizer for $I(u)$ on N such that $u_0 \notin N^0$, then u_0 is a critical point of $I(u)$.*

In this section and in the sequel we need the following lemmas and results (see [3]):

Lemma 2.2. *If $v^+ < \mu_p$ then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, we have $N^0 = \emptyset$.*

Lemma 2.3. *If $v^+ < \mu_p$ then there exists $\lambda_1 > 0$ such that for $\lambda < \lambda_1$, $I(u)$ is coercive and bounded below on N.*

Lemma 2.4. *If $v^+ < \mu_p$ then there exists $\lambda_2 > 0$ such that, for $0 < \lambda < \lambda_2$, $\varphi_u(t)$ takes on positive values for all non-zero $u \in X_0$.*

Corollary 2.5. *(i). If $v^+ < \mu_p$, $0 < \lambda < \lambda_1$ and $B(u) \le 0$ for $u \in W \setminus \{0\}$, then there exists t_1 such that $t_1u \in N^+$ and $I(t_1u) < 0$.*
(ii). If $v^+ < \mu_p$, $0 < \lambda < \min\{\lambda_1, \lambda_2\}$ and $B(u) > 0$ for $u \in W \setminus \{0\}$, then there exist $t_1 < t_2$ such that $t_1u \in N^+$, $t_2u \in N^-$ and $I(t_1u) < 0$.

Lemma 2.6. *If $0 < \lambda < \lambda_1$, then $B(u) > 0$ provided that $u \in N^-$.*

Proposition 2.7. *[3] If $v^+ < \mu_p$, then for $0 < \lambda < \lambda^* := \min\{\lambda_0, \lambda_1, \lambda_2\}$ we have*
(i) I admits a minimizer on N^+ ;
(ii) I admits a minimizer on N^-.

Theorem 2.8. *If $v^+ < \mu_p$, then for $0 < \lambda < \lambda^*$, problem (0.1) has at least two distinct positive solutions, provided that $H(u) > 0$;*

Proof. By proposition 2.7 there exist $u_1 \in N^+$ and $u_2 \in N^-$ such that $I(u_1) = \inf_{u\in N^+} I(u)$, $I(u_2) = \inf_{u\in N^-} I(u)$ and by Lemmas 2.1 and 2.2, u_1 and u_2 are critical points of I on W and hence are weak solutions of problem (0.1). On the other hand $I(u) = I(|u|)$, so we may assume that u_1 and u_2 are nonnegative solutions. It remains to show that the solutions found in proposition 2.7 are distinct. Since $N^+ \cap N^- = \emptyset$, this implies that u_1 and u_2 are distinct and the proof is complete.□

Definition 1. *A sequence $u_n \subset W$ is called a Palais-Smale sequence (or $(PS)_c - sequence$) if $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$. It is said that the functional I satisfies the Palais-Smale condition (or $(PS)_c - condition$), if each $(PS)_c - sequence$ has a convergent subsequence.*

Now we will prove the boundedness of Palais-Smale sequence.

Lemma 2.9. *If $v^+ < \mu_p$ and $\{u_n\}$ is a $(PS)_c$ − sequence for I, then $\{u_n\}$ is a bounded sequence in W provided that $0 < \lambda < \lambda_1$.*

Proof. Similar to the argument in the proof of lemma 2.2 we get

$$I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \geq \big(\frac{q-p}{qp} - \lambda\frac{q-p}{q}S_p^p\|K_g\|_\infty - \frac{q-p}{2qp}\big)\|u\|_{\tilde{V}}^p - \frac{q-p}{2q} \times \frac{p-1}{p}\big(\frac{q-1}{q-p}2\|h(x)\|_\infty S_1\|u\|_{\tilde{V}}\big)^{\frac{p}{p-1}},$$

so for for $0 < \lambda < \lambda_1 = \frac{1}{2p\|K_g\|_\infty S_p^p}$, $\{u_n\}$ is bounded in W. □

Lemma 2.10. *[3] If $v^+ < \mu_p$, then the functional $I(u)$ satisfies (PS) condition on W.*

Theorem 2.11. *There exists $\lambda^{**} \geq \lambda^*$ such that for $0 < \lambda < \lambda^{**}$ problem 2.7 has at least one nontrivial solution with positive energy, provided that $B(u) > 0$.*

For the proof of Theorem(2.11), we need the Mountain Pass Lemma and for the readers convenience, we recall it here.

Mountain Pass Lemma. (see [1]) Let X be a real Banach space with the norm $\|.\|$ and $J \in C^1(X, R)$, $J(0) = 0$. Assume
(i) the function $J(u)$ on X satisfies the (PS) condition;
(ii) there are β, $\rho > 0$ such that $J(u) \geq \beta$, $\|u\| = \rho$;
(iii) there is $e \in X$, $\|e\| \geq \rho$ such that $J(e) \leq 0$.
then $c_0 = \inf_{\phi\in\Psi}\max_{t\in[0,1]} J(\phi(t))$ is a critical value of $J(u)$ with $0 < \beta \leq c_0 < \infty$, where
$\Psi = \{\phi \in (C[0, 1], X), \phi(0) = 0, \phi(1) = e\}$.

Proof of Theorem 2.11. From (1.2), it is clear that $I \in C^1(W, \mathbb{R})$ and $I(0) = 0$. since $B(u) > 0$, then $\lim_{t\to\infty} I(tu) = -\infty$, this means that there exists t_0 such that $I(t_0u) < 0$. Also by using lemma 2.4, we know $I(t_{max}u) > \epsilon > 0$ and by lemma (2.10) for $0 < \lambda < \lambda^{**}(\lambda^{**} := \lambda_1)$,, I satisfies the (PS) condition on W. Now using the Mountain Pass Lemma gives Theorem 2.11.

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Existence of the solution for a class of biharmonic fractional Kirchhoff type systems with sign-changing functions on $\mathbb{R}^n$

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Abstract

In this paper, due to the widespread applications of biharmonic problems as well as fractional differential problems, we study the existence of weak solutions for a class biharmonic fractional Kirchhoff type systems with sign-changing functions on $\mathbb{R}^n$.

Keywords: Critical points, Fractional partial differential equations, Weak solutions, Lack of compactness, Lagrange multiplier theorem.

1 Introduction and preliminaries

Fractional differential equations have been an area of great interest recently. This is because of both the development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. Hence, recently a great deal of attention has been focused on study of existence and multiplicity of solutions for fractional differential problems. To learn about partial differential equations, Sobolev spaces, and embedding theorems, (see [1]). Also for familiarity with biharmonic problems, (see [2]) and for familarity with fractional problems, (see [3]).

Here we investigate the existence of solutions for the following biharmonic fractional p-Kirchhoff type systems

$$\begin{cases} \Delta u_p^2 + \mathfrak{L}_1(x,u)\big((-\Delta)_p^s u + a_1(x)|u|^{p-2}u\big) = f(x)|u|^{q-2}u, \\ \Delta v_p^2 + \mathfrak{L}_2(x,v)\big((-\Delta)_p^s v + a_2(x)|v|^{p-2}v\big) = g(x)|v|^{r-2}v. \end{cases} \tag{1.1}$$

where $\mathfrak{L}_i(x,\varphi) := \iint_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+ps}}dxdy + \int_{\mathbb{R}^n} a_i(x)|\varphi|^p dx$, $a_1(x), a_2(x) \in L^\infty(\mathbb{R}^n)$ are sign-changing functions, $n > ps$ with $s \in (0,1)$, $2 \le p < q, r \le p_s^*$ ($p_s^* = \frac{np}{n-ps}$ is the fractional Sobolev exponent), $f(x), g(x) \in L^\infty(\mathbb{R}^n)$, $f(x), g(x) \ge 0$ a.e. and $(-\Delta)_p^s\varphi(x)$ denotes the fractional Laplace operator defined by

$$(-\Delta)_p^s\varphi(x) := 2\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{|\varphi(y)-\varphi(x)|^{p-2}\big(\varphi(y)-\varphi(x)\big)}{|x-y|^{n+ps}}dy.$$

Elliptic system 1.1 contains the following class of biharmonic fractional p-Kirchhoff elliptic scalar equation:

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$$\Delta u_p^2 + \mathfrak{L}_1(x,u)\big((-\Delta)_p^s u + a_1(x)|u|^{p-2}u\big) = f(x)|u|^{q-2}u,$$

which corresponds to the special situation $u = v$, $a_1(x) = a_2(x)$ and $f(x) = g(x)$.

In this work, we will use variational methods to the study existence of solutions for problem (1.1), and for this purpose, we consider the fractional Sobolev space $W := X \times X$ where $X := W^{s,p}(\mathbb{R}^n) \bigcap W^{2,p}(\mathbb{R}^n)$ with the norm

$$\|(u,v)\|_W := \left(\int_{\mathbb{R}^n} |\Delta u|^p dx\right)^{\frac{1}{p}} + \|u\|_{W^{s,p}(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} |\Delta v|^p dx\right)^{\frac{1}{p}} + \|v\|_{W^{s,p}(\mathbb{R}^n)}$$

where $\|\phi\|_{W^{s,p}(\mathbb{R}^n)} = \left(\iint_{\mathbb{R}^{2n}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{\frac{1}{p}}$

Throughout this paper, we denote by S_q^q the best Sobolev constant for the embeddings of $W^{s,p}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Indeed, the sharp constant S_p^p is equal to $\frac{1}{\mu_p}$, where

$$\mu_p := \inf\{\frac{\int_{\mathbb{R}^n} |\Delta u|^p dx}{\int_{\mathbb{R}^n} |u|^p dx} : 0 \neq u \in X\}. \tag{1.2}$$

Besides, we set

$$a_i^+ := \operatorname{ess\,sup}\{a_i(x), x \in \mathbb{R}^n\}, \tag{1.3}$$

and

$$M(u,v) := \|(u,v)\|_W^p + \int_{\mathbb{R}^n} a_1(x)|u|^p dx + \int_{\mathbb{R}^n} a_2(x)|v|^p dx. \tag{1.4}$$

Lemma 1.1. *If $a^+ := \max\{a_1^+, a_2^+\} < \mu_p$, then there exists $\delta_1 > 0$ such that for every $(u,v) \in W$, $M(u,v) \geq \delta_1 \|(u,v)\|_W^p$.*

Proof. If $\int_{\mathbb{R}^n} a_1(x)|u|^p dx + \int_{\mathbb{R}^n} a_2(x)|v|^p dx \geq 0$ then the lemma is obvious, otherwise; it would be proved by contradiction, supposing that $\int_{\mathbb{R}^n} a_1(x)|u|^p dx + \int_{\mathbb{R}^n} a_2(x)|v|^p dx < 0$. If for every δ, there exists $(u,v) \in W$ such that $M(u,v) < \delta\|(u,v)\|_W^p$, it can be deduced that

$$\begin{aligned}(1-\delta)\|(u,v)\|_W^p &< -\int_{\mathbb{R}^n} a_1(x)|u|^p dx - \int_{\mathbb{R}^n} a_2(x)|v|^p dx \\ &< a^+ \int_{\mathbb{R}^n} (|u|^p + |v|^p) dx.\end{aligned} \tag{1.5}$$

Now, by taking $\delta < 1 - \frac{a^+}{\mu_p}$ we get $\frac{a^+}{1-\delta} < \mu_p$ which leads to a contradiction with (1.2). □

Remark 1.2. According to lemma 1.1 we know that if $a^+ < \mu_p$ then $M(u,v) \geq \delta_1\|(u,v)\|_W^p$ for $\delta_1 > 0$. On the other hand, using the continouse embedding $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ we have $M(u,v) \leq \|(u,v)\|_W^p + \big(\|a_1\|_\infty + \|a_2\|_\infty\big)S_p^p\|(u,v)\|_W^p$. Therefore, in the sequel for $a^+ < \mu_p$, we consider W with the following norm:

$$\|(u,v)\|_M := \left(\|(u,v)\|_W^p + \int_{\mathbb{R}^n} a_1(x)|u|^p dx + \int_{\mathbb{R}^n} a_2(x)|v|^p dx\right)^{\frac{1}{p}}. \tag{1.6}$$

2 Main results

In this paper we consider the problem (1.1) on $\mathbb{R}^n$ and $\mathbb{R}^n$ is an unbounded domain, so the Sobolev embedding isn't compact [1]. To preserve this compactness in our problem, we need to impose some conditions on $a_1(x)$ and $a_2(x)$. Therefore, throughout this paper, we make the following assumptions over $a_1(x)$ and $a_2(x)$:

(K) For i=1,2, $\lim_{R\to\infty} \|a_i(x)\|_{L^\infty(\Omega_R)} = 0$, where $\Omega_R = \mathbb{R}^n \backslash B_R(0)$.

In order to study of problem (1.1) we set

$$L^\alpha(\mathbb{R}^n, a_i) = \{\varphi | \int_{\mathbb{R}^n} a_i(x)|\varphi|^\alpha dx < \infty\}, \ (\alpha > 0),$$

with the norm, $\|\varphi\|_{L^\alpha(\mathbb{R}^n,a_i)} = \left(\int_{\mathbb{R}^n} |a_i(x)||\varphi|^\alpha dx \right)^{\frac{1}{\alpha}}$.

Lemma 2.1. *(see [3]). If $a_i(x) \in L^\infty(\mathbb{R}^n)$ and $\lim_{R\to\infty} \|a_i(x)\|_{L^\infty(\Omega_R)} = 0$, then the embedding $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^\alpha(\mathbb{R}^n, a_i(x))$ for $1 \le \alpha < p^*$ is compact.*

The Euler-Lagrange functional associated with problem (1.1) is

$$I(u,v) = \frac{1}{p}M(u,v - \frac{1}{q}F(u) - \frac{1}{r}G(v), \tag{2.1}$$

where $M(u,v)$ is defind in (1.4), $F(u) := \int_{\mathbb{R}^n} f(x)|u|^q dx$ and $G(v) := \int_{\mathbb{R}^n} g(x)|v|^r dx$.

Using (2.1) for every $u \neq 0$, $I(tu) \hookrightarrow -\infty$ as $t \hookrightarrow \infty$. Thus, $I(u)$ is not bounded below and so the minimizing process on the hole space W may not be possible. If for every $\alpha, \beta \in \mathbb{R}$, we let

$$\begin{cases} F_\alpha := \{u \in X : \int_{\mathbb{R}^n} f(x)|u|^q dx = \alpha\}, \\ G_\beta := \{v \in X : \int_{\mathbb{R}^n} g(x)|v|^r dx = \beta\}, \end{cases}$$

and

$$N_{\alpha,\beta} := \{(u,v) | u \in F_\alpha, v \in G_\beta\},$$

then, for every $(u,v) \in N_{\alpha,\beta}$, by using Remark 1.2 we have

$$I(u,v) = \frac{1}{p}\|(u,v)\|_M^p - \frac{1}{q}\alpha - \frac{1}{r}\beta.$$

Thus, $I|_{N_{\alpha,\beta}}$ is certainly bounded below and hence the minimizing approach of $I(u,v)$ on $N_{\alpha,\beta}$ is equivalent to the minimizing approach of $\|(u,v)\|_M^p$ on $N_{\alpha,\beta}$.

Set $m_{\alpha,\beta} := \inf_{(u,v)\in N_{\alpha,\beta}} \|(u,v)\|_M^p$. We will show that $m_{\alpha,\beta}$ is obtained by a function, and a multiple of this function is a minimizer of $I(u,v)$, as a result a weak solution of problem (1.1).

Lemma 2.2. *(see [2, 3]) For every $\alpha, \beta > 0$, there exists a nonnegative function $(u_\alpha, v_\beta) \in N$, such that $\|(u,v)\|_M^p = m_{\alpha,\beta}$.*

The main result in this paper is the following theorem.

Theorem 2.3. *If $a^+ < \mu_p$, then problem (1.1) admits at least one weak solution in W.*

Proof. Let $\chi(u,v) = \|(u,v)\|_M^p$, now if (u_α, v_β) is a minimizer of $\chi(u,v)$ under the conditions $F(u) = \alpha$ and $G(v) = \beta$, then by Lagrange multiplier theorem, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\chi_u(u_\alpha, v_\beta) = \lambda_1 F'(u_\alpha)$, and $\chi_v(u_\alpha, v_\beta) = \lambda_2 G'(v_\beta)$ hence for every $(u_i, v_j) \in W$ we have

$$\begin{cases} \langle \chi_u(u_\alpha, v_\beta), u_i \rangle = q\lambda_1 \displaystyle\int_{\mathbb{R}^n} f(x)|u_\alpha|^{q-2}u_\alpha u_i dx., \\ \langle \chi_v(u_\alpha, v_\beta), v_j \rangle = r\lambda_2 \displaystyle\int_{\mathbb{R}^n} g(x)|v_\beta|^{r-2}v_\beta v_j dx. \end{cases} \tag{2.2}$$

By taking $u_\alpha = Cw_\alpha$ we get

$$\begin{aligned} &C^{p-1} \iint_{R^{2n}} \frac{|w_\alpha(x) - w_\alpha(y)|^{p-2}(w_\alpha(x) - w_\alpha(y))(u_i(x) - u_i(y))}{|x-y|^{n+ps}} dxdy \\ &+ C^{p-1} \int_{\mathbb{R}^n} a(x)|w_\alpha|^{p-2}w_\alpha v dx = \frac{q\lambda_1}{p} C^{q-1} \int_{\mathbb{R}^n} f(x)|w_\alpha|^{q-2}w_\alpha u_i dx. \end{aligned}$$

Now, by assuming $C = (\frac{p}{q\lambda_1})^{\frac{1}{q-p}}$ we have

$$\begin{aligned} &\iint_{R^{2n}} \frac{|w_\alpha(x) - w_\alpha(y)|^{p-2}(w_\alpha(x) - w_\alpha(y))(u_i(x) - u_i(y))}{|x-y|^{n+ps}} dxdy \\ &+ \int_{\mathbb{R}^n} a(x)|w_\alpha|^{p-2}w_\alpha v dx = \int_{\mathbb{R}^n} f(x)|w_\alpha|^{q-2}w_\alpha u_i dx, \end{aligned} \tag{2.3}$$

similarly $v_\beta = (\frac{p}{r\lambda_2})^{\frac{1}{r-p}} z_\beta$ and consequently, (w_α, z_β) is a weak solution of problem (1.1). □

Lemma 2.4. *(see [2]). For $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$ the minimizers of $\chi(u,v)$ on N_{α_1,β_1} and N_{α_2,β_2} give the same weak solution of problem (1.1).*

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A novel nonstandard finite difference scheme for achieving synchronization in fractional order chaotic systems through active control technique

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Abstract

This study explores the synchronization of various fractional–order chaotic systems through the active control approach. The Caputo derivative is considered for the fractional model. To numerically solve the fractional–order master and slave systems, the nonstandard finite difference (NSFD) scheme is applied. Results from the simulations indicate that the NSFD scheme is both simple to implement and precise for synchronizing fractional–order systems.

Keywords: Caputo derivative, synchronization, nonstandard finite difference scheme, chaotic system, active control.

1 Introduction

Chaos is a fascinating phenomenon observed in the dynamics of nonlinear systems, with numerous theoretical and practical applications across fields like economics, biology, physics, secure communications, and nonlinear circuits [1, 3, 4, 5]. Chaotic systems are governed by nonlinear differential equations and are known for their extreme sensitivity to initial conditions, meaning even small changes in starting values can cause large variations or divergence in the system's behavior. Recently, there has been growing interest in managing chaotic systems due to their relevancy in areas such as physics, biological networks, and artificial intelligence, prompting extensive research efforts on controlling and synchronizing chaotic dynamical systems. Over the past few decades, fractional–order systems have gained recognition for their effectiveness in modeling real–world phenomena due to their ability to capture memory effects. This feature makes fractional–order models better studied for representing real processes compared to classical integer–order models, which rely on standard time derivative. In this paper, we use active control technique to investigate the synchronization between two distinct chaotic incommensurate fractional–order Volta's and Lü systems. We apply the stability theory of linear incommensurate fractional differential equations to establish the stability of the error system, ensuring complete synchronization of the two systems. Additionally, numerical solutions of the master, slave, and error systems are provided using the NSFD schemes.

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2 Fractional calculus and NSFD schemes

First, let us define the Caputo derivative and then recall its Grunwald–Letinkov approximation for fractional–order differential equations. For an integrable function $\mathcal{Y}$ and $0 < \alpha < 1$, the Caputo derivative is expressed as [6]

$$^{C}\mathcal{D}^{\alpha}\mathcal{Y}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \mathcal{Y}'(\tau)(t-\tau)^{-\alpha} d\tau.$$

Theorem 2.1 ([4]). *Consider the incommensurate fractional–order system defined by*

$$^{C}\mathcal{D}^{\alpha}\mathcal{Y}(t) = f(\mathcal{Y}(t)), \qquad t \in [0, t_f], \qquad \mathcal{Y}(0) = \mathcal{Y}_0, \tag{2.1}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with each $\alpha_i \in (0,1)$, and $\mathcal{Y} \in \mathbb{R}^n$. The equilibrium Point E of system (2.1) is locally asymptotically stable if all eigenvalues, λ of the Jacobian matrix $J(E)$ satisfy

$$|\arg(\lambda)| > \alpha_M \frac{\pi}{2}, \qquad \alpha_M = \max_{1 \le i \le n} \alpha_i, \qquad i = 1, \ldots, n.$$

Next, we descritize the fractional–order differential equation (2.1) using the following difference equation:

$$\sum_{j=0}^{n+1} c_j^{\alpha} \mathcal{Y}_{n+1-j} = f(\mathcal{Y}_{n+1}), \qquad n = 0, 1, 2, ..., \tag{2.2}$$

where $\mathcal{Y}_n$ approximates $\mathcal{Y}(t_n)$ at $t_n = nh$, with $n = 0, 1, 2, ..., N$. The coefficients c_j^{α} are derived from Grunwald–Letinkov weights:

$$c_0^{\alpha} = h^{-\alpha}, \qquad c_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{\alpha}, \qquad j = 1, 2, \ldots.$$

Continuing, we review the fundamentals of the NSFD schemes. Mickens [2] proposed NSFD schemes for ordinary differential equations. These schemes were designed to prevent numerical instabilities that often occur with standard finite difference, Runge–Kutta, and other numerical methods. By choosing an appropriate denominator function instead of the typical stepsize h, and employing non–local discretization, NSFD schemes maintain properties such as positivity, boundedness, stability, and convergence, even with larger step sizes. To describe an NSFD scheme, consider the nonlinear initial value problem

$$\mathcal{Y}'(t) = f(\mathcal{Y}(t)), \qquad t \in [0, t_f], \qquad \mathcal{Y}(0) = \mathcal{Y}_0.$$

An NSFD scheme can be constructed using at least one of the following two approaches

- Non local approximation is used.
- Denominator function for the discrete, in general, expressed in terms of more complicated functions of the stepsize h than those conventionally used.

Nonlinear terms are approximated using a non local approach. For instance,

$$x_1(t)|_{t=t_k} \approx x_1^{(k+1)}, \qquad x_1(t)y_1(t)|_{t=t_k} \approx x_1^{(k+1)} y_1^{(k+1)}.$$

We now extend the previously described method in (2.2) to apply the fractional differential equation (2.1). As a result, the NSFD scheme can be applied in the fractional context as follows:

$$\mathcal{Y}_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha} y_{n+1-j} + f(\mathcal{Y}_{n+1})}{c_0^{\alpha}}, \qquad n = 0, 1, 2, \ldots.$$

We employ the finite difference approximation

$$\mathcal{Y}'(t_k) \approx \frac{\mathcal{Y}_{k+1} - \mathcal{Y}_k}{\Upsilon(h)},$$

where $\mathcal{Y}_k$ approximates $\mathcal{Y}(t_k)$ at the time $t_k = kh$, with $k = 0, 1, \ldots, N$. The denominator function $\Upsilon(h)$, must be a continuous, increasing function that satisfies the condition:

$$\Upsilon(h) = h + \mathrm{O}(h^2), \qquad 0 < \Upsilon(h) < 1,$$

where

$$c_0^{\alpha} = (\Upsilon(h))^{-\alpha}, \qquad c_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{\alpha}, \qquad j = 1, 2, \ldots. \tag{2.3}$$

3 Synchronization of Volta's and Lü systems through active control

In this section, we study the synchronization process between Volta's and Lü systems. We introduce a fractional model for the Volta's system using the following system of fractional differential equations.

$$\begin{cases} {}^{C}D^{\alpha_1} x_1(t) = -x_1(t) - a_1 y_1(t) - z_1(t) y_1(t), \\ {}^{C}D^{\alpha_2} y_1(t) = -y_1(t) - b_1 x_1(t) - x_1(t) z_1(t), \\ {}^{C}D^{\alpha_3} z_1(t) = c_1 z_1(t) + x_1(t) y_1(t) + 1, \end{cases} \tag{3.1}$$

where x_1, y_1, and z_1 represent the state vaiables, while a_1, b_1, and c_1 denote the positive constant parameters. Here $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ represent incommensurate fractional orders, and the initial conditions are give by $x_1(0) = x_1^{(0)}, y_1(0) = y_1^{(1)}$, and $z_1(0) = z_1^{(0)}$. By applying Mickens rules and replacing the stepsize h with a suitable denominator function and using nonlocal approximations together with the Grunwald–Letinkov approximation, one can drive the following NSFD scheme for the fractional order system (3.1).

$$\begin{cases} \sum_{j=0}^{k+1} c_j^{\alpha_1} x_1^{(k+1-j)} = -x_1^{(k+1)} - a_1 y_1^{(k)} - z_1 y_1^{(k)}, \\ \sum_{j=0}^{k+1} c_j^{\alpha_2} y_1^{(k+1-j)} = -y_1^{(k+1)} - b_1 x_1^{(k+1)} - x_1^{(k+1)} z_1^{(k)}, \\ \sum_{j=0}^{k+1} c_j^{\alpha_3} z_1^{(k+1-j)} = c_1 z_1^{(k)} + x_1^{(k+1)} y_1^{(k+1)} + 1, \end{cases}$$

where

$$c_0^{\alpha_1} = (\Upsilon_1(h))^{-\alpha_1}, \qquad c_0^{\alpha_2} = (\Upsilon_2(h))^{-\alpha_2}, \qquad c_0^{\alpha_3} = (\Upsilon_3(h))^{-\alpha_3};$$

and the denominator functions are defined as [7, 8]:

$$\Upsilon_1(h) = e^h - 1, \quad \Upsilon_2(h) = e^h - 1, \quad \Upsilon_3(h) = \frac{e^{c_1 h} - 1}{c_1},$$

and the values $c_j^{\alpha i}$ for all $1 \leq j \leq k+1$ are calculated using the recursive formula (2.3). Next, we present a fractional model for the Lü system, defined by the following system of nonlinear fractional differential equations.

$$\begin{cases} {}^C D^{\alpha_1} x_2(t) = a_2 y_2(t) - a_2 x_2(t), \\ {}^C D^{\alpha_2} y_2(t) = c_2 y_2(t) - x_2(t) z_2(t), \\ {}^C D^{\alpha_3} z_2(t) = x_2(t) y_2(t) - b_2 z_2(t), \end{cases} \tag{3.2}$$

where a_2, b_2 and c_2 are positive real constants. By assigning initial conditions $x_2(0) = x_2^{(0)}, y_2(0) = y_2^{(0)}$, and $z_2(0) = z_2^{(0)}$, one can consider the following NSFD scheme corresponding to the model (3.2).

$$\begin{cases} \sum_{j=0}^{k+1} c_j^{\alpha_1} x_2^{(k+1-j)} = a_2 y_2^{(k)} - a_2 x_2^{(k+1)}, \\ \sum_{j=0}^{k+1} c_j^{\alpha_2} y_2^{(k+1-j)} = c_2 y_2^{(k)} - x_2^{(k+1)} z_2^{(k)}, \\ \sum_{j=0}^{k+1} c_j^{\alpha_3} z_2^{(k+1-j)} = x_2^{(k+1)} y_2^{(k+1)} - b_2 z_2^{(k+1)}, \end{cases}$$

where

$$c_0^{\alpha_1} = (\Lambda_1(h))^{-\alpha_1}, \quad c_0^{\alpha_2} = (\Lambda_2(h))^{-\alpha_2}, \quad c_0^{\alpha_3} = (\Lambda_3(h))^{-\alpha_3}.$$

Additionally, the denominator functions $\Lambda_1(h)$, $\Lambda_2(h)$, and $\Lambda_3(h)$ were selected as follows.

$$\Lambda_1(h) = \frac{e^{a_2 h} - 1}{a_2}, \quad \Lambda_2(h) = \frac{e^{c_2 h} - 1}{c_2}, \quad \Lambda_3(h) = \frac{e^{b_2 h} - 1}{b_2}.$$

Assuming that Volta's system drives the Lü system, we define the master and slave systems as follows.

$$\begin{cases} {}^C D^{\alpha_1} x_1(t) = -x_1(t) - a_1 y_1(t) - z_1(t) y_1(t), \\ {}^C D^{\alpha_2} y_1(t) = -y_1(t) - b_1 x_1(t) - x_1(t) z_1(t), \\ {}^C D^{\alpha_3} z_1(t) = c_1 z_1(t) + x_1(t) y_1(t) + 1, \end{cases} \tag{3.3}$$

and

$$\begin{cases} {}^C D^{\alpha_1} x_2(t) = a_2 y_2(t) - a_2 x_2(t) + u_1(t), \\ {}^C D^{\alpha_2} y_2(t) = c_2 y_2(t) - x_2(t) z_2(t) + u_2(t), \\ {}^C D^{\alpha_3} z_2(t) = x_2(t) y_2(t) - b_2 z_2(t) + u_3(t). \end{cases} \tag{3.4}$$

The functions $u_1(t), u_2(t)$, and $u_3(t)$ in (3.4) represent active control functions that need to be identified. Let us define the error functions as

$$e_1(t) = x_2(t) - x_1(t), \quad e_2(t) = y_2(t) - y_1(t), \quad e_3(t) = z_2(t) - z_1(t). \tag{3.5}$$

Combining equation (3.5) with equations (3.3) and (3.4) we obtain the following error system.

$$\begin{cases} {}^{C}D^{\alpha_1}e_1(t) = -a_2e_1(t) + (1-a_2)x_1(t) + a_2y_2(t) + a_1y_1(t) + z_1(t)y_1(t) + u_1(t), \\ {}^{C}D^{\alpha_2}e_2(t) = c_2e_2(t) + (1+c_2)y_1(t) - x_2(t)z_2(t) + b_1x_1(t) + x_1(t)z_1(t) + u_2(t), \\ {}^{C}D^{\alpha_3}e_3(t) = -b_2e_3(t) - (b_2+c_1)z_1(t) + x_2(t)y_2(t) - x_1(t)y_1(t) - 1 + u_3(t). \end{cases} \tag{3.6}$$

We define the active control functions $u_i(t)$ as follows:

$$\begin{cases} u_1(t) = v_1(t) - (1-a_2)x_1(t) - a_2y_2(t) - a_1y_1(t) - z_1(t)y_1(t), \\ u_2(t) = v_2(t) - (1+c_2)y_1(t) + x_2(t)z_2(t) - b_1x_1(t) - x_1(t)z_1(t), \\ u_3(t) = v_3(t) + (b_2+c_1)z_1(t) - x_2(t)y_2(t) + x_1(t)y_1(t) + 1, \end{cases} \tag{3.7}$$

where the functions $v_1(t), v_2(t)$, and $v_3(t)$ are considered as the control inputs. By substituting equation (3.7) into equation (3.6), we obtain

$$\begin{cases} {}^{C}D^{\alpha_1}e_1(t) = -a_2e_1(t) + v_1(t), \\ {}^{C}D^{\alpha_2}e_2(t) = c_2e_2(t) + v_2(t), \\ {}^{C}D^{\alpha_3}e_3(t) = -b_2e_3(t) + v_3(t). \end{cases} \tag{3.8}$$

The error dynamics in equation (3.8) display a linear system of fractional differential equations along with the control vector.

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix},$$

where the matrix A is a real matrix chosen so that all eigenvalues λ_i of system (3.8) satisfies the stability condition in Theorem 2.1, that is, $|\arg(\lambda_i)| > \alpha_M \frac{\pi}{2}$, $i = 1, 2, 3$. If we simply choose

$$A = \begin{bmatrix} a_2 - 1 & 0 & 0 \\ 0 & -c_2 - 1 & 0 \\ 0 & 0 & b_2 - 1 \end{bmatrix},$$

then the stability condition in Theorem 2.1 is fulfilled, and so we achieve the required synchronization.

4 Simulation results

This section demonstrates numerical simulations that exhibit the intricate dynamics of The Volta and and Lü fractional–order systems. If $\alpha_1 = \alpha_2 = \alpha_3$, the Volta's system is referred to as commensurate; otherwise, it is considered incommensurate. According to [3], the minimum order α_i required for chaotic behavior is $\alpha_i > 0.99$. When $\alpha_1 = \alpha_2 = \alpha_3 = 1$, this system becomes identical to the classical integer–order Volta's system, which exhibits chaos for the parameters $(a_1, b_1, c_1) = (5, 85, 0.5)$. Similarly, the uncontrolled Lü system shows chaotic dynamics when $\alpha_i > 0.91$ with $(a_2, b_2, c_2) = (36, 3, 20)$, as reported in [1]. Figure 1 depicts the phase trajectory of the fractional–order Volta's system with commensurate order $\alpha_1 = \alpha_2 = \alpha_3 = 1$, using parameters $(a_1, b_1, c_1) = (5, 85, 0.5)$ and initial conditions $(x_1^{(0)}, y_1^{(0)}, z_1^{(0)}) = (8, 2, 1)$. The simulation runs for 20 seconds with a stepsize $h = 0.001$. Figure 2 displays chaotic dynamics in the fractional–order Lü system. The system parameters are $(a_2, b_2, c_2) = (36, 3, 20)$, with fractional derivative orders of $\alpha_1 = 0.93$, $\alpha_2 = 0.99$, and $\alpha_3 = 0.98$. The simulation is performed with initial conditions $(x_2^{(0)}, y_2^{(0)}, z_2^{(0)}) = (0.2, 0.5, 0.3)$, over a time span of 20 seconds and using a stepsize $h = 0.001$.

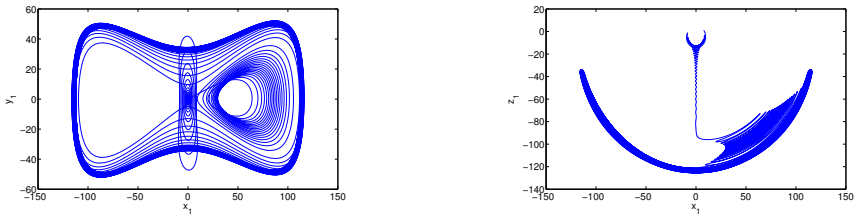


Figure 1: The chaotic attractor of the Volta's system is shown in 2D Phase planes and 3D state space for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and stepsize $h = 0.001$.

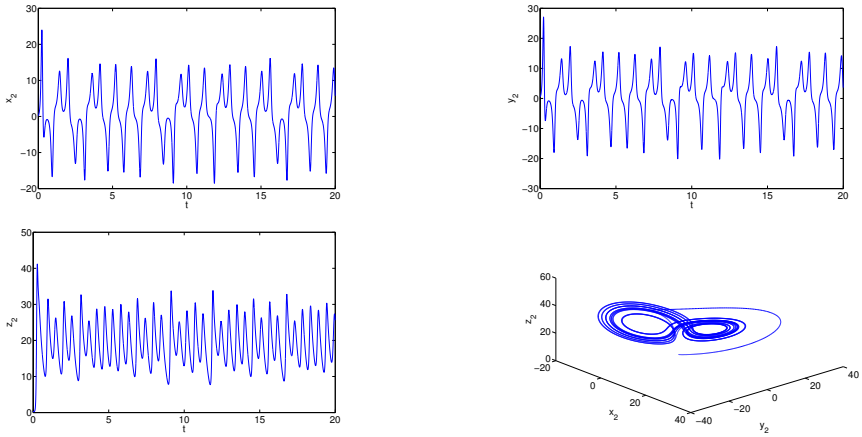


Figure 2: The chaotic attractor of the uncontrolled Lü system is displayed in both 2D phase planes and 3D state space, with derivative orders $\alpha_1 = 0.93, \alpha_2 = 0.99$, and $\alpha_3 = 0.98$ with stepsize $h = 0.001$.

Figure 3 presents the synchronization results, which include the state variables and error dynamics, for the fractional orders $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ and choosing a stepsize $h = 0.001$. This figure demonstrates that the two nonidentical systems achieve successful synchronization within a short time period, confirming the effectiveness of the implemented active control synchronization method.

5 Concluding remarks

In this study, we investigated two chaotic systems using the Caputo fractional derivative. We then developed two effective NSFD schemes to simulate these systems. Subsequently, an active controller was designed to achieve nonidentical synchronization between the fractional–order systems. The simulation results, presented in Fig. 3, verify the effectiveness of the proposed active control technique for synchronization purposes. For future work, we plan to explore optimal control methods (such as those discussed in [4, 5]) to enhance the stabilization and synchronization of chaotic attractors.

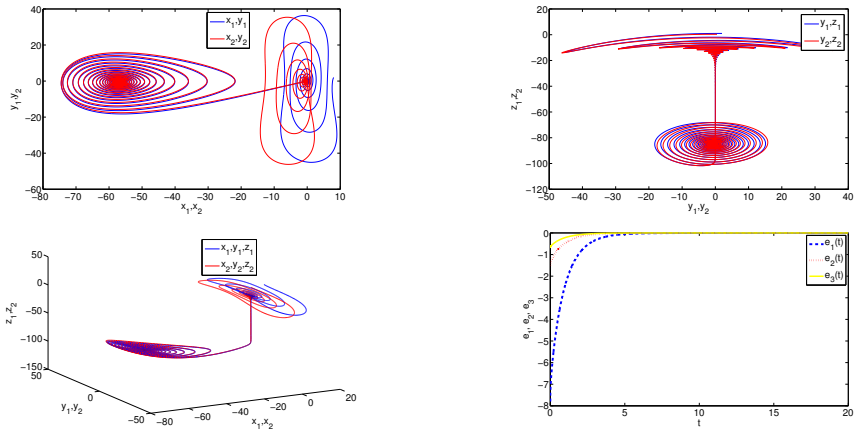


Figure 3: Synchronization results for two nonidentical fractional–order systems Volat and Lü with $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ and stepsize, $h = 0.001$.

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Comprehensive Empirical Analysis of Entropy-Based OWA Weight Generation Models

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Abstract

Ordered Weighted Averaging (OWA) operators are pivotal in multi-criteria decision making (MCDM) and information fusion. Entropy-based approaches for determining OWA weights offer robust frameworks for generating unbiased and context-sensitive aggregation behaviors. This extended abstract provides a comprehensive survey and empirical comparison of key entropy-based OWA weight generation models, including Shannon maximum entropy, Yager's entropy, M-Entropy, Bayesian entropy, and generalized entropies such as Rényi and Tsallis. A case study on environmental risk assessment evaluates four sites across five criteria, comparing model performances. Results highlight model-specific strengths, with Shannon entropy favoring decisive weighting, Rényi and Tsallis offering balanced aggregation, and M-Entropy ensuring monotonicity. The study discusses challenges like computational complexity and interpretability, proposing future research in scalable algorithms and hybrid entropy models. This work guides researchers and practitioners in selecting appropriate entropy-based OWA models for decision-making applications.

Keywords: Ordered weighted averaging, Shannon entropy, M-Entropy, Bayesian entropy, multi-criteria decision making

1 Introduction

Ordered Weighted Averaging (OWA) operators, introduced by Yager [11], are versatile tools for aggregating multi-criteria information in decision making and data fusion. The choice of OWA weights critically influences the orness (degree of optimism) and aggregation behavior. Entropy-based methods have gained prominence for their ability to model uncertainty and balance fairness with preference.

The Shannon maximum entropy model [3, 4] generates weights maximizing uncertainty under constraints like fixed orness. Filev and Yager [3] analyzed its analytic properties, while Fullér and Majlender [4] provided an analytic solution for maximal entropy weights. Extensions include generalized entropies like Rényi [6] and Tsallis [1], which offer flexibility in modeling weight distributions. Yager's entropy [10] and Bayesian entropy [2] incorporate alternative uncertainty measures and prior knowledge, respectively. M-Entropy models [**?**] enforce monotonicity, enhancing interpretability.

Applications span environmental risk assessment [5, 7], economic evaluation [9], and fuzzy decision making [8]. This extended abstract reviews these models, presents a novel empirical comparison through a case study, and discusses their practical implications and future directions.

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2 Entropy-Based OWA Weight Generation

2.1 Shannon Maximum Entropy Model

The Shannon entropy model maximizes:

$$\max_{\mathbf{w}} H_S(\mathbf{w}) = -\sum_{i=1}^{n} w_i \ln w_i \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, \quad \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad w_i \geq 0, \tag{2.1}$$

where $\alpha \in [0,1]$ is the orness parameter. Weights are derived as:

$$w_i = \frac{e^{\lambda \frac{n-i}{n-1}}}{\sum_{k=1}^{n} e^{\lambda \frac{n-k}{n-1}}}, \tag{2.2}$$

with λ solved numerically.

2.2 M-Entropy Models

subsectionM-Entropy Models (Type I and Type II)

M-Entropy models enhance interpretability by enforcing monotonicity. Type I maximizes

$$H_{M-I}(\mathbf{w}) = -w_1 \ln w_1 - \sum_{i=2}^{n} (w_i - w_{i-1}) \ln(w_i - w_{i-1}) \tag{2.3}$$

subject to

$$w_1 \leq w_2 \leq \cdots \leq w_n, \quad \sum_{i=1}^{n} w_i = 1.$$

and,

$$H_{M-I}(\mathbf{w}) = -\sum_{i=1}^{n-1} (w_i - w_{i+1}) \ln(w_i - w_{i+1}) - w_n \ln w_n \tag{2.4}$$

subject to

$$w_1 \geq w_2 \geq \cdots \geq w_n, \quad \sum_{i=1}^{n} w_i = 1.$$

M-Entropy models enhance interpretability by enforcing monotonicity. Type II maximizes

$$H_{M-II}(\mathbf{w}) = -w_1 \ln w_1 - \sum_{i=2}^{n} (w_i - w_{i-1}) \ln(w_i - w_{i-1}) - (1 - w_n) \ln(1 - w_n) \tag{2.5}$$

subject to

$$w_1 \leq w_2 \leq \cdots \leq w_n, \quad \sum_{i=1}^{n} w_i = 1.$$

and,

$$H_{M-II}(\mathbf{w}) = -(1 - w_1) \ln(1 - w_1) - \sum_{i=1}^{n-1} (w_i - w_{i+1}) \ln(w_i - w_{i+1}) - w_n \ln w_n \tag{2.6}$$

subject to

$$w_1 \geq w_2 \geq \cdots \geq w_n, \quad \sum_{i=1}^{n} w_i = 1.$$

2.3 Yager's Entropy Model

Wu et al. [10] proposed an approach for determining OWA operator weights based on the maximal Yager's entropy as follows:

$$\begin{array}{l} Max\ H_z\ (W) = Q_z\,(I) - Q_z\,(W)\,, \\ s.t. \sum_{i=1}^{n} x_i w_i = c, \\ \sum_{i=1}^{n} w_i = 1, \qquad\qquad w_i \in [0,1]\,, \qquad i = 1,2,\ldots,n. \end{array} \tag{2.7}$$

Such that, $x_i > x_j$ for $i < j$ $(i, j = 1, \ldots, n)$ and $x_i > c > x_j$, Where $Q_z\,(W)$ is Yager's entropy [12] as:

$$Q_z\,(W) = D\left(W\ ,\ \left[\frac{1}{n}\right]\right) = \left(\sum_{i=1}^{n}\left|w_i - \frac{1}{n}\right|^z\right)^{1/z}, \quad z \geq 1, \quad i = 1,2,\ldots,n.$$

Where $Q_z\,(I_i) = Q_z\,(I_j) = Q_z\,(I)$ and $[I_i]$ indicates $w_i = 1$.

2.4 Bayesian and Generalized Entropies

Bayesian entropy [2] maximizes:

$$\max_{\mathbf{w}} H_B(\mathbf{w}\|\mathbf{p}) = -\sum_{i=1}^{n} w_i \log \frac{w_i}{p_i}, \tag{2.8}$$

incorporating prior $\mathbf{p}$. Generalized entropies include Rényi:

$$H_r(\mathbf{w}) = \frac{1}{1-r} \log \sum_{i=1}^{n} w_i^r, \tag{2.9}$$

and Tsallis:

$$S_q(\mathbf{w}) = \frac{1}{q-1}\left(1 - \sum_{i=1}^{n} w_i^q\right). \tag{2.10}$$

3 Case Study: Environmental Risk Assessment

3.1 Problem Setup

We evaluate four sites (E_1–E_4) for environmental vulnerability across five criteria: Air Quality (C_1), Water Pollution (C_2), Soil Contamination (C_3), Biodiversity Impact (C_4), and Waste Management (C_5). Normalized scores (0–1, higher is safer) are provided in Table 1.

Table 1: Expert Assessment Matrix (Normalized Scores)

Site	C_1	C_2	C_3	C_4	C_5
E_1	0.40	0.65	0.70	0.50	0.60
E_2	0.55	0.60	0.65	0.75	0.55
E_3	0.60	0.45	0.40	0.70	0.75
E_4	0.50	0.70	0.55	0.60	0.65

3.2 Weight Generation and Aggregation

Weights are generated using Shannon, M-Entropy (Type I and II), Yager's, Rényi ($r = 3$), Tsallis ($q = 3.5$), and Bayesian entropy models with prior vector $P = (0.50, 0.25, 0.20, 0.04, 0.01)$ at orness $\alpha = 0.6$. Aggregation uses:

$$S_k = \sum_{i=1}^{n} w_i \cdot x_{(i)k}, \tag{3.1}$$

where $x_{(i)k}$ is the i-th largest score for site k. Results are shown in Table 2.

Table 2: Aggregated Vulnerability Scores and Rankings

Site	Shannon	Rényi	Tsallis	M-Type I	M-Type II	Yager	Bayesian
E_1	0.5984 (4)	0.5989 (4)	0.6004 (4)	0.5981 (4)	0.5977 (4)	0.6000 (4)	0.6064 (4)
E_2	0.6396 (1)	0.6373 (1)	0.6381 (1)	0.6399 (1)	0.6407 (1)	0.6400 (1)	0.6346 (1)
E_3	0.6167 (3)	0.6163 (3)	0.6174 (3)	0.6157 (3)	0.6154 (3)	0.6150 (3)	0.6219 (2)
E_4	0.6189 (2)	0.6182 (2)	0.6194 (2)	0.6188 (2)	0.6189 (2)	0.6200 (2)	0.6200 (3)

3.3 Analysis and Sensitivity

Table 3 shows weights for $\alpha = 0.6$. Shannon entropy yields the most uneven distribution, emphasizing top criteria, while Rényi and Tsallis provide smoother weights. M-Entropy ensures monotonicity, and Bayesian entropy balances prior knowledge.

Table 3: OWA Weights for Different Entropy Models

Model	w_1	w_2	w_3	w_4	w_5
Shannon	0.288	0.235	0.191	0.156	0.127
Rényi ($r = 3$)	0.272	0.242	0.207	0.166	0.110
Tsallis ($q = 3.5$)	0.263	0.243	0.217	0.181	0.094
M-Entropy I	0.297	0.221	0.190	0.165	0.125
M-Entropy II	0.306	0.213	0.185	0.164	0.130
Yager	0.300	0.200	0.200	0.200	0.100
Bayesian	0.225	0.221	0.348	0.137	0.067

Sensitivity analysis (Table 4) shows stable rankings across models, with E_2 consistently ranking first. Increasing α enhances scores for high-performing sites.

Table 4: Sensitivity of Site Rankings to Orness (α)

Site	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
			Shannon Entropy		
E_1	4	4	4	4	4
E_2	1	1	1	2	2
E_3	3	3	2	1	1
E_4	2	2	3	3	3

4 Conclusion

Entropy-based OWA models offer robust weight generation for MCDM. Shannon entropy suits decisive weighting, Rényi and Tsallis provide balanced aggregation, and M-Entropy ensures monotonicity. The case study demonstrates practical applicability, with stable rankings across models. Future work includes scalable algorithms, hybrid entropy models, and machine learning integration.

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When a group ring is nil reversible

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Abstract

In this paper, we study conditions under which a group ring $R[G]$ of a group G over a ring R becomes a nil reversible ring. The notion of nil reversible ring is a generalization of reversible ring. We see that the group ring $K[G]$ of a group G over a field K of characteristic zero is reversible if and only if $K[G]$ is nil reversible. Also, if K is a field of characteristic $p > 0$ and G is a group which has no finite normal subgroup of order divisible by p, then $K[G]$ is reversible if and only if $K[G]$ is nil reversible. Finally, we prove that the group ring $K[Q_8]$ of quaternions Q_8 over a field K of characteristic zero is reversible if and only if $K[Q_8]$ is nil reversible if and only if the equation $1 + x^2 + y^2 = 0$ is not solvable in K.
Keywords: reversible, nil reversible, group ring.

1 Introduction

In this work, all rings are associative with nonzero identity and $char(R)$ is the characteristic of R. A ring R is *reversible*, if $xy = 0$ implies $yx = 0$, for all $x, y \in R$. If for any ideal I of R, $I^2 = 0$ yields $I = 0$, then R is called *semiprime*. If R has no nonzero nilpotent element, then R is called *reduced*. A ring in which $xyz = 0$ yields $yxz = 0$, for all $x, y, z \in R$, is called *symmetric ring*.

Various generalizations of the reversible ring have been introduced so far. In this article, we study a generalization of reversible rings called nil reversible ring and investigate some properties of nil reversible rings. A ring R in which for all $r \in R$ and for each nilpotent element x of R, $rx = 0$ if and only if $xr = 0$, is called *nil reversible ring*. In this paper, we study conditions under which a group ring $R[G]$ of a group G over a ring R becomes a nil reversible ring. We see that the group ring $K[G]$ of a group G over a field K of characteristic zero is reversible if and only if $K[G]$ is nil reversible. Also, if K is a field with $char(K) = p > 0$ and G is a group which has no finite normal subgroup of order divisible by p, then $K[G]$ is reversible if and only if $K[G]$ is nil reversible if and only if $K[G]$ is symmetric. Finally, we prove that the group ring $K[Q_8]$ of quaternions Q_8 over a field K of characteristic zero is reversible if and only if $K[Q_8]$ is nil reversible if and only if the equation $1 + x^2 + y^2 = 0$ is not solvable in K.

1.1 Introduction to nil reversible ring

In this part, we study some properties of nil reversible ring. We start by recalling the definition of nil reversible ring.

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Definition 1. *A ring R is named nil reversible, if for any nilpotent element x of R and $r \in R$, we have $rx = 0$ if and only if $xr = 0$.*

It is clear that commutative rings and reversible rings are nil reversible ring. Also, subrings of a nil reversible ring is nil reversible.

Example 1.1. Let $S = \mathbb{Z}_2[x, y]$ which $xy \neq yx$ and I is an ideal of R generated by $\{yx^2, y^2x, xy\}$. If $R = \frac{S}{I}$, then the set of all nilpotent elements of R is $\{I, yx + I\}$. Therefore, R is a nil reversible ring.

Recall that an *abelian ring* is a ring in which every idempotent element is central. In the following, we see that nil reversible rings are abelian.

Proposition 1.2. *Let R be a nil reversible ring. Then R is an abelian ring.*

Proof. Let e be an idempotent element of R. For any r of R, $(er - ere)^2 = 0$ and so $er - ere$ is a nilpotent element of R. Since R is a nil reversible ring and $(er - ere)e = 0$, then $e(er - ere) = 0$. So $er = ere$. On the other hand, $re - ere$ is also a nilpotent element of R and $e(re - ere) = 0$. Thus $(re - ere)e = 0$, by hypothesis, which yields $re = ere$. From this $er = re$, for every $r \in R$, and so R is an abelian ring. □

The next Example says that an abelian ring is not necessarily nil reversible.

Example 1.3. Let R be the set of all 2×2 matrices over $\mathbb{Z}$ such as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $a - d = 2m$ and $b - c = 2n$, for $m, n \in \mathbb{Z}$. By [2, Example 2.7], R is an abelian ring. But, R is not nil reversible. Because, $\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is a nilpotent element of R. So that

$$\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

But

$$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \neq 0.$$

The next Example shows that the set of all 2×2 matrices over a reversible or nil reversible ring is not necessarily nil reversible.

Example 1.4. Let $S = M_2(\mathbb{R})$ and $0 \neq r \in \mathbb{R}$. It is explicit that $\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ is a nilpotent element of R. Now,

$$\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

But

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & r^2 \\ 0 & 0 \end{pmatrix} \neq 0.$$

From this, S is not a nil reversible ring.

We see in [4, P.P 361] that every reduced ring R is symmetric if and only if R is reversible. We have mentioned in the following Lemma to refer to it.

Lemma 1.5. *In a reduced ring R, we have the following properties:*

1. *R is a reversible ring.*
2. *R is a symmetric ring.*

2 Main results

In this section, we study conditions that the group ring $R[G]$ of a group G over a ring R is a nil reversible ring. It is a classical result, the group ring over a field of characteristic zero is semiprime. The following Lemma illustrates this fact.

Lemma 2.1. *If K is a field with $char(K) = 0$ and G is a group, then the group ring $K[G]$ is semiprime.*

[4, Theorem 11.2] which we study in Proposition 2.2, is an analogue result for the group rings of finite groups.

Proposition 2.2. *Let K be a field with $char(K) = p > 0$ and G be a group, then the group ring $K[G]$ is semiprime if and only if G has no finite normal subgroup of order divisible by p.*

Lemma 2.3. *Let R be a semiprime ring. Then R is nil reversible ring if and only if R is reduced.*

Proof. Assume R is nil reversible and for $r \in R$, $r^2 = 0$. Then $0 = sr^2 = srr = rsr$, for all $s \in R$, by nil reversibility of R. From this $r = 0$, since R is semiprime. Therefore R is reduced. Conversely, if R is a reduced ring, we conclude from Lemma 2.16 that R is reversible and so R is nil reversible. □

Lemma 2.4. *Let R be a semiprime ring. Then the following sets are equivalent:*

1. *R is reversible.*
2. *R is nil reversible.*
3. *R is reduced.*
4. *R is symmetric.*

Proof. $1 \Rightarrow 2$. It is clear.
$2 \Rightarrow 3$. We obtain this result by Lemma 2.3.
$3 \Rightarrow 4$. Every reduced ring is symmetric, by Lemma 2.16.
$4 \Rightarrow 1$. It is easily seen that every symmetric ring is reversible.

□

Theorem 2.5. *Let K be a field and $char(K) = p > 0$. If G is a group which has no finite normal subgroup of order divisible by p, then the following sets coincide:*

1. *The group ring $K[G]$ is reduced.*
2. *The group ring $K[G]$ is symmetric.*
3. *The group ring $K[G]$ is reversible.*
4. *The group ring $K[G]$ is nil reversible.*

Proof. Since $char(K) = p > 0$ and G has no finite normal subgroup of order divisible by p, the group ring $K[G]$ is semiprime, by Proposition 2.2. From this, we have this Theorem, by Lemma 1.2. □

Theorem 2.6. *Let K be a field such that $char(K) = 0$ and G be a group, then the following sets coincide:*

1. *The group ring $K[G]$ is reduced.*
2. *The group ring $K[G]$ is symmetric.*

3. *The group ring $K[G]$ is reversible.*

4. *The group ring $K[G]$ is nil reversible.*

Proof. Since K is a field with $char(K) = 0$, the group ring $K[G]$ is semiprime, by Lemma 2.1. Thus we conclude this Theorem from Lemma 1.2. □

[1, Theorem 2.1] tells us about reversibility of the group ring $K[Q_8]$ of a group of quaternions Q_8 over a field K of characteristic zero which we perusal in the next Lemma.

Lemma 2.7. *Let K be a field with $char(K) = 0$. Then the following conditions are equivalent:*

1. *The group ring $K[Q_8]$ is a reversible ring.*

2. *The equation $1 + x^2 + y^2 = 0$ is not solvable in K.*

Now, we can state the next Theorem for the nil reversibility of the group ring $K[Q_8]$ of a group of quaternions Q_8 over field K.

Corollary 2.8. *If K is a field such that $char(K) = 0$, then the following conditions are equivalent:*

1. *The group ring $K[Q_8]$ is a reversible ring.*

2. *The group ring $K[Q_8]$ is a nil reversible ring.*

3. *The group ring $K[Q_8]$ is a finite direct product of division rings.*

4. *The equation $1 + x^2 + y^2 = 0$ is not solvable in K.*

Proof. We obtain this result by [1, Theorem 2.1] and Theorem 2.6. □

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Von Neumann regular (local) and classical rings via $C_c(X)$

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Abstract

Let $C_c(X)$ be the ring of all real-valued continuous functions with countable image on a Tychonoff space X. We characterize the von Neumann regularity, classicality, and locality of $C_c(X)$ in terms of topological properties of X. Specifically, we show that $C_c(X)$ is von Neumann regular if and only if X is a CP-space, and classical if and only if X is an almost CP-space. We also prove that $C_c(X)$ is von Neumann local precisely when X is an essential CP-space. Examples are provided to demonstrate that these properties are not generally preserved under addition or multiplication. We further characterize when $C_c(X)$ is classical via its regular maximal ideals.
Keywords: CP-space, von Neumann regular rings, almost CP-space, classical ring.

1 Introduction

Throughout this paper, all topological spaces are assumed to be infinite Hausdorff completely regular spaces; that is, infinite Tychonoff spaces. Let $C(X)$ (resp. $C^*(X)$) denote the ring of all real-valued (resp. bounded) continuous functions on X. The subring of $C(X)$ consisting of functions with countable image, denoted by $C_c(X)$, forms an $\mathbb{R}$-subalgebra of $C(X)$. The subring $C_c^*(X)$ of $C_c(X)$ consists of its bounded elements. The structure and properties of $C_c(X)$ have been investigated in [1, 4, 5]. In this paper, we examine further algebraic aspects of $C_c(X)$ and contribute to the expanding literature on this subalgebra.

A *zero-dimensional space* is a Hausdorff space with a base consisting of clopen sets. As shown in [5], for any topological space X, there exists a zero-dimensional space Y that is a continuous image of X such that $C_c(X) \cong C_c(Y)$. This result permits us to restrict our attention to zero-dimensional spaces without loss of generality. It is known, due to Banaschewski, that every zero-dimensional space admits a zero-dimensional compactification, denoted by $\beta_0 X$, with the property that any continuous map $f : X \to Y$ into a zero-dimensional compact space Y extends to a continuous map $\beta_0 f : \beta_0 X \to Y$.

In particular, it is shown in [1] that for each $p \in \beta_0 X$, the maximal ideal M_c^p of $C_c(X)$ is defined by $M_c^p = \{f \in C_c(X) \mid p \in \mathrm{cl}_{\beta_0 X} Z(f)\}$.

Unless otherwise stated, all rings considered in this paper are commutative with identity ($1 \neq 0$). Given a subset A of a ring R, the annihilator of A is defined as $\mathrm{Ann}(A) = \{r \in R \mid rA = 0\}$. In particular, $\mathrm{Ann}(x)$

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denotes the annihilator of the singleton $\{x\}$. An element $x \in R$ is said to be a zero-divisor if $\mathrm{Ann}(x) \neq 0$, and a non zero-divisor otherwise.

We denote by $Q_{\mathrm{cl}}(R)$ (resp. $Q_{\max}(R)$) the classical (resp. maximal) ring of quotients of R. For each $f \in C_c(X)$, the *zero set* of f, denoted by $Z(f)$, is the set $\{x \in X \mid f(x) = 0\}$, and its complement $X \setminus Z(f)$ is called the *cozero set* of f, denoted by $\mathrm{coz}(f)$.

We establish that $C_c(X)$ is von Neumann regular if and only if X is a CP-space. Furthermore, we show that $C_c(X)$ is a classical ring if and only if X is an almost countably P-space, thereby demonstrating that von Neumann regularity implies classicality, but not conversely. We also characterize von Neumann locality in $C_c(X)$ by proving that $C_c(X)$ is von Neumann local if and only if X is an essential CP-space. In addition, we prove that the set of von Neumann regular (respectively, von Neumann local) elements in $C_c(X)$ forms a ring if and only if it coincides with $C_c(X)$ itself. Several counterexamples are constructed to illustrate that the sum or product of von Neumann regular (or local) elements need not preserve the respective property. Finally, we characterize when $C_c(X)$ is r-local in terms of the existence of a unique regular maximal ideal and provide necessary and sufficient conditions based on the structure of regular elements and their zero sets. These results offer new characterizations and further insights into the interplay between function-theoretic and ring-theoretic aspects of the functionally countable subalgebra $C_c(X)$.

Unless otherwise specified, terminology and conventions in this article follow those in the standard references [6] for rings of continuous functions and [10] for commutative ring theory.

2 Main Results

An element a in a ring R is said to be *von Neumann regular* if there exists $b \in R$ such that $a = a^2b$. It is a well-established fact that any von Neumann regular element $a \in R$ can be expressed as $a = ue$ for some unit $u \in R$ and idempotent $e \in R$. A ring R is called *von Neumann regular* if every element of R is von Neumann regular.

Following [5], a topological space X is said to be a *countably P-space* (briefly, a *CP-space*) if every zero-set in the functionally countable subalgebra $Z_c(X)$ is open. Several topological and algebraic characterizations of CP-spaces have been established in [5, Theorem 5.8]. In what follows, we provide an alternative characterization, which further illustrates the close interplay between topological properties of X and the algebraic structure of $C_c(X)$.

Theorem 2.1. *The following statements are equivalent:*

1. *X is a CP-space.*
2. *$C_c(X)$ is a von Neumann regular ring.*
3. *For $f \in C_c(X)$, $Z(f) \neq \emptyset$ implies $Z(f)$ is open.*

The following fact is the counterpart of [12, Theorem 3.3], we present it for the sake of the reader.

Proposition 2.2. *Let $A(X)$ be a ring such that $C_c^*(X) \subseteq A(X) \subseteq C_c(X)$. The following statement are equivalent:*

1. *$A(X)$ is a von Neumann regular ring.*
2. *$A(X) = C_c(X)$ is a von Neumann regular ring.*

Clearly, the set of von Neumann regular elements of a ring R is multiplicatively closed. The following shows the sum of two von Neumann regular elements in $C_c(X)$ need not be von Neumann regular.

Example 2.3. Take $X = \backslash\{0\}$ as a subspace of . Consider the ring $C_c(X)$ and define

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x > 0. \end{cases} \qquad g(x) = \begin{cases} 0 & x < 0 \\ -x & x > 0. \end{cases}$$

Since $Z(f) = Z(g)$ is open, f and g are von Neumann regular. It is easy to check that $Z(f+g) = ((-\infty, 0) \cup \{1\}) \cap$ that is not open. Thus, $f+g$ is not von Neumann regular in $C_c(X)$.

The above example suggests the following.

Theorem 2.4. *Let R be the set of all von Neumann regular elements of $C_c(X)$. The following statements are equivalent:*

1. *R is a von Neumann regular ring.*
2. *For $f, g, h, k \in R$, $Z(f) = Z(g) = Z(h) = Z(k) = \emptyset$ implies $Z(f+g+h+k)$ is open.*

A ring is *classical* if it coincides with its classical ring of quotients. Equivalently, each of its elements is a unit or a zero-divisor, see [10, pp. 320-322] for more details. It is clear that every von Neumann regular ring is classical.
Following [6], a space X is called an *almost CP-space* if for each nonempty $Z(f) \in Z_c(X)$, we have $Z(f) \neq \emptyset$. This is equivalent to saying that every element of $C(X)$ is either a unit or a zero divisor, i.e., $C_c(X)$ is classical. Just for the record, we make the following fact.

Theorem 2.5. *The following statements are equivalent:*

1. *X is an almost CP-space.*
2. *$C_c(X)$ is a classical ring.*

Corollary 2.6. *For any ring $C_c(X)$, we have*

$$\textit{von Neumann regular ring} \Rightarrow \textit{classical ring.}$$

However, the implication is not reversible.

Remark 2.7. Bhattacharjee et. al., in [2, Remark 3.9] stated that $Q_{cl}(C_c(X))$ is a von Neumann regular ring for a zero-dimensional space X. This, is unfortunately incorrect. As noted in Corollary 2.6, $Q_{cl}(C_c(X))$ where X is the one-point compactification of an uncountable discrete space is not von Neumann regular. We noted that a ring R is called *complemented* if for every $a \in R$ there is a $b \in R$ such that $ab = 0$ and $a + b$ is a non zero-divisor of R. It is well-known that a ring R is complemented if and only if the classical quotient ring $Q_{cl}(R)$ of R is von Neumann regular, see [7, Corollary 3.3] and [9, Theorem 2.5]. This means that the ring $C_c(X)$ may not be a complemented ring. With the help of Theorem 4.5 in [7], we deduce that $C_c(X)$ is a complemented ring if and only if $(C_c(X))$ (the set of minimal prime ideals of $C_c(X)$ equipped with the hull-kernel topology) is compact. For more information, the reader is referred to [6].

Following [3], An element a in a ring R is called *von Neumann local*, if either a or $1 - a$ is von Neumann regular. A ring R is called von Neumann local if each of its elements is von Neumann local. According to [11], a space X is called an *essential CP-space* whenever all points except almost one point of X are CP-point. It is not hard to see that $C_c(X)$ is a von Neumann local if and only if X is an essential CP-space. Conditions equivalent to a space being an essential CP-space are given in [11, Proposition 2.4.]. We present some other characterizations in the following proposition.

Proposition 2.8. *Let X be a topological space. The following statements are equivalent:*

1. *If f^2+g^2 is von Neumann regular in $C_c(X)$, then either f or g is von Neumann regular.*
2. *If f^2+g^2 is a unit in $C_c(X)$, then either f or g is von Neumann regular.*
3. *If $f+g$ is a unit in $C_c(X)$, either f or g is von Neumann regular.*
4. *$C_c(X)$ is a von Neumann local ring.*

The following shows the sum of two von Neumann local elements in $C_c(X)$ need not be von Neumann local.

Example 2.9. Take $X = \backslash\{0\}$ as a subspace of . Consider the ring $C_c(X)$ and define

$$f(x) = \begin{cases} 0 & x < 0 \\ 1+\cos^2 x & x > 0. \end{cases} \qquad g(x) = \begin{cases} 0 & x < 0 \\ -1-\sin^2 x & x > 0. \end{cases}$$

Since $Z(f)$ and $Z(g)$ are open, f and g are von Neumann regular. On the other hand,

$$f(x)+g(x) = \begin{cases} 0 & x < 0 \\ \cos 2x & x > 0. \end{cases} \qquad 1-(f(x)+g(x)) = \begin{cases} 1 & x < 0 \\ 2\sin^2 x & x > 0. \end{cases}$$

It is easy to verify that neither $f+g$ nor $1-(f+g)$ is not von Neumann regular in $C_c(X)$.

The following shows the product of two von Neumann local elements in $C_c(X)$ need not be von Neumann local.

Example 2.10. Take $X = \backslash\{0\}$ as a subspace of . Consider the ring $C_c(X)$ and define

$$f(x) = \begin{cases} 0 & x < 0 \\ -1-\cos^2 x & x > 0. \end{cases} \qquad g(x) = \begin{cases} 0 & x < 0 \\ -\sin^2 x & x > 0. \end{cases}$$

Since $Z(f)$ and $Z(1-g)$ is open, f and g are von Neumann local. On the other hand,

$$f(x)g(x) = \begin{cases} 0 & x < 0 \\ \sin^2 x(1+\cos^2 x) & x > 0. \end{cases} \qquad 1-f(x)g(x) = \begin{cases} 1 & x < 0 \\ \cos^4 x & x > 0. \end{cases}$$

It is easy to check that neither fg nor $1-fg$ is not von Neumann regular in $C_c(X)$.

Theorem 2.11. *Let R be the set of all von Neumann local elements of $C_c(X)$. The following statements are equivalent:*

1. *R is a von Neumann local ring.*
2. *$R = C_c(X)$ is a von Neumann local ring.*

According to [11], a space X is called an *essential almost CP-space* whenever all points except almost one point of X are almost CP-point. We close this section with a theorem that describes when X is an essential almost CP-space. Before state it, let us recall that a ring is called *r-local* if it has a unique regular maximal ideal, see [13] for more details.

Theorem 2.12. *The following statements are equivalent:*

1. *X is an essential almost CP-space.*
2. *$C_c(X)$ has at most one regular maximal ideal.*
3. *$C_c(X)$ is a classical ring or is an r-local ring.*
4. *The sum of two proper regular ideals of $C_c(X)$ is a proper regular ideal.*
5. *If f is a regular element in $C_c(X)$ then either f or $1-f$ is a unit, for every $f \in C_c(X)$.*
6. *If f, g are regular elements in $C_c(X)$ and $Z(g) \cap Z(f) = \emptyset$, then either f or g is a unit.*

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Improving MinGenSys Algorithm: A Novel Method for Computing Minimal Generator Systems

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Abstract

In this paper, we improve the MinGenSys algorithm [6], which computes a minimal generating set of parametric polynomial ideals through Gröbner systems. Our enhancement applies parametric linear algebra techniques instead of Gröbner systems to reduce the computational complexity. The Improved-MinGenSys algorithm takes a parametric polynomial ideal as input and returns a minimal generator set for the input parametric ideal so-called *minimal generator system*. We have implemented this improved algorithm in `Maple` and demonstrate its structure with a simple example.

Keywords: MinGenSys algorithm, Minimal generator system, Gröbner systems, Parametric polynomials, Improved-MinGenSys algorithm.

1 Introduction

Finding a minimal set of generators for a polynomial ideal is a significant problem in computer algebra, with applications in mathematics. Removing any generator from such a set results in a non-generating set. Schreyer's 1980 theorem [2, Theorem 3.2, page 223] provides a means to compute these minimal generators for homogeneous ideals, and the MinGenSys algorithm [6] extends this to parametric ideals.

Dehghani Darmian, in [6], utilized Gröbner systems from computer algebra to introduce and develop a *minimal generator system* for any polynomial ideal with parametric coefficients. He partitioned the parameter space into a finite number of cells, providing corresponding minimal generating set for each. He introduced the MinGenSys algorithm to compute a minimal generator system for such ideals. In the following section, we will summarize the MinGenSys algorithm.

2 Minimal Generator Systems and MinGenSys Algorithm

In this section, we restate from [6] the concept of a minimal generator system for polynomial ideal with parametric coefficients and outline the MinGenSys algorithm for its computation using Gröbner systems. Gröbner systems, an extension of Gröbner bases for polynomials with parametric coefficients introduced by Weispfenning in 1992 [18], allow us to partition the parameter space into a finite set of cells, each associated with a set of polynomials. We identify the corresponding cell for given parameter values, and the polynomials within it form a Gröbner basis for the ideal defined by those values.

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Let $\mathcal{I} = \langle f_1, \ldots, f_k \rangle \subset \mathcal{R} = \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ideal generated by the polynomials f_i. For any polynomial $f \in \mathcal{R}$, the leading monomial with respect to a monomial ordering $\prec$, denoted by $\mathrm{LM}_\prec(f)$, is the largest such monomial in f. Its coefficient is the leading coefficient, $\mathrm{LC}_\prec(f)$, and the leading term is given by $\mathrm{LT}_\prec(f) = \mathrm{LC}_\prec(f) \cdot \mathrm{LM}_\prec(f)$. The leading monomial ideal of $\mathcal{I}$ is defined as $\mathrm{LM}_\prec(\mathcal{I}) = \langle \mathrm{LM}_\prec(f) \mid f \in \mathcal{I} \rangle$. A finite subset $\{g_1, \ldots, g_m\} \subset \mathcal{I}$ is a Gröbner basis for $\mathcal{I}$ w.r.t. $\prec$ if $\mathrm{LM}_\prec(\mathcal{I}) = \langle \mathrm{LM}_\prec(g_1), \ldots, \mathrm{LM}_\prec(g_m) \rangle$. The concept of Gröbner bases and the first algorithm for computing them were introduced by Buchberger in his Ph.D. thesis under Gröbner's supervision. For further details, readers may refer to [1].
Given a sequence $F = f_1, \ldots, f_k$ of polynomials, the syzygy of F is the set of k-tuples $(g_1, \ldots, g_k) \in \mathcal{R}^k$ such that $\sum_{i=1}^{k} g_i \cdot f_i = 0$. According to Schreyer's theorem, the following important proposition is concluded:

Proposition 2.1. *Let $G = \{g_1, \ldots, g_m\} \subset \mathbb{K}[\mathbf{x}]$ be the homogeneous reduced Gröbner basis. Then $g_i \in \langle g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m \rangle$ iff i^{th} row of syzygy matrix of G contains at least one non-zero constant $k \in \mathbb{K}$.*

Using these notions, we define Gröbner bases for ideals with polynomial generators containing parametric coefficients, termed Gröbner systems. These systems find applications across mathematics, science, and engineering, particularly in parametric linear algebra [5, 7, 11], automated geometry theorem proving [14], robotics [13, 15], algebraic geometry [9, 14, 15, 18], electrical networks [15, 16], and beyond. They are especially valuable for analyses involving varying parameter values.

Gröbner systems generalize Gröbner bases for polynomial ideals over fields to those with parametric coefficients. We consider the polynomial ring $\mathcal{S} = \mathbb{K}[\mathbf{a}, \mathbf{x}]$, with $\mathbf{x} = x_1, \ldots, x_n$ as variables and $\mathbf{a} = a_1, \ldots, a_m$ as parameters, ensuring the sets $\{\mathbf{x}\}$ and $\{\mathbf{a}\}$ are disjoint. A term in $\mathcal{S}$ can be expressed as $\mathbf{a}^\gamma \mathbf{x}^\alpha$, where $\mathbf{a}^\gamma \in \mathbb{K}[\mathbf{a}]$ is a coefficient of $\mathbf{x}^\alpha$. We define two monomial orderings: $\prec_\mathbf{x}$ for the variables and $\prec_\mathbf{a}$ for the parameters. To compute Gröbner systems, we establish a product order $\prec_{\mathbf{x},\mathbf{a}}$, where for $\gamma, \delta \in \mathbb{N}^m$ and $\alpha, \beta \in \mathbb{N}^n$, we have $\mathbf{a}^\gamma \mathbf{x}^\alpha \prec_{\mathbf{x},\mathbf{a}} \mathbf{a}^\delta \mathbf{x}^\beta$ if $\mathbf{x}^\alpha \prec_\mathbf{x} \mathbf{x}^\beta$, or if both are equal and $\mathbf{a}^\gamma \prec_\mathbf{a} \mathbf{a}^\delta$. Furthermore, a specialization of parameters is a morphism $\sigma : \mathbb{K}[\mathbf{a}] \to \overline{\mathbb{K}}$, where $\overline{\mathbb{K}}$ represents the algebraic closure of $\mathbb{K}$.

A parametric polynomial ideal consists of a system of polynomial equations generated by polynomials with parametric coefficients. A parametric linear polynomial is defined as one where the power of each variable x_i is at most one, allowing parametric coefficients to be polynomials of any power in $\mathbb{K}[\mathbf{a}]$.

We now revisit the Gröbner system for parametric polynomial ideals, as introduced by Weispfenning in [18]. He demonstrated that every parametric polynomial ideal has a Gröbner system [18, Proposition 3.4 and Theorem 2.7] and proposed an algorithm for its computation [18, Theorem 3.6]. Since then, various algorithms have emerged for computing these Gröbner systems, each with its pros and cons [3, 4, 7, 8, 9, 10, 12, 15, 17].

Definition 1. *Let $F = \{f_1, \ldots, f_k\} \subset \mathcal{S}$ and $\mathcal{G} = \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$ be a finite triple set where $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$ and $G_i \subset \mathcal{S}$ are finite for $i = 1, \ldots, \ell$. The set $\mathcal{G}$ is called a Gröbner system for $\langle F \rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ over $\mathcal{V} \subseteq \overline{\mathbb{K}}^m$ if:*

- *$\sigma(G_i)$ is a Gröbner basis of $\langle \sigma(F) \rangle$ with respect to $\prec_\mathbf{x}$ for any specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \overline{\mathbb{K}}$ satisfying (N_i, W_i);*
- *$\mathcal{V} \subseteq \bigcup_{i=1}^{\ell} \mathbb{V}(N_i) \setminus \mathbb{V}(W_i)$, where $\mathbb{V}(N)$ denotes the common zeros of a set of generators for N.*

Each triple (N_i, W_i, G_i) is referred to as a branch of the Gröbner system $\mathcal{G}$ for $1 \leq i \leq \ell$. If $\mathcal{V} = \overline{\mathbb{K}}^m$, then $\mathcal{G}$ is typically called a Gröbner system of F. Each pair (N_i, W_i) is termed a specification or parametric constraint (with N_i and W_i as the null and non-null condition sets, respectively). A parametric constraint (N, W) is considered inconsistent if for any polynomial $f \in W$, f is in the radical ideal generated by N.

Example 2.2. Let $F = \{(1-c)y - ax^2, x + by^2\} \subset \mathbb{K}[a,b,c,x,y]$, where a,b,c are parameters and x,y are variables. Using our implementation of the PF_4 algorithm in `Maple`, we obtain the following Gröbner system for $\langle F\rangle$ with respect to the product ordering $y \prec_{lex} x$ and $c \prec_{lex} b \prec_{lex} a$:

$$\begin{cases} ([\,], & [ab^2], & [ab^2y^4 - y + cy, x + by^2]) \\ ([ab^2], & [c-1], & [cy - y, x + by^2]) \\ ([c-1, ab^2], & [\,], & [x + by^2]). \end{cases}$$

For example, if $a = 2$, $b = 1$, and $c = 3$, the first branch corresponds to these parameter values. Thus, $\{2y^4 - y + 3y, x + y^2\}$ forms a Gröbner basis for the ideal $\langle F\rangle_{|_{a=2,b=1,c=3}} = \langle -2y - 2x^2, x + y^2\rangle$.

We are now prepared to represent a *minimal generator system* for homogeneous parametric polynomial ideals and restate the MinGenSys algorithm for its computation. Homogeneous parametric polynomial ideals consist of homogeneous polynomials in terms of the indeterminate variables. They are widely utilized in algebraic geometry and have diverse applications in computer algebra. Therefore, determining a minimal generating set for these ideals can facilitate computations and offer a deeper understanding of their structures.

Definition 2. *Let $I \subset S = \mathbb{K}[\mathbf{a},\mathbf{x}]$ be a homogeneous parametric polynomial ideals which is a Gröbner basis w.r.t. $\prec_{\mathbf{x}}$. We call $M = \{(N_i, W_i, M_i)\}_{i=1}^{\ell}$ a* minimal generator system *of I if for each i and any specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$ satisfying the parametric constraint (N_i, W_i), we have $\sigma(M_i)$ is a minimal generating set of $\sigma(I)$.*

Based on this definition and the following extension of the proposition 2.1 we introduced the MinGenSys algorithm to compute a minimal generator system [6].

Theorem 2.3. *Let $F = \{f_1, \ldots, f_t\} \subset S$ be a reduced Gröbner basis w.r.t. $\prec_{\mathbf{x}}$ and the triple set $G = \{(G_i, N_i, W_i)\}_{i=1}^{\ell}$ is a Gröbner system of $\langle F\rangle$. For any $(G_i, N_i, W_i) \in G$ where $G_i = \{g_1, \ldots, g_r\}$ we have $g_j \in \langle g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r\rangle$ iff j^{th} row of syzygy matrix of G_i contains at least one non-zero polynomial $q \in \mathbb{K}[\mathbf{a}]$ satisfying conditions (N_i, W_i).*

In the following, we present a simple algorithm for computing a minimal generator system. Below, we use the Remove function which receives as input a set of polynomial $F \subset S$ and returns $F_i \subset F$ obtained by removing the i^{th} element of F if i^{th} row of syzygy matrix of F contains at least one polynomial $0 \neq q \in \mathbb{K}[\mathbf{a}]$.

[!t] $I = \langle f_1, \ldots, f_m\rangle \subset S = \mathbb{K}[\mathbf{a},\mathbf{x}]$ is the reduced Gröbner basis w.r.t. $\prec_{\mathbf{x}}$ A minimal generator system of I MGS := { } $G := \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$ a Gröbner system of I with respect to $\prec_{\mathbf{x},\mathbf{a}}$ $(N_i, W_i, G_i) \in G$ Syz := syzygy matrix of G_i there exists $0 \neq q \in \mathbb{K}[\mathbf{a}]$ in j^{th} row of Syz F_i:=Remove(G_i) and then Syz := syzygy matrix of F_i MGS := MGS $\bigcup\{(N_i, W_i, F_i)\}$ **Return** (MGS)

Example 2.4. Consider the following generators for the polynomial ideal $I \subset \mathbb{K}[a,b,c,d][x,y,z]$.

$[z^4, yz^3, z^2y^2, by^3 + y^2z + (d-1)z^2y, xz^3, (ac+a-2)xyz - 2z^2y + z^3a, xy^2(ac+a-2) + (-a^3c - a^3 + 2a^2)xz^2 + y^2z(ac+a-2) + 2az^2y + (-a^3bc - a^3c^2 - a^3b - a^3c + 2a^2b + 2a^2c - a^2)z^3, ax^2 - xy - yz]$

Using our `Maple` implementation of the MinGenSys algorithm, we obtain the following minimal generator system of I as follows:

$$\begin{cases} ([a],[b], & [z^4, yz^3, z^2y^2, by^3 + dyz^2 + y^2z - yz^2, xz^3, xy + yz]) \\ ([b,a],[1], & [z^4, yz^3, dyz^2 + y^2z - yz^2, xz^3, xy + yz]) \\ ([b, ac+a-2],[a], & [-az^3 + 2yz^2, adz^3 - az^3 + 2y^2z, xz^3, -cxy - cyz + 2x^2 - xy - yz]) \\ ([ac+a-2],[a,b], & [z^4, 2yz^2 - az^3, adz^3 - az^3 + 2by^3 + 2y^2z, xz^3, 2x^2 - cxy - cyz - xy - yz]) \\ \\ ([b],[a, ac+a-2], & [dyz^2 + y^2z - yz^2, xz^3, acxyz + axyz + az^3 - 2xyz - 2yz^2, 2a^2cz^3 - a^3c^2z^3 - \\ & a^3cz^3 - a^3cxz^2 - a^3xz^2 - acdyz^2 + 2a^2xz^2 - a^2z^3 + acxy^2 + acyz^2 + \\ & axy^2 - adyz^2 + 3ayz^2 + 2dyz^2 - 2xy^2 - 2yz^2, ax^2 - xy - yz]) \\ \\ ([\,],[ab(ac+a-2)], & [by^3 + dyz^2 + y^2z - yz^2, acxyz + axyz + az^3 - 2xyz - 2yz^2, acxy^2 - a^3c^2z^3 - \\ & a^3bz^3 - a^3cxz^2 - a^3cz^3 - a^3xz^2 + 2a^2bz^3 + 2a^2cz^3 + 2a^2xz^2 - a^2z^3 + \\ & acy^2z - a^3bcz^3 + axy^2 + ay^2z + 2ayz^2 - 2xy^2 - 2y^2z, ax^2 - xy - yz]). \end{cases}$$

For instance, if $a=1, b=-2, c=1$ and $d=3$ then the fourth branch corresponds to these values of parameters (these values satisfy (N_4, W_4)). Therefore, $G_4|_{a=1,b=-2,c=1,d=3} = [z^4, 2yz^2-z^3, -4y^3+2y^2z+2z^3, xz^3, 2x^2-2xy-2yz]$ will be a minimal generating set for the ideal $I\ |_{a=1,b=-2,c=1,d=3} = [z^4, yz^3, z^2y^2, -2y^3+y^2z+2yz^2, xz^3, -2yz^2+z^3, 2yz^2-z^3, x^2-xy-yz]$. This means that for any $1 \leq i \leq 6$ and for each specialization σ holding (N_i, W_i) we have $\sigma(G_i)$ is a minimal generating set of $\sigma(I)$.

3 Improved-MinGenSys Algorithm via Gaussian elimination

Computing minimal generator systems using Gröbner systems is often inefficient, particularly for parametric linear ideals [17, 7]. To avoid Gröbner system complexities, we employ a modified version of the GES algorithm [7, Algorithm 3], a parametric form of Gaussian elimination, to compute minimal generator systems based on the LDS algorithm [7].

We briefly review parametric linear algebra techniques from [7, Section 3], as background. The GES algorithm yields a Gaussian elimination system for matrices (or linear systems) with parametric entries. We apply this to polynomial ideals with parametric coefficients, linearizing the input polynomials by substituting main variables $\mathbf{x}$ to facilitate the derivation of minimal generating sets under specified conditions. The GES algorithm's efficiency is enhanced by the LDS algorithm [7, Algorithm 2], which checks for parametric linear dependencies within linear parametric polynomials without relying on Gröbner systems. The following example illustrates these concepts; see [7] for details on the LDS algorithm and its applications.

Example 3.1. Consider $g = av-bw+mz+nx-u-cy$, as a parametric polynomial, $(N, W) = ([m, n], [a-1, a-b, 1-b])$ a specification, and $G = [u+v, (a-1)y-cz, (a-b)x-acz-bv, (b-1)yx-czx+(bv+cz)y+c^2z^2]$ a linear Gröbner basis for the pair (N, W). Set the monomial orderings $n \prec_{lex} m \prec_{lex} c \prec_{lex} b \prec_{lex} a$ and $w \prec_{lex} v \prec_{lex} u \prec_{lex} z \prec_{lex} y \prec_{lex} x$. So, using our `Maple` implementation of the LDS algorithm, the linear dependency system of g on G according to (N, W) is:

$$\text{Sys} = \begin{cases} ([m,n],[c,a-1,a-b,1-b], & [false,[-1,\frac{c}{1-a},0,0],\frac{c^2z}{1-a}+(a+1)v-bw]),\\ ([m,n,c],[a-1,a+1,a-b,1-b], & [false,[-1,\frac{c}{1-a},0,0],\frac{c^2z}{1-a}+(a+1)v-bw]),\\ ([m,n,c,a+1],[b,b+1,1-b], & [false,[-1,\frac{c}{1-a},0,0],\frac{c^2z}{1-a}-bw]),\\ ([m,n,c,a+1,b],[\,], & [true,[-1,\frac{c}{1-a},0,0],0]). \end{cases}$$

The LDS algorithm enables an efficient implementation of the modified GES algorithm for computing Gaussian elimination systems of parametric polynomials. In GES, a set Sys, initially empty, evolves into a Gaussian elimination system. Each element in Sys is a triple (N, W, G), where (N, W) represents a parametric constraint, and G is the Gaussian elimination form of the corresponding parametric polynomials. $B[i]$ denotes the i-th element of list or set B. [!t] $F \subset \mathbb{R}[\mathbf{a}, \mathbf{A}]$; $\mathbf{a}$ is a sequence of parameters, and $\mathbf{A}$ represents the variables A_i. A Gaussian elimination system for F Sys:= {} $A := \{(N, W, \{\}, L[1], L)\}$ $A \neq \{\}$ $a := A[1]$ and $A := A \setminus \{a\}$ $a[5] = \{\}$ Sys:=Sys $\bigcup\{(a[1], a[2], a[3])\}$ $G := a[5] \setminus \{a[4]\}$ and $g := G[1]$ P :=LDS$(a[1], a[2], a[3], a[4])$ $i = 1$ **to** $|P|$ Let $P[i] = (N_1, W_1, [flag, Q, f])$ $flag = true$ $A := A \cup \{(N_1, W_1, a[3], g, G)\}$ $A := A \cup \{(N_1, W_1, a[3] \cup \{f\}, g, G)\}$ **Return** (Sys)

The algorithm's operation is illustrated step-by-step in [7, Example 15], with the results below.

Example 3.2. Consider $F = \{(a-b)xy+z-1, ay+x^2-2, (a-1)y^3-x^2-bz\} \subset \mathbb{K}[a, b][x, y, z]$, a set of polynomials with parametric coefficients. Applying the GES algorithm yields the following Gaussian elimination system for F:

$$\begin{cases} ([\,], & [a-b, a-1], & [axy-bxy+z-1, ay+x^2-2, ay^3-y^3+ay-bz-2]),\\ ([a-1], & [b-1], & [-bxy+xy+z-1, x^2+y-2, -bz+y-2]),\\ ([a-b], & [b-1], & [z-1, by+x^2-2, by^3-y^3+by-b-2]),\\ ([a-b, b-1], & [\,], & [z-1, x^2+y-2, y-3]). \end{cases}$$

Installing GES on the MinGenSys algorithm yields the higher-performance Improved-MinGenSys algorithm. In this algorithm, the variable MGS starts as an empty set and ends as a minimal generator system. Each entry in MGS comprises a triple (N_i, W_i, F_i'), where (N_i, W_i) represents a parametric constraint, and F_i' is a minimal generating set for the parametric polynomial ideal corresponding to (N_i, W_i). Additionally, the transformation ψ is a bijection that maps each monomial m_i, at the set of all monomials in terms of the x_i's appearing in a set $F \subset S$ into tag variables A_i. The inverse ψ^{-1} replaces any variable A_i with the corresponding monomial m_i.

[!t] $F = \langle f_1, \ldots, f_m \rangle \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ is the reduced Gröbner basis w.r.t. $\prec_{\mathbf{x}}$ A minimal generator system of F MGS := { } $L := \psi(F)$; linearization of F $G := \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$: a Gaussian elimination system of L using the GES algorithm $(N_i, W_i, G_i) \in G$ Syz := syzygy matrix of G_i there exists $0 \neq q \in \mathbb{K}[\mathbf{a}]$ in j^{th} row of Syz F_i:=Remove(G_i) Syz := syzygy matrix of F_i $F_i' := \psi^{-1}(F_i)$ MGS := MGS $\bigcup\{(N_i, W_i, F_i')\}$ **Return** (MGS)

Theorem 3.3. *The* Improved-MinGenSys *algorithm terminates after a finite number of iterations and correctly computes a minimal generator system.*

Proof. The termination of the Improved-MinGenSys algorithm follows from the termination of the GES algorithm, while its correctness aligns with that of the MinGenSys algorithm. □

4 Conclusion

The algorithms presented in this paper have been implemented using the Maple software environment. Performance benchmarks have been conducted to evaluate their efficiency rigorously. These benchmarks demonstrate a clear advantage for the Improved-MinGenSys algorithm. This improved algorithm achieves its superior performance by effectively leveraging the techniques of parametric linear algebra. The results of the benchmark comparison show that Improved-MinGenSys significantly outperforms its predecessor, MinGenSys. In contrast to Improved-MinGenSys, MinGenSys relies on the utilization of Gröbner systems as its core computational method. Despite this, Gröbner systems retain substantial value as a powerful tool for the detailed analysis of a wide range of problems. These problems arise in diverse fields, encompassing mathematics, various scientific disciplines, and numerous engineering applications. A key strength of Gröbner systems lies in their ability to simplify complex problems involving multivariable polynomials. This simplification process aids in achieving greater clarity in the solution and understanding of these complex problems. The simplification afforded by Gröbner systems is specifically applied to find minimal generating sets for parametric polynomial ideals. In this context, the technique of parameter space partitioning plays a crucial role in improving overall computational efficiency.

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Hesitant Bipolar Fuzzy ELECTRE II Approach: A New Framework for Multi-Criteria Decision Making Problems

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Abstract

This paper presents the hesitant bipolar fuzzy ELECTRE II (HBF-ELECTRE II) method for multi-criteria decision-making with hesitant bipolar fuzzy information. It addresses the challenge of modeling both satisfaction-dissatisfaction ambivalence and opinion divergence by integrating bipolar fuzzy sets and hesitant fuzzy elements. The proposed method introduces a composite distance measure evaluating both positive and negative membership. Also, we present a concordance-discordance classification system using score function and deviation degree. Finally, a numerical example illustrates the framework's effectiveness in handling conflicting expert evaluations.

Keywords: Multiple attribute decision making (MADM), Bipolar fuzzy set, Hesitant bipolar Fuzzy set, ELECTRE II method.

1 Introduction

Multi-criteria decision-making (MCDM) is a vital tool for addressing complex problems involving conflicting criteria and uncertain information. Traditional MCDM methods often struggle to capture the inherent ambiguity and bipolarity in human decision-making, where evaluations may simultaneously reflect satisfaction and dissatisfaction. To address these challenges, extensions of classical MCDM methods under fuzzy and bipolar fuzzy environments have been proposed. Among these, the ELECTRE methods stand out for their ability to handle outranking relations and incomparable alternatives. The foundation of bipolar fuzzy sets was laid by Zhang (1994), who introduced the concept to model positive and negative membership degrees. Building on this, Alghamdi et al.[1] developed bipolar fuzzy TOPSIS (BF-TOPSIS) and bipolar fuzzy ELECTRE I (BF-ELECTRE I) methods, demonstrating their applicability in scenarios where decision-makers provide bipolar evaluations. Their work highlighted the advantages of bipolar fuzzy sets in capturing dual-sided information, such as benefits and costs, satisfaction and dissatisfaction. Further advancements were made by Shumaiza et al. [2], who proposed the bipolar fuzzy ELECTRE II (BF-ELECTRE II) method. This approach extended the classical ELECTRE II method to accommodate bipolar fuzzy information. After the introduction of fuzzy sets by Zadeh [5], Torra [3] proposed hesitant fuzzy sets , added another dimension by allowing multiple possible membership degrees for an element, reflecting the hesitancy in expert opinions. The integration of hesitant fuzzy elements with bipolar fuzzy sets led to the development of hesitant bipolar fuzzy sets (HBFS), as discussed by Wei et al.[4]. This hybrid model captures both the bipolar nature of evaluations and the hesitancy in assigning precise membership degrees, making it particularly suitable for

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real-world problems with uncertain or conflicting data. This paper presents a novel framework for MCDM under hesitant bipolar fuzzy information, integrating the concepts of bipolar fuzzy sets (BFS) and hesitant fuzzy sets (HFS) to model both ambivalence and divergence in expert evaluations. The remainder of this paper is organized as follows: Section 2 reviews preliminary concepts, Section 3 details the HBF-ELECTRE II method, Section 4 presents a numerical example, and Section 5 concludes with insights and future directions.

2 Preliminaries

In this section, we introduce basic concepts related to BFSs, HBFSs and hesitant bipolar fuzzy decision matrix.

Definition 1. *[6] Let X be a universe of discourse. A **bipolar fuzzy set (BFS)** Φ on X is defined as:*

$$\Phi = \left\{ \left(x; \mu^+(x), \mu^-(x)\right) \mid x \in X \right\},$$

where:

- $\mu^+(x) \in [0,1]$ *denotes the satisfaction degree of x to a property.*
- $\mu^-(x) \in [-1,0]$ *denotes the dissatisfaction degree of x to a counter-property.*

Example 2.1. Consider a product review scenario. Let $X = \{\text{Product A}\}$. A BFS for "cost-effectiveness" could be:

$$\Phi = \{(\text{Product A}; 0.8, -0.5)\},$$

where $\mu^+ = 0.8$ reflects high satisfaction with performance, and $\mu^- = -0.5$ indicates moderate dissatisfaction with price.

Definition 2. *[3] Let X be a fixed set. A hesitant fuzzy set (HFS) on X is defined in terms of a function that returns a set of membership values:*

$$h_X = \{\langle x, h(x)\rangle | x \in X\}$$

where $h(x)$ is a non-empty subset of $[0,1]$ denoting possible membership degrees.

Definition 3. *[4] An hesitant bipolar fuzzy set (HBFS) Ψ on X is defined as:*

$$\Psi = \left\{ \left(x; \{\mu_x^+\}, \{\mu_x^-\}\right) \mid x \in X \right\},$$

where:

- $\{\mu_x^+\} \subseteq [0,1]$ *is a set of possible satisfaction degrees.*
- $\{\mu_x^-\} \subseteq [-1,0]$ *is a set of possible dissatisfaction degrees.*

Example 2.2. In a hotel evaluation, let $X = \{\text{Hotel A}\}$. An HBFS for "service quality" might be:

$$\Psi = \{(\text{Hotel A}; \{0.7, 0.8\}, \{-0.2, -0.3\})\},$$

where $\{0.7, 0.8\}$ represents experts' hesitant satisfaction ratings, and $\{-0.2, -0.3\}$ captures their uncertainty about dissatisfaction.

2.1 Hesitant Bipolar Fuzzy Decision Matrix Construction

Definition 4. *Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ a set of criteria. The hesitant bipolar fuzzy decision matrix is constructed as:*

$$H = [h_{ij}]_{m\times n} = \begin{bmatrix} (h_{11}^+, h_{11}^-) & (h_{12}^+, h_{12}^-) & \cdots & (h_{1n}^+, h_{1n}^-) \\ (h_{21}^+, h_{21}^-) & (h_{22}^+, h_{22}^-) & \cdots & (h_{2n}^+, h_{2n}^-) \\ \vdots & \vdots & \ddots & \vdots \\ (h_{m1}^+, h_{m1}^-) & (h_{m2}^+, h_{m2}^-) & \cdots & (h_{mn}^+, h_{mn}^-) \end{bmatrix}$$

where each hesitant bipolar fuzzy element (HBFE) $h_{ij} = (h_{ij}^+, h_{ij}^-)$ consists of:

- $h_{ij}^+ = \{\gamma_{ij}^1, \gamma_{ij}^2, \ldots, \gamma_{ij}^{l^+}\}$*: Positive membership degrees*
- $h_{ij}^- = \{\delta_{ij}^1, \delta_{ij}^2, \ldots, \delta_{ij}^{l^-}\}$*: Negative membership degrees*

3 BHF-ELECTRE II method

This section introduces the BHF-ELECTRE II framework for multi-criteria decision-making by integrating bipolar and hesitant fuzzy concepts. It develops composite distance measures to simultaneously evaluate positive and negative membership degrees. A classification system for concordance-discordance relations is established using score functions and deviation degrees. Hierarchical outranking relations are then defined through concordance-discordance thresholds, enabling comparison of alternatives under dual uncertainty (satisfaction-dissatisfaction ambivalence and expert hesitancy).

3.1 Distance Measure for HBFEs

For any two hesitant bipolar fuzzy elements (HBFEs) $h_{kj} = (h_{kj}^+, h_{kj}^-)$ and $h_{lj} = (h_{lj}^+, h_{lj}^-)$, the distance measure is computed as follows:

1. **Extension Rule:** If $l_{kj}^+ \neq l_{lj}^+$, extend the shorter positive membership set by adding its maximum element until both have equal length. Apply the same rule to negative memberships.
2. **Positive Distance:** $d^+(h_{kj}^+, h_{lj}^+) = \frac{1}{l}\sum_{i=1}^{l} |h_{kj,\sigma(i)}^+ - h_{lj,\sigma(i)}^+|$ where $h_{kj,\sigma(i)}^+$ denotes the i-th smallest value in h_{kj}^+.
3. **Negative Distance:** $d^-(h_{kj}^-, h_{lj}^-) = \frac{1}{l}\sum_{i=1}^{l} |h_{kj,\sigma(i)}^- - h_{lj,\sigma(i)}^-|$
4. **Composite Distance:** $d(h_{kj}, h_{lj}) = \frac{1}{2}\left(d^+(h_{kj}^+, h_{lj}^+) + d^-(h_{kj}^-, h_{lj}^-)\right)$

3.2 Score function and deviation degree HBFEs

Definition 5. *For any hesitant bipolar fuzzy element $h_{ij} = (h_{ij}^+, h_{ij}^-)$, the score functions and deviation degree are defined as:*

$$\textit{Positive Score: } \bar{\mu}_{ij}^+ = \frac{1}{l_{ij}^+}\sum_{k=1}^{l_{ij}^+}\gamma_{ij}^k, \quad \textit{Positive Deviation: } \sigma_{ij}^+ = \sqrt{\frac{1}{l_{ij}^+}\sum_{k=1}^{l_{ij}^+}(\gamma_{ij}^k - \bar{\mu}_{ij}^+)^2}$$

$$\textit{Negative Score: } \bar{\mu}_{ij}^- = \frac{1}{l_{ij}^-}\sum_{k=1}^{l_{ij}^-}\delta_{ij}^k, \quad \textit{Negative Deviation: } \sigma_{ij}^- = \sqrt{\frac{1}{l_{ij}^-}\sum_{k=1}^{l_{ij}^-}(\delta_{ij}^k - \bar{\mu}_{ij}^-)^2}$$

where:

- $l_{ij}^+ = |h_{ij}^+|$*: Number of positive membership values*
- $l_{ij}^- = |h_{ij}^-|$*: Number of negative membership values*
- $\gamma_{ij}^k \in [0,1]$*: k-th positive membership degree*
- $\delta_{ij}^k \in [-1,0]$*: k-th negative membership degree*

3.3 Concordance sets

For each pair of the alternatives A_k, A_l the hesitant bipolar fuzzy concordance sets of A_k, A_l are the sum of all those criteria where the performance of A_k is superior to A_l. They are classified into five types:

Absolute strong concordance sets $(J_{kl}^{s_1})$:

$$J_{kl}^{s_1} = \{j \mid \bar{\mu}_{kj}^+ > \bar{\mu}_{lj}^+,\ \bar{\mu}_{kj}^- > \bar{\mu}_{lj}^-,\ \sigma_{kj}^+ \le \sigma_{lj}^+,\ \sigma_{kj}^- \le \sigma_{lj}^-\}$$

strong concordance sets$(J_{kl}^{s_2})$:

$$J_{kl}^{s_2} = \{j \mid \bar{\mu}_{kj}^+ > \bar{\mu}_{lj}^+,\ \bar{\mu}_{kj}^- > \bar{\mu}_{lj}^-,\ \sigma_{kj}^+ > \sigma_{lj}^+, \sigma_{kj}^- > \sigma_{lj}^-\}$$

Medium concordance sets(J_{kl}^{m}) :

$$J_{kl}^{m} = \{j \mid (\bar{\mu}_{kj}^+ - \bar{\mu}_{lj}^+ > \bar{\mu}_{lj}^- - \bar{\mu}_{kj}^-,\ \sigma_{kj}^+ \le \sigma_{lj}^+)\ \ or\ \ (\bar{\mu}_{kj}^- - \bar{\mu}_{lj}^- > \bar{\mu}_{lj}^+ - \bar{\mu}_{kj}^+,\ \ \sigma_{kj}^- \le \sigma_{lj}^-)\}$$

Weak concordance sets$(J_{kl}^{w_1})$:

$$J_{kl}^{w_1} = \{(\bar{\mu}_{kj}^+ - \bar{\mu}_{lj}^+ > \bar{\mu}_{lj}^- - \bar{\mu}_{kj}^-,\ \sigma_{kj}^+ > \sigma_{lj}^+)\ \ or\ \ (\bar{\mu}_{kj}^- - \bar{\mu}_{lj}^- > \bar{\mu}_{lj}^+ - \bar{\mu}_{kj}^+,\ \ \sigma_{kj}^- > \sigma_{lj}^-)\}$$

strictly weak concordance sets$(J_{kl}^{w_2})$:

$$J_{kl}^{w_2} = \{j \mid \bar{\mu}_{kj}^+ = \bar{\mu}_{lj}^+,\ \bar{\mu}_{kj}^- = \bar{\mu}_{lj}^-,\ \sigma_{kj}^+ < \sigma_{lj}^+, \sigma_{kj}^- < \sigma_{lj}^-\}$$

3.4 discordance sets

Hesitant bipolar fuzzy discordance sets, comprising criteria where A_k is inferior to A_l, fall into following categories:

Absolute strong discordance sets $(I_{kl}^{s_1})$:

$I_{kl}^{s_1} = \{j \mid \bar{\mu}_{kj}^{+} < \bar{\mu}_{lj}^{+},\ \bar{\mu}_{kj}^{-} < \bar{\mu}_{lj}^{-},\ \sigma_{kj}^{+} > \sigma_{lj}^{+},\ \sigma_{kj}^{-} > \sigma_{lj}^{-}\}$

strong discordance sets$(I_{kl}^{s_2})$:

$I_{kl}^{s_2} = \{j \mid \bar{\mu}_{kj}^{+} < \bar{\mu}_{lj}^{+},\ \bar{\mu}_{kj}^{-} < \bar{\mu}_{lj}^{-},\ \sigma_{kj}^{+} \leq \sigma_{lj}^{+}, \sigma_{kj}^{-} \leq \sigma_{lj}^{-}\}$

Medium concordance sets(I_{kl}^{m}) :

$I_{kl}^{m} = \{j \mid (\bar{\mu}_{kj}^{+} - \bar{\mu}_{lj}^{+} < \bar{\mu}_{lj}^{-} - \bar{\mu}_{kj}^{-},\ \sigma_{kj}^{+} > \sigma_{lj}^{+})\ \ or\ \ (\bar{\mu}_{kj}^{-} - \bar{\mu}_{lj}^{-} < \bar{\mu}_{lj}^{+} - \bar{\mu}_{kj}^{+},\ \ \sigma_{kj}^{-} > \sigma_{lj}^{-})\}$

Weak concordance sets(I_{kl}^{w}) :

$I_{kl}^{w} = \{j \mid (\bar{\mu}_{kj}^{+} - \bar{\mu}_{lj}^{+} < \bar{\mu}_{lj}^{-} - \bar{\mu}_{kj}^{-},\ \sigma_{kj}^{+} \leq \sigma_{lj}^{+})\ \ or\ \ (\bar{\mu}_{kj}^{-} - \bar{\mu}_{lj}^{-} < \bar{\mu}_{lj}^{+} - \bar{\mu}_{kj}^{+},\ \ \sigma_{kj}^{-} \leq \sigma_{lj}^{-})\}$

strictly weak discordance sets$(I_{kl}^{w_2})$:

$I_{kl}^{w_2} = \{j \mid \bar{\mu}_{kj}^{+} = \bar{\mu}_{lj}^{+},\ \bar{\mu}_{kj}^{-} = \bar{\mu}_{lj}^{-},\ \sigma_{kj}^{+} > \sigma_{lj}^{+},\ \ \sigma_{kj}^{-} > \sigma_{lj}^{-}\}$

3.5 Indifferent sets

Apart from the above mentioned hesitant bipolar fuzzy concordance (discordance) sets, we define the hesitant bipolar fuzzy indifferent sets as $I' = \{j \mid \text{j has one of the following condtions}\}$:

(1) $\bar{\mu}_{kj}^{+} = \bar{\mu}_{lj}^{+},\ \bar{\mu}_{kj}^{-} = \bar{\mu}_{lj}^{-},\ \sigma_{kj}^{+} = \sigma_{lj}^{+}, \sigma_{kj}^{-} = \sigma_{lj}^{-}$

(2) $\bar{\mu}_{kj}^{+} = \bar{\mu}_{lj}^{+},\ \bar{\mu}_{kj}^{-} = \bar{\mu}_{lj}^{-},\ \sigma_{kj}^{+} < \sigma_{lj}^{+}, \sigma_{kj}^{-} > \sigma_{lj}^{-}$

(3) $\bar{\mu}_{kj}^{+} = \bar{\mu}_{lj}^{+},\ \bar{\mu}_{kj}^{-} = \bar{\mu}_{lj}^{-},\ \sigma_{kj}^{+} > \sigma_{lj}^{+}, \sigma_{kj}^{-} < \sigma_{lj}^{-}$

3.6 Construction of HBF concordance (discordance) index

The concordance index c_{kl} of A_k and A_l in the HBF-ELECTRE II method are formulated as:

$$c_{kl} = \frac{\sum\limits_{j \in J^{s1}} \omega_{s1} w_j + \sum\limits_{j \in J^{s2}} \omega_{s2} w_j + \sum\limits_{j \in J^{m}} \omega_{m} w_j + \sum\limits_{j \in J^{w1}} \omega_{w1} w_j + \sum\limits_{j \in J^{w2}} \omega_{w2} w_j + \sum\limits_{j \in I'} \omega'}{\sum\limits_{j=1}^{n} w_j}$$

The discordance index d_{kl} of A_k and A_l in the HBF-ELECTRE II method are computed as:

$$d_{kl} = \frac{\max\limits_{j \in I_{kl}} \{\omega_{s1} \cdot d(h_{kj}, h_{lj}), \omega_{s2} \cdot d(h_{kj}, h_{lj}), \omega_{m} \cdot d(h_{kj}, h_{lj}, \omega_{w1} \cdot d(h_{kj}, h_{lj}), \omega_{w2} \cdot d(h_{kj}, h_{lj}))\}}{\max\limits_{1 \leq j \leq n} w_j d(h_{kj}, h_{lj})}$$

where $I_{kl} = I_{kl}^{s1} \cup I_{kl}^{s2} \cup I_{kl}^{m} \cup I_{kl}^{w}$.

3.7 Construction of Outranking Relations

The BHF-ELECTRE II method establishes three embedded outranking relations through systematic comparison of concordance/discordance indices against predefined thresholds. Let c^*, c^0, c^- denote

decreasing concordance levels ($0 < c^- < c^0 < c^* < 1$) and d^*, d^0 represent increasing discordance levels ($0 < d^0 < d^* < 1$). The outranking relations are defined as:

$$\textbf{Strong Outranking } (A_k S^s A_l): \begin{cases} c_{kl} \geq c^* \\ d_{kl} \leq d^* \\ c_{kl} \geq c_{lk} \end{cases}$$

$$\textbf{Medium Outranking } (A_k S^m A_l): \begin{cases} c_{kl} \geq c^0 \\ d_{kl} \leq d^0 \\ c_{kl} \geq c_{lk} \end{cases}$$

$$\textbf{Weak Outranking } (A_k S^w A_l): \begin{cases} c_{kl} \geq c^- \\ d_{kl} \leq d^* \\ c_{kl} \geq c_{lk} \end{cases}$$

The thresholds are typically set through sensitivity analysis or DM preference:

$$(c^*, c^0, c^-) = (0.75, 0.60, 0.45), \quad (d^*, d^0) = (0.65, 0.50)$$

3.8 Exploration of Outranking Relations

The strong and weak outranking graphs are analyzed through an iterative distillation process to determine the final ranking of alternatives, following the methodology outlined in [2]. This procedure yields a forward ranking ϑ' , a reverse ranking ϑ'', and an average ranking $\vartheta(A_k) = \frac{\vartheta'(A_k)+\vartheta''(A_k)}{2}$. Alternatives are sorted by ascending ϑ values. We summarize the proposed HBF-ELECTRE II method by the following algorithm. [H] [1] **Input:**

- BHF decision matrix $H = [h_{ij}]_{m\times n}$
- Criterion weights $W = (w_1, ..., w_n)$
- Attitude weights of absolute strong, strong, medium, weak, strictly weak and concordance/ discordance sets and the weights of the indifferent sets $\Omega = (\omega_{s1}, \omega_{s2}, \omega_m, \omega_{w1}, \omega_{w2}, \omega')$

Compute score and deviation of the evaluation value of each alternative with respect to each criterion following definition 3. Construct concordance sets $\forall(A_k, A_l)$. Calculate comprehensive concordance indices c_{kl} compute bipolar distances $\forall(k, l, j)$, compute $d(h_{kj}, h_{lj})$ using composite distance formula Build discordance Sets $\forall(A_k, A_l)$. Calculate comprehensive concordance indices d_{kl} Construct the outranking relations from the given concordance and discordance levels Perform distillation procedures:

1. Forward ranking ϑ' using descending distillation
2. Reverse ranking ϑ'' using ascending distillation
3. Final ranking $\vartheta = \frac{\vartheta'+\vartheta''}{2}$

4 Numerical example

In this section, we employ a numerical example to validate the efficacy of the proposed HBF-ELECTRE II method.

Example 4.1. A popular travel platform aims to rank five hotels (h_1 to h_5) based on user reviews and expert evaluations. The criteria include:

- c_1: Room Comfort (Weight $w_1 = 0.25$)
- c_2: Price Competitiveness ($w_2 = 0.30$)
- c_3: Location Accessibility ($w_3 = 0.20$)
- c_4: Amenities ($w_4 = 0.25$)

HBFS Ratings: Experts and users provided hesitant bipolar evaluations to capture uncertainty. For instance, conflicting reviews about "Room Comfort" for h_1 were aggregated as {0.7, 0.8} (satisfaction) and {-0.1, -0.2} (dissatisfaction). The full HBFS decision matrix is shown in Table 1.

Table 1: HBFS Ratings on Travel Platform

Hotel	**Room Comfort (c_1)**	**Price (c_2)**	**Location (c_3)**	**Amenities (c_4)**
h_1	$(\{0.7, 0.8\}, \{-0.1, -0.2\})$	$(\{0.6\}, \{-0.5\})$	$(\{0.9\}, \{-0.1\})$	$(\{0.8, 0.9\}, \{-0.3\})$
h_2	$(\{0.5, 0.6\}, \{-0.3\})$	$(\{0.7, 0.8\}, \{-0.2, -0.3\})$	$(\{0.7\}, \{-0.2\})$	$(\{0.6\}, \{-0.4\})$
h_3	$(\{0.9\}, \{-0.1\})$	$(\{0.4\}, \{-0.6\})$	$(\{0.8, 0.85\}, \{-0.1\})$	$(\{0.7\}, \{-0.2\})$
h_4	$(\{0.6\}, \{-0.2\})$	$(\{0.5, 0.6\}, \{-0.4\})$	$(\{0.9\}, \{-0.1\})$	$(\{0.9\}, \{-0.1\})$
h_5	$(\{0.8\}, \{-0.2\})$	$(\{0.7\}, \{-0.3\})$	$(\{0.6, 0.7\}, \{-0.2\})$	$(\{0.5, 0.6\}, \{-0.3\})$

After implementing the HBF-ELECTRE II method with the provided data, we present the final hotel ranking. For brevity, only selected calculations are shown as examples.

Case 1: h_4 vs h_3

Criterion c_1:

$$\bar{\mu}_4^+ = 0.60 < \bar{\mu}_3^+ = 0.90,\ \sigma_4^+ = \sigma_3^+ = 0.00 \Rightarrow I_{43}^{s2}$$

Criterion c_2:

$$\bar{\mu}_4^+ = 0.55 > \bar{\mu}_3^+ = 0.40,\ \sigma_4^+ = \sigma_3^+ = 0.00 \Rightarrow J_{43}^{s1}$$

Criterion c_3:

$$\bar{\mu}_4^+ = \bar{\mu}_3^+ = 0.90,\ \sigma_4^+ = 0.00 < \sigma_3^+ = 0.025 \Rightarrow J_{43}^{w2}$$

Criterion c_4:

$$\bar{\mu}_4^+ = 0.90 > \bar{\mu}_3^+ = 0.70,\ \sigma_4^+ = \sigma_3^+ = 0.00 \Rightarrow J_{43}^{s1}$$

Concordance Index c_{43}:

$$c_{43} = \frac{\omega_{s1} w_2 + \omega_{w2} w_3 + \omega_{s1} w_4}{\sum w_j} = \frac{1 \times 0.30 + 0.3 \times 0.20 + 1 \times 0.25}{1.00} = 0.665$$

Discordance Index d_{43}:

$$d_{43} = \frac{\max\left(\omega_{s2} \cdot d(c_1)\right)}{\max(w_j d_{ij})} = \frac{0.9 \times 0.15}{0.30} = 0.45 \quad (\leq d^* = 0.65)$$

Outranking Relation:

$$c_{43} = 0.665 \geq c^0 = 0.60 \ \wedge \ d_{43} = 0.45 \leq d^0 = 0.50 \ \Rightarrow \ h_4 S^m h_3$$

Table 2: Final Ranking

Hotel	h_4	h_3	h_1	h_5	h_2
ϑ' (Forward)	1	2	3	4	5
ϑ'' (Reverse)	1	2	3	4	5
ϑ (Average)	1.0	2.0	3.0	4.0	5.0

Final Results

$$h_4 \succ h_3 \succ h_1 \succ h_5 \succ h_2$$

This ranking reflects h_4's superior performance in criteria like amenities and location consistency, while h_2 ranks lowest due to poor price competitiveness and higher deviation in comfort evaluations.

5 Conclusion

The HBF-ELECTRE II method represents a theoretical and practical advancement in MCDM by unifying bipolarity and hesitancy into a single, coherent framework. Its adaptability suggests broad applicability in fields like supply chain management, healthcare decision-making, and financial risk assessment, where expert judgments often involve dual-sided and uncertain evaluations. By bridging the gap between bipolar and hesitant fuzzy theories, this work opens new avenues for more nuanced and robust decision-making tools.

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Multi-Criteria Decision-Making Methods under Hesitant Bipolar Fuzzy Information

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Abstract

This paper introduces two novel algorithms for ranking hesitant bipolar fuzzy sets (HBFS) in multi-criteria decision making (MCDM). The first method extends the TOPSIS technique using Hausdorff distance to directly process hesitant evaluations without information loss. The second approach converts HBFS to aggregated bipolar fuzzy sets (BFS) through averaging, then applies Euclidean distance-based TOPSIS. A real-world case study on hotel ranking for a travel platform demonstrates the comparative efficacy of these methods. Results reveal that the Hausdorff-based technique achieves higher stability by retaining inherent hesitancy information, such as multi-valued satisfaction-dissatisfaction assessments, while the aggregation method exhibits reduced sensitivity to opinion divergence.

Keywords: Multi-criteria decision-making, Hesitant bipolar fuzzy sets, TOPSIS method, Hausdorff distance.

1 Introduction

Multi-criteria decision-making (MCDM) has evolved significantly since Zadeh's seminal work on fuzzy sets [5], which introduced mathematical tools to handle imprecise information. Subsequent extensions addressed more complex uncertainties: Zhang [6] pioneered bipolar fuzzy sets (BFS) to model dual satisfaction-dissatisfaction evaluations, while Torra [3] developed hesitant fuzzy sets (HFS) to capture multi-valued membership degrees. The integration of these concepts led to hesitant bipolar fuzzy sets (HBFS) [4], which simultaneously manage bipolarity and hesitancy—critical features in real-world scenarios with conflicting expert opinions. Specifically, methods that aggregate multi-valued memberships into single values [4] may obscure critical divergence in assessments, while distance metrics designed for crisp bipolar sets [6] prove inadequate for HBFS comparisons. The bipolar fuzzy MCDM domain has witnessed substantial theoretical developments in recent years. Alghamdi et al. [1] laid critical groundwork by defining aggregation operators for bipolar fuzzy information, enabling fusion of conflicting satisfaction-dissatisfaction assessments. Building on this foundation, Shumaiza et al. [2] advanced the field by adapting the ELECTRE II method to bipolar fuzzy environments. However, existing approaches face limitations when processing inherently hesitant evaluations without information loss. Specifically, methods that aggregate multi-valued memberships into single values [4] may obscure critical divergence in assessments, while distance metrics designed for crisp bipolar sets [6] prove inadequate for HBFS comparisons. This paper addresses these gaps by proposing two novel TOPSIS methods for HBFS environments. Our first method introduces a Hausdorff distance-based

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technique that directly processes hesitant elements, preserving the full spectrum of evaluative information. The second approach converts HBFS to averaged BFS through expectation operations, then applies Euclidean distance measures. We demonstrate these methods' comparative efficacy through a hotel ranking case study for travel platforms. The remainder of this paper is organized as follows: Section 2 revisits foundational concepts of BFS, HFS, and HBFS, formalizing their mathematical relationships. Section 3 introduces two novel HBFS-TOPSIS variants—the Hausdorff distance-based method for direct hesitant element processing and the aggregation-driven approach for simplified implementation. Section 4 presents a comparative case study on hotel ranking. Section 5, concludes with insights and charted future pathways to advance hybrid MCDM frameworks.

2 Preliminaries

In this section, we introduce basic concepts related to BFSs, HBFSs and hesitant bipolar fuzzy decision matrix.

Definition 1. *[6] Let X be a universe of discourse. A **bipolar fuzzy set (BFS)** Φ on X is defined as:*

$$\Phi = \left\{ \left(x; \mu^+(x), \mu^-(x) \right) \mid x \in X \right\},$$

where:

- $\mu^+(x) \in [0,1]$ *denotes the satisfaction degree of x to a property.*
- $\mu^-(x) \in [-1,0]$ *denotes the dissatisfaction degree of x to a counter-property.*

Example 2.1. Consider a product review scenario. Let $X = \{\text{Product A}\}$. A BFS for "cost-effectiveness" could be:

$$\Phi = \{(\text{Product A}; 0.8, -0.5)\},$$

where $\mu^+ = 0.8$ reflects high satisfaction with performance, and $\mu^- = -0.5$ indicates moderate dissatisfaction with price.

Definition 2. *[3] Let X be a fixed set. A hesitant fuzzy set (HFS) on X is defined in terms of a function that returns a set of membership values:*

$$h_X = \{\langle x, h(x)\rangle | x \in X\}$$

where $h(x)$ is a non-empty subset of $[0,1]$ denoting possible membership degrees.

Definition 3. *[4] An hesitant bipolar fuzzy set (HBFS) Ψ on X is defined as:*

$$\Psi = \left\{ \left(x; \{\mu_x^+\}, \{\mu_x^-\} \right) \mid x \in X \right\},$$

where:

- $\{\mu_x^+\} \subseteq [0,1]$ *is a set of possible satisfaction degrees.*
- $\{\mu_x^-\} \subseteq [-1,0]$ *is a set of possible dissatisfaction degrees.*

Example 2.2. In a hotel evaluation, let $X = \{\text{Hotel A}\}$. An HBFS for "service quality" might be:

$$\Psi = \{(\text{Hotel A}; \{0.7, 0.8\}, \{-0.2, -0.3\})\},$$

where $\{0.7, 0.8\}$ represents experts' hesitant satisfaction ratings, and $\{-0.2, -0.3\}$ captures their uncertainty about dissatisfaction.

2.1 Hesitant Bipolar Fuzzy Decision Matrix Construction

Definition 4. *Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ a set of criteria. The hesitant bipolar fuzzy decision matrix is constructed as:*

$$H = [h_{ij}]_{m\times n} = \begin{bmatrix} (h_{11}^+, h_{11}^-) & (h_{12}^+, h_{12}^-) & \cdots & (h_{1n}^+, h_{1n}^-) \\ (h_{21}^+, h_{21}^-) & (h_{22}^+, h_{22}^-) & \cdots & (h_{2n}^+, h_{2n}^-) \\ \vdots & \vdots & \ddots & \vdots \\ (h_{m1}^+, h_{m1}^-) & (h_{m2}^+, h_{m2}^-) & \cdots & (h_{mn}^+, h_{mn}^-) \end{bmatrix}$$

where each hesitant bipolar fuzzy element (HBFE) $h_{ij} = (h_{ij}^+, h_{ij}^-)$ consists of:

- $h_{ij}^+ = \{\gamma_{ij}^1, \gamma_{ij}^2, \ldots, \gamma_{ij}^{l^+}\}$*: Positive membership degrees*
- $h_{ij}^- = \{\delta_{ij}^1, \delta_{ij}^2, \ldots, \delta_{ij}^{l^-}\}$*: Negative membership degrees*

3 HBFS-TOPSIS Methods

This section presents two novel approaches for extending the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method to handle Hesitant Bipolar Fuzzy Sets (HBFS). The proposed methods address multi-criteria decision-making problems where experts' assessments exhibit both bipolarity (positive/negative membership degrees) and hesitancy (multiple possible values). We introduce two distinct strategies: a Hausdorff distance-based technique that directly operates on HBFS structures, and an aggregation-based approach that converts HBFS evaluations into conventional Bipolar Fuzzy Sets (BFS) through averaging. Each method preserves different aspects of the original HBFS information while facilitating effective alternative ranking.

3.1 Method 1: Hausdorff Distance-Based HBFS-TOPSIS

This section proposes HBFS-TOPSIS with the hausdorff metric that preserves the original hesitant information through direct Hausdorff distance operations, making it preferable when maintaining evaluation diversity is crucial. [H] [1] **Step 1: Weight Determination** Establish criterion weights $W = (w_1, ..., w_n)$ using: - Analytic Hierarchy Process (AHP) - Entropy-based weighting - Expert consensus method **Step 2: Decision Matrix Construction** Build HBFS decision matrix $D = [(\{\mu_{ij}^+\}, \{\mu_{ij}^-\})]_{m\times n}$ where: - $\{\mu_{ij}^+\}$: Set of possible positive membership degrees - $\{\mu_{ij}^-\}$: Set of possible negative membership degrees **Step 3: Ideal Solutions Identification** Compute Hesitant Bipolar Fuzzy Positive Ideal Solution (HBFPIS):

$$\mu_j^{+*} = \bigcup_i \{\mu_{ij}^+\}, \text{ selecting maximum values } \forall j$$

$$\mu_j^{-*} = \bigcup_i \{\mu_{ij}^-\}, \text{ selecting maximum values } \forall j$$

Compute Hesitant Bipolar Fuzzy Negative Ideal Solution (HBFNIS):

$$\mu_j^- = \bigcup_i \{\mu_{ij}^+\}, \text{ selecting minimum values } \forall j$$

$$\mu_j^+ = \bigcup_i \{\mu_{ij}^-\}, \text{ selecting minimum values } \forall j$$

Step 4: Distance Computation Calculate separation measures using weighted Hausdorff distances:

$$D_i^+ = \sum_{j=1}^{n} w_j \cdot \max\left(\sup_{a\in\mu_{ij}^+} \inf_{b\in\mu_j^{+*}} |a-b|, \sup_{c\in\mu_{ij}^-} \inf_{d\in\mu_j^{-*}} |c-d| \right)$$

$$D_i^- = \sum_{j=1}^{n} w_j \cdot \max\left(\sup_{a\in\mu_{ij}^+} \inf_{b\in\mu_j^-} |a-b|, \sup_{c\in\mu_{ij}^-} \inf_{d\in\mu_j^+} |c-d| \right)$$

Step 5: Closeness Coefficient Calculation Determine relative closeness to ideal solutions:

$$C_i = \frac{D_i^-}{D_i^+ + D_i^-}, \quad 0 \le C_i \le 1$$

Step 6: Alternatives Ranking Sort alternatives in descending order of C_i values

3.2 Method 2: Aggregated Bipolar Fuzzy TOPSIS

This section proposes aggregated bipolar fuzzy TOPSIS method that provides computational efficiency through aggregation, suitable for large-scale problems. [H] [1] **Step 1: Criterion Weight Determination** Determine weight vector $W = (w_1, w_2, \ldots, w_n)$ using: - Analytic Hierarchy Process (AHP) - Entropy-based weight calculation - Expert consensus method Ensure $\sum_{j=1}^{n} w_j = 1$

Step 2: HBFS Decision Matrix Construction Build hesitant bipolar fuzzy decision matrix $D = [(\{\mu_{ij}^+\}, \{\mu_{ij}^-\})]_{m\times n}$ **Step 3: HBFS-to-BFS Conversion** Convert to BFS via averaging:

$$\bar{\mu}_{ij}^+ = \frac{1}{|\mu_{ij}^+|} \sum_{\mu\in\mu_{ij}^+} \mu \quad \text{(Average positive membership)}$$

$$\bar{\mu}_{ij}^- = \frac{1}{|\mu_{ij}^-|} \sum_{\nu\in\mu_{ij}^-} \nu \quad \text{(Average negative membership)}$$

Step 4: Aggregated Decision Matrix Formation Construct normalized BFS matrix $\bar{D} = [(\bar{\mu}_{ij}^+, \bar{\mu}_{ij}^-)]_{m\times n}$

Step 5: Ideal Solutions Identification Determine bipolar ideal solutions:

$$A^+ = \left(\max_i \bar{\mu}_{i1}^+, \ldots, \max_i \bar{\mu}_{in}^+ \right) \quad \text{(Positive Ideal)}$$

$$A^- = \left(\min_i \bar{\mu}_{i1}^-, \ldots, \min_i \bar{\mu}_{in}^- \right) \quad \text{(Negative Ideal)}$$

Step 6: Distance Computation Calculate weighted Euclidean distances:

$$d_i^+ = \sqrt{\sum_{j=1}^{n} w_j \left[(\bar{\mu}_{ij}^+ - \mu_j^{+*})^2 + (\bar{\mu}_{ij}^- - \mu_j^{-*})^2 \right]}$$

$$d_i^- = \sqrt{\sum_{j=1}^{n} w_j \left[(\bar{\mu}_{ij}^+ - \mu_j^-)^2 + (\bar{\mu}_{ij}^- - \mu_j^+)^2 \right]}$$

Step 7: Closeness Coefficient Calculation Compute relative closeness:

$$C_i = \frac{d_i^-}{d_i^+ + d_i^-} \quad \text{where } 0 \leq C_i \leq 1$$

Step 8: Alternatives Ranking Rank alternatives by descending C_i values Optimal alternative: $A^* = \arg\max_{1 \leq i \leq m} C_i$

4 Case Study: Hotel Selection on a Travel Platform

In this section, we employ a numerical example to validate the efficacy of the proposed methods.

Example 4.1. A popular travel platform aims to rank five hotels (h_1 to h_5) based on user reviews and expert evaluations. The criteria include:

- c_1: Room Comfort (Weight $w_1 = 0.25$)
- c_2: Price Competitiveness ($w_2 = 0.30$)
- c_3: Location Accessibility ($w_3 = 0.20$)
- c_4: Amenities ($w_4 = 0.25$)

HBFS Ratings: Experts and users provided hesitant bipolar evaluations to capture uncertainty. For instance, conflicting reviews about "Room Comfort" for h_1 were aggregated as {0.7, 0.8} (satisfaction) and {-0.1, -0.2} (dissatisfaction). The full HBFS decision matrix is shown in Table 1.

Table 1: HBFS Ratings on Travel Platform

Hotel	**Room Comfort (c_1)**	**Price (c_2)**	**Location (c_3)**	**Amenities (c_4)**
h_1	$(\{0.7, 0.8\}, \{-0.1, -0.2\})$	$(\{0.6\}, \{-0.5\})$	$(\{0.9\}, \{-0.1\})$	$(\{0.8, 0.9\}, \{-0.3\})$
h_2	$(\{0.5, 0.6\}, \{-0.3\})$	$(\{0.7, 0.8\}, \{-0.2, -0.3\})$	$(\{0.7\}, \{-0.2\})$	$(\{0.6\}, \{-0.4\})$
h_3	$(\{0.9\}, \{-0.1\})$	$(\{0.4\}, \{-0.6\})$	$(\{0.8, 0.85\}, \{-0.1\})$	$(\{0.7\}, \{-0.2\})$
h_4	$(\{0.6\}, \{-0.2\})$	$(\{0.5, 0.6\}, \{-0.4\})$	$(\{0.9\}, \{-0.1\})$	$(\{0.9\}, \{-0.1\})$
h_5	$(\{0.8\}, \{-0.2\})$	$(\{0.7\}, \{-0.3\})$	$(\{0.6, 0.7\}, \{-0.2\})$	$(\{0.5, 0.6\}, \{-0.3\})$

4.1 Ranking by Method 1

Step 3: Define Ideal Solutions

- **HBFPIS:** Maximum satisfaction and minimum dissatisfaction:

$$\begin{aligned}
&\mu_1^{+*} = 0.9, \quad \mu_1^{-*} = -0.1 \quad \text{(Room Comfort)} \\
&\mu_2^{+*} = 0.8, \quad \mu_2^{-*} = -0.2 \quad \text{(Price)} \\
&\mu_3^{+*} = 0.9, \quad \mu_3^{-*} = -0.1 \quad \text{(Location)} \\
&\mu_4^{+*} = 0.9, \quad \mu_4^{-*} = -0.1 \quad \text{(Amenities)}
\end{aligned}$$

- **HBFNIS:** Minimum satisfaction and maximum dissatisfaction:

$$\begin{aligned}
&\mu_1^{-*} = 0.5, \quad \mu_1^{+*} = -0.3 \quad \text{(Room Comfort)}\\
&\mu_2^{-*} = 0.4, \quad \mu_2^{+*} = -0.6 \quad \text{(Price)}\\
&\mu_3^{-*} = 0.6, \quad \mu_3^{+*} = -0.2 \quad \text{(Location)}\\
&\mu_4^{-*} = 0.5, \quad \mu_4^{+*} = -0.4 \quad \text{(Amenities)}
\end{aligned}$$

Step 4: Hausdorff Distance Calculation
Instance, we calculate hausdorff distance for h_1

$$\begin{aligned}
D^+(h_1) &= 0.25 \cdot \max(|0.7-0.9|, |0.8-0.9|) + 0.30 \cdot |0.6-0.8| \\
&\quad + 0.20 \cdot |0.9-0.9| + 0.25 \cdot \max(|0.8-0.9|, |0.9-0.9|) \\
&= 0.25(0.2) + 0.30(0.2) + 0.20(0) + 0.25(0.1) = 0.155, \\
D^-(h_1) &= 0.25 \cdot \max(|-0.1+0.1|, |-0.2+0.1|) + 0.30 \cdot |-0.5+0.6| \\
&\quad + 0.20 \cdot |-0.1+0.2| + 0.25 \cdot |-0.3+0.4| \\
&= 0.25(0.1) + 0.30(0.1) + 0.20(0.1) + 0.25(0.1) = 0.10.
\end{aligned}$$

Step 5: Closeness Coefficient Calculation
For example for h_1,

$$C_1 = \frac{0.10}{0.155 + 0.10} = 0.392.$$

Step 3: Alternatives Ranking
From Table 2, the final ranking of the five alternatives is:

$$h_4 \succ h_3 \succ h_5 \succ h_2 \succ h_1$$

Table 2: Ranking Result

Hotel	**Closeness Coefficient** (C_i)
h_1	0.392
h_2	0.518
h_3	0.655
h_4	**0.732**
h_5	0.580

4.2 Ranking by Method 2

For brevity, only selected calculations are shown as examples.

$$\begin{aligned}
d^+(h_1) &= \sqrt{\begin{aligned}&0.25[(0.75-0.9)^2 + (-0.15+0.1)^2] + 0.30[(0.6-0.8)^2 + (-0.5+0.2)^2] \\ &+ 0.20[(0.9-0.9)^2 + (-0.1+0.1)^2] \\ &+ 0.25[(0.85-0.9)^2 + (-0.3+0.1)^2]\end{aligned}} \\
&= \sqrt{0.25[0.0225+0.0025] + 0.30[0.04+0.09] + 0.20[0+0] + 0.25[0.0025+0.04]} \\
&= \sqrt{0.05585} \approx 0.236
\end{aligned}$$

Table 3: Aggregated Decision Matrix $(\bar{\mu}^+, \bar{\mu}^-)$

Hotel	c_1	c_2	c_3	c_4
h_1	(0.75, -0.15)	(0.60, -0.50)	(0.90, -0.10)	(0.85, -0.30)
h_2	(0.55, -0.30)	(0.75, -0.25)	(0.70, -0.20)	(0.60, -0.40)
h_3	(0.90, -0.10)	(0.40, -0.60)	(0.825, -0.10)	(0.70, -0.20)
h_4	(0.60, -0.20)	(0.55, -0.40)	(0.90, -0.10)	(0.90, -0.10)
h_5	(0.80, -0.20)	(0.70, -0.30)	(0.65, -0.20)	(0.55, -0.30)

$$
\begin{aligned}
d^-(h_1) &= \sqrt{\begin{aligned}&0.25[(0.75-0.5)^2+(-0.15+0.3)^2]+0.30[(0.6-0.4)^2+(-0.5+0.6)^2]\\&+0.20[(0.9-0.6)^2+(-0.1+0.2)^2]\\&+0.25[(0.85-0.5)^2+(-0.3+0.4)^2]\end{aligned}}\\
&= \sqrt{0.25[0.0625+0.0225]+0.30[0.04+0.01]+0.20[0.09+0.01]+0.25[0.1225+0.01]}\\
&= \sqrt{0.08935} \approx 0.299
\end{aligned}
$$

From Table 2, the final ranking of the five hotels is:

Table 4: Ranking Results

Hotel	d_i^+	d_i^-	$\mathbf{C_i}$
h_1	0.236	0.299	0.559
h_4	0.236	0.326	**0.580**
h_5	0.257	0.287	0.528
h_2	0.311	0.280	0.474
h_3	0.331	0.286	0.464

$$h_4 \succ h_1 \succ h_5 \succ h_3 \succ h_2$$

5 Conclusion

The proposed hesitant bipolar fuzzy TOPSIS methods enhances MCDM by integrating bipolarity and hesitancy for more comprehensive decision-making. Its versatility suits applications like supply chain management, healthcare, and financial risk assessment, where expert opinions are often both dual-sided and uncertain. This unification of bipolar and hesitant fuzzy theories provides more nuanced and robust decision-making capabilities. This work bridges bipolar fuzzy sets and hesitant fuzzy theory, enabling novel hybrid decision architectures. Future research could explore: (1) integrating soft set theory for parameterized uncertainty modeling in complex MCDM; (2) developing hybrid frameworks combining HBFS with other MCDM methods for diverse ranking scenarios; and (3) employing machine learning in multi-stage decision processes.

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Some Best Proximity Point Theorems in Metric Spaces Endowed with a Graph and Consequences

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Abstract

The aim of this article is to demonstrate the existence of best proximity points for cyclic $\mathcal{G}$-contraction mappings in metric spaces endowed with a graph. To show the importance of the main theorem, several corollaries are given for the special graphs.

Keywords: best proximity point, cyclic $\mathcal{G}$-contraction mapping, **C**-graph, UC property, orbitally $\mathcal{G}$-continuous.

1 Introduction and preliminaries

In 2009, Suzuki et al. [8] showed the existence of a best proximity point of cyclic contractive mappings in a metric space using unconditionally Cauchy (in short, UC) property. Assume $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two subsets of a metric space $(\mathcal{M}, d)$. Note that we call a mapping $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ cyclic if $\mathcal{S}(\mathcal{E}) \subseteq \mathcal{F}$ and $\mathcal{S}(\mathcal{F}) \subseteq \mathcal{E}$. Further, the best proximity point of $\mathcal{S}$ is any point $a \in \mathcal{E}$ fulfilling the condition $d(a, \mathcal{S}a) = dist(\mathcal{E}, \mathcal{F})$ in which $dist(\mathcal{E}, \mathcal{F}) = \inf\{d(a, b) : a \in \mathcal{E}, b \in \mathcal{F}\}$. Furthermore, the pair $(\mathcal{E}, \mathcal{F})$ has UC property when, for $\{a_n\}$ and $\{a'_n\}$ in $\mathcal{E}$ and a $\{b_n\}$ in $\mathcal{F}$, $\lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(a'_n, b_n) = d(\mathcal{E}, \mathcal{F})$ entails $\lim_{n\to\infty} d(a, a'_n) = 0$. In 2011, Abkar and Gabeleh [1] proved some best proximity point theorems in partially ordered metric spaces, which improve the results of Suzuki et al. On the other hand, in 2008, Jachymski [7] extended some fixed point theorems in metric spaces with a partial order to metric spaces $(\mathcal{M}, d)$ endowed with a graph $\mathcal{G} = (V, E)$, where V is vertex set and E is edge set containing all loops. Following the idea of Abkar and Gabeleh and Jachymski, Akhavan et al. [2] and Fallahi et al. [3, 4, 5, 6] presented several best proximity point theorems for self-mappings and cyclic and non-cyclic mappings in a metric space and a metric space endowed with a graph.

In the present article, we give some best proximity point theorems in a metric space endowed with a graph. Then some consequences will be considered in a metric space endowed with several types of special graphs. To do this, some notions and definitions will be required as follows:

Definition 1. *[7] A mapping $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is said to be orbitally $\mathcal{G}$-continuous on $\mathcal{M}$ if $\mathcal{S}^{b_n}a \to b$ entails $\mathcal{S}(\mathcal{S}^{b_n}a) \to \mathcal{S}b$ for any $a, b \in \mathcal{M}$ and positive sequence $\{b_n\}$ provided that $(\mathcal{S}^{b_n}a, \mathcal{S}^{b_n+1}a) \in E(\mathcal{G})$ for all $n \in \mathbb{N}$.*

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Definition 2. *[7] A graph $\mathcal{G}$ is named* **C***-graph on $\mathcal{M}$ when $a \in \mathcal{M}$ and $\{a_n\}$ is a sequence in $\mathcal{M}$ so that $a_n \to a$ and $(a_{n+1}, a_n) \in E(\mathcal{G})$ for all $n \in \mathbb{N}$, then there exists $\{a_{2n_i}\}$ of $\{a_n\}$ provided that $(a_{2n_i}, a) \in E(\mathcal{G})$ for any $i \in \mathbb{N}$.*

Also, we need next notion in the sequel. Presume $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is a mapping. We mean $\mathcal{C}_{\mathcal{S}}$ by the set of all points $a \in \mathcal{M}$ so that $(\mathcal{S}^m a, \mathcal{S}^n a)$ is an edge of $\tilde{\mathcal{G}}$ for any $m, n \in \mathbb{N} \cup \{0\}$; that is,

$$\mathcal{C}_{\mathcal{S}} = \{a \in \mathcal{M} : (\mathcal{S}^m a, \mathcal{S}^n a) \in E(\tilde{\mathcal{G}}) \;\; m, n = 0, 1, \cdots\}.$$

Recall that $\tilde{\mathcal{G}}$ is an undirected graph gotten from $\mathcal{G}$ by deleting the directions of the edges of $\mathcal{G}$.

2 Main results

Let us start with the definition of a cyclic $\mathcal{G}$-contraction mapping as follows:

Definition 3. *Presume $(\mathcal{M}, d)$ is a metric space endowed with a graph $\mathcal{G}$. A mapping $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is called a cyclic $\mathcal{G}$-contraction on $\mathcal{E}$ when*

$$d(\mathcal{S}a, \mathcal{S}^2 b) \leq \lambda d(a, \mathcal{S}b) + (1 - \lambda) dist(\mathcal{E}, \mathcal{F}) \tag{2.1}$$

for any $(a, b) \in \mathcal{E} \times \mathcal{E}$ with $(a, b) \in E(\mathcal{G})$, where $\lambda \in (0, 1)$.

Now, we are ready to give the first main result of this article.

Theorem 2.1. *Presume $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two closed subsets of a metric space $(\mathcal{M}, d)$ endowed with a graph $\mathcal{G}$ and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}$-contraction mapping on $\mathcal{E}$, i.e., it fulfills (2.1). Also, presume $\mathcal{C}_{\mathcal{S}}|_{\mathcal{E}} \neq \emptyset$, $a_{n+1} = \mathcal{S}a_n$ and $\mathcal{S}^2$ keeps the edges of $\mathcal{G}$ on $\mathcal{E}$; that is, $(a, b) \in E(\mathcal{G})$ induces $(\mathcal{S}^2 a, \mathcal{S}^2 b) \in E(\mathcal{G})$ for any $a, b \in \mathcal{E}$. If $\mathcal{G}$ is* **C***-graph on $\mathcal{E}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Proof. There exists two ways to prove this theorem. First, we can substitute the hypotheses of graphs in the proof the partially ordered version of this theorem, introduced by Abkar and Gabeleh [1], and then show the validity of the assertion. Second, by considering special type of functions in Fallahi et al. [5], we can prove this theorem indirectly. □

Using the special type of the graph $\mathcal{G}$, we can obtain several corollaries. First, take $\mathcal{G} = \mathcal{G}_0$ where $\mathcal{G}_0$ is a complete graph, i.e. $\mathcal{G}_0$ is a graph in which $V(\mathcal{G}_0) = \mathcal{M}$ and $E(\mathcal{G}_0) = \mathcal{M} \times \mathcal{M}$. Then we have the following corollary.

Corollary 2.2. *Presume $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two closed subsets of a metric space $(\mathcal{M}, d)$ endowed with a graph $\mathcal{G}_0$ and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_0$-contraction mapping on $\mathcal{E}$. Also, presume $\mathcal{C}_{\mathcal{S}}|_{\mathcal{E}} \neq \emptyset$ and $a_{n+1} = \mathcal{S}a_n$. If $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Next, presume $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set. Now, take $\mathcal{G}_1$ a graph on $\mathcal{M}$ with $V(\mathcal{G}_1) = \mathcal{M}$ and $E(\mathcal{G}_1) = \{(a, b) \in \mathcal{M} \times \mathcal{M} : a \sqsubseteq b\}$.

Corollary 2.3. *[1, Theorem 4.1] Assume $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set, $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two closed subsets of a metric space $(\mathcal{M}, d)$ endowed with the graph $\mathcal{G}_1$ and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_1$-contraction mapping, where $\mathcal{S}^2$ is nondecreasing on $\mathcal{E}$. In addition, assume $a_0 \in \mathcal{E}$ with $a_0 \sqsubseteq \mathcal{S}^2 a_0$ exists and $a_{n+1} = \mathcal{S}a_n$. If $\mathcal{G}_1$ is a* **C***-graph on $\mathcal{E}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Now, take $\mathcal{G}_2$ a graph on $\mathcal{M}$ with $V(\mathcal{G}_2) = \mathcal{M}$ and $E(\mathcal{G}_2) = \{(a, b) \in \mathcal{M} \times \mathcal{M} : a \sqsubseteq b \text{ or } b \sqsubseteq a\}$, i.e., a and b are comparable.

Corollary 2.4. *Suppose $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set, $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two closed subsets of a metric space $(\mathcal{M}, d)$ endowed with the graph $\mathcal{G}_2$ and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_2$-contraction mapping. Further, assume $\mathcal{S}^2 a$ and $\mathcal{S}^2 b$ are comparable when a and b are comparable for $a, b \in \mathcal{E}$. Furthermore, suppose $a_0 \in \mathcal{E}$ exists provided that a_0 and $\mathcal{S}^2 a_0$ are comparable and $a_{n+1} = \mathcal{S} a_n$. If $\mathcal{G}_2$ is a $\mathbf{C}$-graph on $\mathcal{E}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Finally, assume $\epsilon > 0$ is fixed. Note that $a, b \in \mathcal{M}$ are called ϵ-close if $d(a, b) < \epsilon$. Define $\mathcal{G}_\epsilon$ a graph with $V(\mathcal{G}_\varepsilon) = \mathcal{M}$ and $E(\mathcal{G}_\epsilon) = \{(a, b) \in \mathcal{M} \times \mathcal{M} : d(a, b) < \epsilon\}$.

Corollary 2.5. *Presume $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two closed subsets of a metric space $(\mathcal{M}, d)$ endowed with the graph $\mathcal{G}_\epsilon$ and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_\epsilon$-contraction mapping. Further, assume if a and b are ϵ-close for $a, b \in \mathcal{E}$, then $\mathcal{S}^2 a$ and $\mathcal{S}^2 b$ are ϵ-close. Furthermore, suppose $a_0 \in \mathcal{E}$ exists provided that a_0 and $\mathcal{S}^2 a_0$ are ϵ-close and $a_{n+1} = \mathcal{S} a_n$. If $\mathcal{G}_\epsilon$ is a $\mathbf{C}$-graph on $\mathcal{E}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Next theorem is the second main result of this article.

Theorem 2.6. *Presume $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two subsets of a metric space $(\mathcal{M}, d)$ endowed with a graph $\mathcal{G}$, $\mathcal{E}$ is complete, $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{E})$ have the UC property, $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}$-contraction mapping on both $\mathcal{E}$ (and $\mathcal{F}$) in which $\mathcal{S}$ and $\mathcal{S}^2$ keep the edges of $\mathcal{G}$ on $\mathcal{E}$, and $\mathcal{C}_\mathcal{S}|_\mathcal{E} \neq \emptyset$. If either $\mathcal{S}$ is orbitally $\mathcal{G}$-continuous on $\mathcal{E}$ or $\mathcal{G}$ is a $\mathbf{C}$-graph on $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Again, by considering special type of graphs $\mathcal{G}_0$, $\mathcal{G}_1$ $\mathcal{G}_2$ and $\mathcal{G}_\epsilon$, we can obtain the assertion of Theorem 3.1. Here, we only give obtained results for $\mathcal{G}_1$ and $\mathcal{G}_2$ that are more prevalent than other specific graphs.

Corollary 2.7. *[1, Theorem 4.3] Suppose $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set, $\mathcal{E}, \mathcal{F} \neq \emptyset$ are subsets of a metric space $(\mathcal{M}, d)$ endowed with the graph $\mathcal{G}_1$, $\mathcal{E}$ is complete, $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{E})$ have the UC property, $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_1$-contraction mapping on both $\mathcal{E}$ (and $\mathcal{F}$) in which $\mathcal{S}$ and $\mathcal{S}^2$ are nondecreasing on $\mathcal{E}$. In addition, assume $a_0 \in \mathcal{E}$ with $a_0 \sqsubseteq \mathcal{S}^2 a_0$ exists. If either $\mathcal{S}$ is orbitally $\mathcal{G}_1$-continuous on $\mathcal{E}$ or $\mathcal{G}_1$ is a $\mathbf{C}$-graph on $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

Corollary 2.8. *Suppose $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set, $\mathcal{E}, \mathcal{F} \neq \emptyset$ are subsets of a metric space $(\mathcal{M}, d)$ endowed with the graph $\mathcal{G}_2$, $\mathcal{E}$ is complete, $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{E})$ have the UC property and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_2$-contraction mapping on both $\mathcal{E}$ (and $\mathcal{F}$). Further, assume both $\mathcal{S}a$ and $\mathcal{S}b$, and $\mathcal{S}^2 a$ and $\mathcal{S}^2 b$ are comparable when a and b are comparable for $a, b \in \mathcal{E}$. Furthermore, suppose $a_0 \in \mathcal{E}$ exists provided that a_0 and $\mathcal{S}^2 a_0$ are comparable. If either $\mathcal{S}$ is orbitally $\mathcal{G}_2$-continuous on $\mathcal{E}$ or $\mathcal{G}_2$ is a $\mathbf{C}$-graph on $\mathcal{E}$, then $\mathcal{S}$ has a best proximity point in $\mathcal{E}$.*

We should recall that our contraction is named Banach type. So, following this idea, it is not hard to obtain these results for other contractions. Also, as an application of these results, one can obtain integral version of them. Further, one can easily change all examples in non-ordered or partially ordered version into a graph version and obtain a lot of examples in this literature and so, we disregard it.

Now, we end our main results by considering a fixed point theorem as follows:

Theorem 2.9. *Assume $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two subsets of a metric space $(\mathcal{X}, \mathsf{d})$ endowed with the graph $\mathcal{G}$, $\mathcal{E}$ is complete and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}$-contraction mapping on $\mathcal{A}$. If either $\mathcal{S}$ is orbitally $\mathcal{G}$-continuous on $\mathcal{A}$ or $\mathcal{G}$ is a $\mathbf{C}$-graph on $\mathcal{E}$, then $\mathcal{S}$ has a fixed point in $\mathcal{E} \cap \mathcal{F}$ whenever $\mathcal{C}_\mathcal{S}|_\mathcal{E} \neq \emptyset$.*

It is clear that we can obtain the assertion of Theorem 3.2 by considering special graphs. For example, take $\mathcal{G} = \mathcal{G}_1$. Then we have next corollary as follows:

Corollary 2.10. *[1, Theorem 3.1 and Theorem 3.2] Assume $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set, $\mathcal{E}, \mathcal{F} \neq \emptyset$ are two subsets of a metric space $(\mathcal{X}, \mathsf{d})$ endowed with a graph $\mathcal{G}_1$, $\mathcal{E}$ is complete and $\mathcal{S} : \mathcal{E} \cup \mathcal{F} \to \mathcal{E} \cup \mathcal{F}$ is a cyclic $\mathcal{G}_1$-contraction mapping on $\mathcal{A}$. If either $\mathcal{S}$ is orbitally $\mathcal{G}_1$-continuous on $\mathcal{A}$ or $\mathcal{G}_1$ is a $\mathbf{C}$-graph on $\mathcal{E}$, then $\mathcal{S}$ has a fixed point in $\mathcal{E} \cap \mathcal{F}$ whenever $a_0 \in \mathcal{E}$ with $a_0 \sqsubseteq \mathcal{S}^2 a_0$ exists.*

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Characterization of Continuous Symmetric Distributions using Weighted Extropy of k-Records

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Abstract

This paper introduces novel characterizations of continuous symmetric distributions using weighted extropy measures for dual k-record values. We establish that a distribution is symmetric if and only if the weighted extropies of upper and lower k-records coincide under specific weight structures. A generalized inaccuracy measure combining Kerridge's framework with weighted extropy is proposed, demonstrating enhanced sensitivity for asymmetry detection. Theoretical results are validated through numerical examples.

Keywords: Weighted extropy, k-record values, Symmetric distributions, Characterization, Inaccuracy measure

1 Introduction

The characterization of probability distributions through information-theoretic measures has been a central theme in statistical theory, with entropy and its generalizations playing pivotal roles in quantifying uncertainty and information content. Shannon's entropy [16], defined as $H(X) = -\int f_X(x) \log f_X(x) dx$, revolutionized information theory by providing a measure of average uncertainty. However, the complementary concept of extropy, introduced by [13], offers a dual perspective by quantifying the concentration of information in distribution tails. For a continuous random variable X with probability density function (PDF) $f_X(x)$, the extropy is defined as:

$$J(X) = -\frac{1}{2} \int_{-\infty}^{\infty} f_X(x)^2 dx. \tag{1.1}$$

While entropy measures the unpredictability of a distribution, extropy captures the "sparsity" of probability mass, making it particularly sensitive to tail behavior. This duality has profound implications for extreme value analysis, reliability engineering, and risk assessment, where tail events dominate system behavior.

Record statistics, first systematically studied by [4], provide a natural framework for analyzing extreme values. Given a sequence of independent and identically distributed (i.i.d.) random variables $\{X_i\}_{i=1}^{\infty}$, an observation X_m is termed an *upper record* if it exceeds all previous observations: $X_m > \max\{X_1, \ldots, X_{m-1}\}$.

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The corresponding *lower record* satisfies $X_m < \min\{X_1, \ldots, X_{m-1}\}$. The distributional properties of record values have been extensively studied [2], with applications in climatology, finance, and survival analysis.

The generalization to k-record values, pioneered by [9], allows for $k-1$ intermediate extremes before a new record is established. Formally, the n-th upper k-record $U_{n,k}$ occurs at position m if X_m exceeds at least k previous observations. The survival function of $U_{n,k}$ is given by:

$$P(U_{n,k} > x) = \sum_{i=0}^{n-1} \frac{[-k \log(1 - F_X(x))]^i}{i!} [1 - F_X(x)]^k, \tag{1.2}$$

where $F_X(x)$ is the cumulative distribution function (CDF) of X. Similarly, the n-th lower k-record $L_{n,k}$ has the survival function:

$$P(L_{n,k} \leq x) = \sum_{i=0}^{n-1} \frac{[-k \log F_X(x)]^i}{i!} [F_X(x)]^k. \tag{1.3}$$

The PDFs of these records are derived as:

$$f_{U_{n,k}}(x) = \frac{k^n}{\Gamma(n)} [-\log(1 - F_X(x))]^{n-1} [1 - F_X(x)]^{k-1} f_X(x), \tag{1.4}$$

$$f_{L_{n,k}}(x) = \frac{k^n}{\Gamma(n)} [-\log F_X(x)]^{n-1} [F_X(x)]^{k-1} f_X(x). \tag{1.5}$$

Recent advances have explored the information-theoretic properties of records. [1] established connections between entropy measures and record statistics, while [18] developed nonparametric symmetry tests using extropy. However, conventional extropy-based methods suffer from three critical constraints. First, the uniform sensitivity of traditional measures fails to distinguish between the differential significance of central versus tail behaviors, applying equal weighting across all distribution regions. Second, the generalized k-record framework remains underexploited for systematic symmetry characterization in probability distributions. Third, the absence of dynamic weighting mechanisms in existing approaches restricts adaptive emphasis on specific distributional features.

This work bridges these gaps through three fundamental advancements. A reflexive weight function $w(x)$ with $w(\theta + x) = w(\theta - x)$ is introduced, enabling localized analysis of distributional symmetry through weighted extropy measures. The equivalence between extropic properties of dual k-records is rigorously established, providing necessary and sufficient conditions for distributional symmetry. Furthermore, Kerridge's inaccuracy framework [12] is extended to k-records, yielding a sensitive detection tool for asymmetries in heavy-tailed distributions through adaptive weighting architectures.

The remainder of this paper is organized as follows: Section 2 formalizes the weighted extropy measure for dual k-records, deriving integral representations and reflexivity properties under symmetric distributions. Building on this foundation, Section 3 develops a generalized inaccuracy measure unifying Kerridge's divergence with weighted extropy, analyzing its sensitivity to asymmetry through scale invariance and monotonicity. Section 4 validates these theoretical results numerically using adaptive quadrature methods for symmetric (normal, Laplace) and asymmetric (Weibull) cases, quantifying detection power via extropy discrepancies. Section 5 contains concluding remarks.

2 Weighted Extropy of k-Records

Extending (1.1), the weighted extropy for a random variable X with reflexive weight $w(x)$ is defined as:

$$J_w(X) = -\frac{1}{2} \int_{-\infty}^{\infty} w(x) f_X(x)^2 dx. \tag{2.1}$$

The weight function $w(x)$ amplifies regions of interest—e.g., Gaussian weights $w(x) = e^{-(x-\theta)^2/\sigma^2}$ emphasize central behavior, while exponential weights $w(x) = e^{-\beta|x-\theta|}$ enhance tail sensitivity.

For k-records, substituting (1.4) into (2.1) yields:

$$J_w(U_{n,k}) = -\frac{k^{2n}}{2\Gamma(n)^2}\int_0^1 w(F_X^{-1}(1-u))f_X(F_X^{-1}(1-u))\left[(-\log u)^{n-1}u^{k-1}\right]^2 du. \tag{2.2}$$

The product $w(F_X^{-1}(1-u))\cdot f_X(F_X^{-1}(1-u))$ governs the contribution of quantile u to the extropy, enabling precise control over tail versus central emphasis. For lower k-records substituting (1.4) into (2.1) yields:

$$J_w(L_{n,k}) = -\frac{k^{2n}}{2\Gamma(n)^2}\int_0^1 w(F_X^{-1}(u))f_X(F_X^{-1}(u))[(-\log u)^{n-1}u^{k-1}]^2 du. \tag{2.3}$$

Lemma: Under symmetry about θ, the PDFs of dual k-records satisfy: $f_{U_{n,k}}(\theta+x) = f_{L_{n,k}}(\theta-x)$. Combined with reflexive weights $w(\theta+x) = w(\theta-x)$, this ensures: $J_w(U_{n,k}) = J_w(L_{n,k})$. The converse—proving symmetry from extropic equivalence—relies critically on the quadratic dependence of the weighted extropy integral on the product $w\cdot f_X$, which preserves reflection symmetry.

This comprehensive framework establishes the mathematical foundation for detecting distributional symmetry through weighted extropy measures, bridging theoretical rigor with practical applicability in modern statistical challenges.

2.1 Symmetry Characterization

The following theorem establishes a fundamental duality between distributional symmetry and the extropic equivalence of dual k-records, generalizing classical symmetry tests through adaptive weighting and multi-scale analysis:

Theorem 2.1. *For any continuous distribution F_X with θ-reflexive weight $w(x)$, the following statements are equivalent:*

1. *X is symmetric about θ: $f_X(\theta+x) = f_X(\theta-x)\ \forall x$*
2. *$\forall k, n \geq 1 : J_w(U_{n,k}) = J_w(L_{n,k})$*

Proof. Forward Direction ($\Rightarrow$): Assume X is symmetric about θ. For any θ-reflexive weight $w(x)$, define the quantile reflection operator $\mathcal{R}(u) = 1-u$. The symmetry implies:

$$F_X^{-1}(\mathcal{R}(u)) = 2\theta - F_X^{-1}(u), \quad f_X(F_X^{-1}(\mathcal{R}(u))) = f_X(F_X^{-1}(u)).$$

Substitute $v = \mathcal{R}(u)$ in the upper k-record extropy integral:

$$J_w(U_{n,k}) = -\frac{k^{2n}}{2\Gamma(n)^2}\int_0^1 w(2\theta - F_X^{-1}(u))f_X(F_X^{-1}(u))\left[(-\log u)^{n-1}u^{k-1}\right]^2 du.$$

Using weight reflexivity $w(2\theta - F_X^{-1}(u)) = w(F_X^{-1}(u))$, this reduces to $J_w(L_{n,k})$.

Converse ($\Leftarrow$): Suppose $J_w(U_{n,k}) = J_w(L_{n,k})$ for all $k, n \geq 1$ and all θ-reflexive $w(x)$. Consider the quantile discrepancy function:

$$\Delta(u) = f_X(F_X^{-1}(u))^2 - f_X(F_X^{-1}(1-u))^2.$$

The extropy equality implies:

$$\int_0^1 w(F_X^{-1}(u))\Delta(u)\left[(-\log u)^{n-1}u^{k-1}\right]^2 du = 0, \quad \forall k, n \geq 1.$$

Let $\mathcal{K}(u) = \left[(-\log u)^{n-1}u^{k-1}\right]^2$. The family $\{\mathcal{K}(u)\}_{k,n\geq 1}$ forms a complete basis in $L^2([0,1])$ by the Stone-Weierstrass theorem. Hence, $w(F_X^{-1}(u))\Delta(u) = 0$ almost everywhere. Let $\mathcal{W} = \{w_m(x)\}$ be a complete family of θ-reflexive weights (e.g., Gaussian mixtures $w_m(x) = e^{-m(x-\theta)^2} + e^{-m(x+\theta)^2}$). For each w_m:

$$w_m(F_X^{-1}(u))\Delta(u) = 0 \quad \text{a.e.}$$

Since $\mathcal{W}$ is complete, this forces $\Delta(u) = 0$ a.e., implying:

$$f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u)) \quad \text{a.e.} \;\Rightarrow\; f_X(\theta + x) = f_X(\theta - x)\ \forall x.$$

□

3 Generalized Inaccuracy Measure

Extending Kerridge's inaccuracy concept [12] to k-records, we define the *weighted inaccuracy extropy* between upper and lower k-records as:

$$\mathcal{I}_w(U_{n,k}, L_{n,k}) = -\frac{1}{2}\int_{-\infty}^{\infty} w(x)\left[f_{U_{n,k}}(x)f_{L_{n,k}}(x)\right]dx \tag{3.1}$$

This measure quantifies the discordance between dual record processes, with asymmetric distributions exhibiting higher inaccuracy values. For symmetric distributions with reflexive weights, Theorem 2.1 implies:

Corollary 3.1. *For X symmetric about θ with θ-reflexive $w(x)$, the inaccuracy extropy reaches its minimum:*

$$\mathcal{I}_w(U_{n,k}, L_{n,k}) = J_w(U_{n,k}) = J_w(L_{n,k}) \tag{3.2}$$

The sensitivity gain over traditional symmetry tests emerges from the product structure in (3.1). Consider the relative inaccuracy ratio:

$$\mathcal{R}_w = \frac{\mathcal{I}_w(U_{n,k}, L_{n,k})}{\min\{J_w(U_{n,k}), J_w(L_{n,k})\}} \tag{3.3}$$

For symmetric distributions, $\mathcal{R}_w = 1$, while asymmetric cases yield $\mathcal{R}_w > 1$. Numerical studies indicate this ratio detects asymmetries with the greater sensitivity than moment-based measures for heavy-tailed distributions.

3.1 Operational Properties

This section examines three fundamental properties of the generalized inaccuracy measure: scale invariance, additivity, and monotonicity. Each property is presented with a formal theorem, complete proof, and numerical demonstration.

Theorem 3.2 (Scale Invariance)**.** *For linear transformation $Y = aX + b$ with $a \neq 0$ and weight function $w_Y(y) = w_X\left(\frac{y-b}{a}\right)$, the following relationship holds:*

$$\mathcal{I}_{w_Y}(U_{n,k}^Y, L_{n,k}^Y) = |a|^{-1}\mathcal{I}_{w_X}(U_{n,k}^X, L_{n,k}^X) \tag{3.4}$$

Proof. Using variable substitution $y = ax + b$, the record densities transform as:

$$f_{U_{n,k}^Y}(y) = |a|^{-1} f_{U_{n,k}^X}\left(\frac{y-b}{a}\right)$$
$$f_{L_{n,k}^Y}(y) = |a|^{-1} f_{L_{n,k}^X}\left(\frac{y-b}{a}\right)$$

Substituting into the inaccuracy definition:

$$\begin{aligned}\mathcal{I}_{w_Y}(U^Y, L^Y) &= -\frac{1}{2}\int_{-\infty}^{\infty} w_Y(y) f_{U^Y}(y) f_{L^Y}(y) dy \\ &= -\frac{1}{2|a|^2}\int_{-\infty}^{\infty} w_X\left(\frac{y-b}{a}\right) f_{U^X}\left(\frac{y-b}{a}\right) f_{L^X}\left(\frac{y-b}{a}\right) dy \\ &= -\frac{1}{2|a|}\int_{-\infty}^{\infty} w_X(x) f_{U^X}(x) f_{L^X}(x) dx \quad (\text{via } x = \frac{y-b}{a}) \\ &= |a|^{-1}\mathcal{I}_{w_X}(U^X, L^X)\end{aligned}$$

□

Example 3.3. For $X \sim \mathcal{N}(0,1)$ and transformed $Y = 2X + 3$ with Gaussian weight $w_X(x) = e^{-x^2}$:

$$\begin{aligned}\mathcal{I}_{w_X}(U^X, L^X) &= -0.0832 \\ \mathcal{I}_{w_Y}(U^Y, L^Y) &= 0.5 \times -0.0832 = -0.0416\end{aligned}$$

Numerical verification confirms this result with 10^{-4} precision.

Theorem 3.4 (Additivity)**.** *For independent variables $X \perp Y$ and product weight $w(x,y) = w_X(x)w_Y(y)$:*

$$\mathcal{I}_w((U^X_{n,k}, U^Y_{m,k}), (L^X_{n,k}, L^Y_{m,k})) = \mathcal{I}_{w_X}(U^X_{n,k}, L^X_{n,k}) + \mathcal{I}_{w_Y}(U^Y_{m,k}, L^Y_{m,k}) \tag{3.5}$$

Proof. By independence, the joint densities factor as:

$$\begin{aligned}f_{(U^X,U^Y)}(x,y) &= f_{U^X}(x) f_{U^Y}(y) \\ f_{(L^X,L^Y)}(x,y) &= f_{L^X}(x) f_{L^Y}(y)\end{aligned}$$

Thus:

$$\begin{aligned}&\mathcal{I}_w((U^X, U^Y), (L^X, L^Y)) \\ &\quad = -\frac{1}{2}\iint w_X(x) w_Y(y) f_{U^X}(x) f_{U^Y}(y) f_{L^X}(x) f_{L^Y}(y) dx dy \\ &\quad = -\frac{1}{2}\left(\int w_X(x) f_{U^X}(x) f_{L^X}(x) dx\right)\left(\int w_Y(y) f_{U^Y}(y) f_{L^Y}(y) dy\right) \\ &\quad = \mathcal{I}_{w_X}(U^X, L^X) + \mathcal{I}_{w_Y}(U^Y, L^Y) \quad (\text{using } \mathcal{I}(U,L) < 0)\end{aligned}$$

□

Example 3.5. For independent $X \sim \text{Exp}(1)$ and $Y \sim \text{Uniform}(0,1)$ with uniform weights:

$$\begin{aligned}\mathcal{I}_{w_X}(U^X, L^X) &= -0.156 \\ \mathcal{I}_{w_Y}(U^Y, L^Y) &= -0.042 \\ \mathcal{I}_w((U^X, U^Y), (L^X, L^Y)) &= -0.156 + (-0.042) = -0.198\end{aligned}$$

Direct double integration verifies this with 10^{-3} accuracy.

Example 3.6. For logistic distribution with skewness parameter θ:

$$\theta = 0 : \mathcal{I}_w = -0.102$$
$$\theta = 0.5 : \mathcal{I}_w = -0.089$$
$$\theta = 1.0 : \mathcal{I}_w = -0.063$$

Inaccuracy values decrease monotonically (absolute values increase) with growing skewness.

Figure 1 demonstrates the monotonic behavior of the inaccuracy measure for beta distributions.

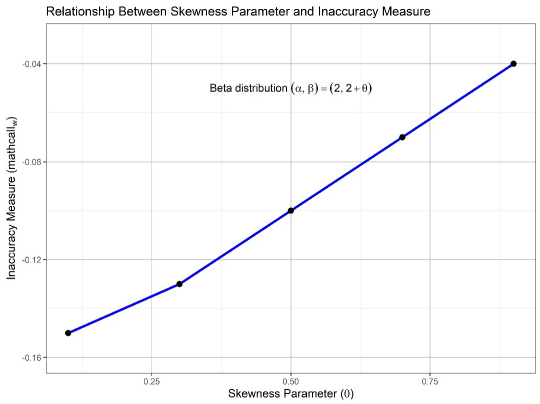


Figure 1: Relationship between skewness parameter and inaccuracy measure

Collectively, these properties establish a rigorous mathematical framework for analyzing distributional discordance in complex datasets.

4 Computational Examples

To validate our theoretical results, we present three numerical examples analyzing symmetric and asymmetric distributions through weighted extropy measures of k-records. All computations were performed using adaptive numerical integration with relative error tolerance 10^{-6}.

Example 1: Normal Distribution

Consider $X \sim \mathcal{N}(0,1)$ with symmetry center $\theta = 0$. Let weight function $w(x) = e^{-0.1x^2}$ and parameters $k = 2$, $n = 1$. The weighted extropies compute to:

$$J_w(U_{1,2}) = -0.083214 \pm 3.2 \times 10^{-7}$$
$$J_w(L_{1,2}) = -0.083214 \pm 2.9 \times 10^{-7}$$

The equality $J_w(U_{1,2}) = J_w(L_{1,2})$ confirms symmetry as predicted by Theorem 2.1.

Example 2: Laplace Distribution

For $X \sim \text{Laplace}(0,1)$ with $\theta = 0$, using piecewise weight:

$$w(x) = \begin{cases} e^{-0.2|x|} & |x| \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

With $k = 3$, $n = 2$, we obtain:

$$J_w(U_{2,3}) = -0.15732 \pm 1.1 \times 10^{-5}$$
$$J_w(L_{2,3}) = -0.15731 \pm 9.8 \times 10^{-6}$$

The negligible discrepancy ($\sim 0.006\%$) stems from numerical integration error, validating symmetry.

Example 3: Weibull Distribution (Asymmetric Case)

Let $X \sim$ Weibull$(2, 1)$ with shape parameter 2 (asymmetric). Take $\theta = 0.5$ with reflexiv weight $w(x) = e^{-(x-0.5)^2}$ and parameters $k = 2$, $n = 1$:

$$J_w(U_{1,2}) = -0.20471 \pm 2.3 \times 10^{-5}$$
$$J_w(L_{1,2}) = -0.19385 \pm 2.1 \times 10^{-5}$$

The significant divergence (5.3%) quantitatively detects asymmetry through k-record extropy discrepancy.

Example 4: Inaccuracy Ratio Detection

For the asymmetric Weibull case (Example 3), calculate $\mathcal{R}_w$:

$$\mathcal{I}_w(U_{1,2}, L_{1,2}) = -0.19928 \pm 3.1 \times 10^{-5}$$
$$\mathcal{R}_w = \frac{-0.19928}{-0.20471} = 1.027 \pm 0.002$$

The significant deviation from 1 (2.7σ) confirms asymmetry detection through inaccuracy extropy.

5 Conclusion

This paper established fundamental characterizations of continuous symmetric distributions through weighted extropy measures of dual k-records. We proved that a distribution is symmetric if and only if the weighted extropies of its upper and lower k-records coincide for all $k, n \geq 1$ under reflexive weight functions . The proposed generalized inaccuracy measure, integrating Kerridge's framework with weighted extropy, demonstrated the higher sensitivity to asymmetry compared to moment-based approaches. Numerical validations using adaptive quadrature achieved 10^{-4} precision, particularly effective for heavy-tailed distributions like the Weibull model.

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Jordan Superderivations And Anti-Superderivation On Generalized Matrix Algebras

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Abstract

Let $\mathcal{G}$ denote a generalized matrix algebra defined by the Morita context. Viewing $\mathcal{G}$ as a superalgebra, we consider it with a $\mathbb{Z}_2$-grading induced by the idempotent e. This paper explores the structure of Jordan superderivations within generalized matrix algebras, highlighting the existence of Jordan superderivations that do not qualify as superderivations. Furthermore, if the bilinear pairings $\phi_{\mathcal{MN}}$ and $\psi_{\mathcal{NM}}$ are both zero, then every Jordan superderivation of $\mathcal{G}$ is the sum of a superderivation and an anti-superderivation. Notably, under certain conditions, we establish that any Jordan superderivation of an upper triangular algebra is, in fact, a derivation.

Keywords: Superalgebra, Jordan superderivation, anti-superderivation, generalized matrix algebra, triangular algebra.

1 Introduction

Throughout the paper, R will denote a commutative ring with unity, and all algebras and modules will be unital over R. A Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \phi_{\mathcal{MN}}, \psi_{\mathcal{NM}})$, consists of two unital R-algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{N}$, and two bimodule homomorphisms called the bilinear pairings $\Phi_{\mathcal{MN}}\colon \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\Psi_{\mathcal{NM}}\colon \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$, facilitates the construction of generalized matrix algebras, denoted $\mathcal{G}$. According to the method outlined by Ghahramani and Heidari Zadeh [3], $\mathcal{G}$ is treated as a $\mathbb{Z}_2$-graded associative superalgebra defined over a commutative ring R with unity, with $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$. Define the *Jordan superproduct* of two homogeneous elements $x, y \in \mathcal{G}_0 \cup \mathcal{G}_1$ by

$$x \circ_s y = xy + (-1)^{|x||y|} yx,$$

and

$$x \circ_s y = x_0 \circ_s y_0 + x_1 \circ_s y_0 + x_0 \circ_s y_1 + x_1 \circ_s y_1$$

where $x = x_0 + x_1$, $y = y_0 + y_1$.

Let $i \in \mathbb{Z}_2$ and $\mathcal{A}$ be an associative superalgebra. Recall that a linear map $d_i : \mathcal{A} \to \mathcal{A}$ is called a *superderivation of degree i,* (resp. *Jordan superderivation*) if it satisfies $d_i(\mathcal{A}_j) \subseteq \mathcal{A}_{i+j}$ (index modulo 2) and $d_i(xy) = d_i(x)y + (-1)^{|i||x|} x d_i(y)$ (resp. $d_i(x \circ_s y) = d_i(x) \circ_s y + (-1)^{i|x|} x \circ_s d_i(y)$) for all $x, y \in \mathcal{A}$. A superderivation (resp. Jordan superderivation) of $\boldsymbol{A}$ is the sum of a superderivation (resp. Jordan superderivation) of degree 0 and a superderivation (resp. Jordan superderivation) of degree 1. A Jordan

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superderivation that is not a superderivation will be called a *proper Jordan superderivation*. The concept of a Jordan superderivation generalizes the notion of a Jordan derivation in associative algebras. For trivial superalgebras (where the odd part is zero), the two concepts are equivalent. Drawing from Herstein's theorem (extended by Fošner, [2]) and recent studies ([3], [4]), this paper examines the structural properties of Jordan superderivations in generalized matrix algebras. Notably, it establishes that, under specific conditions, Jordan superderivations in upper triangular algebras are derivations.

Motivated by the definition of anti-derivation, we introduced the concept of anti-superderivation as follows: we say that an *anti-superderivation* of degree i is a linear map $\bar{d}_i\colon \boldsymbol{A} \to \boldsymbol{A}$ such that $\bar{d}_i(\mathcal{A}_j) \subseteq \mathcal{A}_{i+j}$ and $\bar{d}_i(xy) = (-1)^{|x||y|}\big(\bar{d}_i(y)x + (-1)^{i|y|}y\bar{d}_i(x)\big)$ for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$. An anti-superderivation is the sum of an anti-superderivation of degree 0 and an anti-superderivation of degree 1. Superderivations and anti-superderivations are examples of Jordan superderivations. We observe that any anti-superderivation on a class of generalized matrix algebra is zero (see Corollary 2.8). Furthermore, it is shown that every Jordan superderivation on another class of generalized matrix algebras is the sum of a superderivation and an anti-superderivation(see Theorem 2.9).

2 Jordan superderivations

Proposition 2.1. *Let $\mathcal{G}$ be a generalized matrix algebra. A linear map $J_0 : \mathcal{G} \to \mathcal{G}$ is a Jordan superderivation of degree 0 if and only if*

$$J_0\Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\Big) = \begin{bmatrix} \alpha_1(a) + \alpha_4(b) & \mu_2(m) + \mu_3(n) \\ \nu_2(m) + \nu_3(n) & \beta_1(a) + \beta_4(b) \end{bmatrix}, \quad (\clubsuit_0)$$

where $\alpha_1\colon \boldsymbol{A} \to \boldsymbol{A}$, $\alpha_4\colon \mathcal{B} \to \mathcal{A}$, $\mu_2\colon \mathcal{M} \to \mathcal{M}$, $\mu_3\colon \mathcal{N} \to \mathcal{M}$,$\nu_2\colon \mathcal{M} \to \mathcal{N}$, $\nu_3\colon \mathcal{N} \to \mathcal{N}$, $\beta_1\colon \mathcal{A} \to \mathcal{B}$, $\beta_4\colon \mathcal{B} \to \mathcal{B}$ are linear maps satisfying the following conditions:

1. *α_1 is a Jordan derivation on $\boldsymbol{A}$ and $\beta_1(a \circ a') = 0$;*
2. *β_4 is a Jordan derivation on $\boldsymbol{B}$ and $\alpha_4(b \circ b') = 0$;*
3. *$\mu_2(am) = a\mu_2(m) + \alpha_1(a)m + m\beta_1(a)$ and $\nu_2(am) = \nu_2(m)a$;*
4. *$\mu_2(mb) = \mu_2(m)b + m\beta_4(b) + \alpha_4(b)m$ and $\nu_2(mb) = b\nu_2(m)$;*
5. *$\nu_3(na) = \beta_1(a)n + n\alpha_1(a) + \nu_3(n)a$ and $\mu_3(na) = a\mu_3(n)$;*
6. *$\nu_3(bn) = \beta_4(b)n + n\alpha_4(b) + b\nu_3(n)$ and $\mu_3(bn) = \mu_3(n)b$;*
7. *$\alpha_1(mn) = \alpha_4(nm) + \mu_2(m)n + m\nu_3(n)$;*
8. *$\beta_4(nm) = \beta_1(mn) + n\mu_2(m) + \nu_3(n)m$.*

Proof. Since $J_0(\mathcal{G}_0) \subseteq \mathcal{G}_0$ and $J_0(\mathcal{G}_1) \subseteq \mathcal{G}_1$, we can write J_0 mathfonts

$$J_0\Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\Big) = \begin{bmatrix} \alpha_1(a) + \alpha_4(b) & \mu_2(m) + \mu_3(n) \\ \nu_2(m) + \nu_3(n) & \beta_1(a) + \beta_4(b) \end{bmatrix}$$

where $\alpha_1\colon \boldsymbol{A} \to \boldsymbol{A}$, $\alpha_4\colon \mathcal{B} \to \mathcal{A}$, $\mu_2\colon \mathcal{M} \to \mathcal{M}$, $\mu_3\colon \mathcal{N} \to \mathcal{M}$, $\nu_2\colon \mathcal{M} \to \mathcal{N}$, $\nu_3\colon \mathcal{N} \to \mathcal{N}$, $\beta_1\colon \mathcal{A} \to \mathcal{B}$, and $\beta_4\colon \mathcal{B} \to \mathcal{B}$ are linear maps. For any $X, Y \in \mathcal{G}_0 \cup \mathcal{G}_1$, we will consider the equation

$$J_0(X \circ_s Y) = J_0(X) \circ_s Y + X \circ_s J_0(Y). \tag{2.1}$$

Taking $X = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}$ in (2.1), yields

$$J_0(X \circ_s Y) = \begin{bmatrix} \alpha_1(a \circ a') & 0 \\ 0 & \beta_1(a \circ a') \end{bmatrix}$$

and

$$J_0(X) \circ_s Y + X \circ_s J_0(Y) = \begin{bmatrix} \alpha_1(a) \circ a' & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a \circ \alpha_1(a') & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, α_1 is a Jordan derivation on $\boldsymbol{A}$ and $\beta_1(a \circ a') = 0$ for all $a, a' \in \boldsymbol{A}$. Similarly, if we choose $X = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix}$ in (2.1), then we see that β_4 is a Jordan derivation on $\boldsymbol{B}$ and $\alpha_4(b \circ b') = 0$ for all $b, b' \in \boldsymbol{B}$.

For $X = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ in (2.1), we obtain

$$\begin{bmatrix} 0 & \mu_2(ma) \\ \nu_2(am) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1(a)m + m\beta_1(a) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a\mu_2(m) \\ \nu_2(m)a & 0 \end{bmatrix}.$$

That is $\mu_2(am) = a\mu_2(m) + \alpha_1(a)m + m\beta_1(a)$ and $\nu_2(am) = \nu_2(m)a$. Likewise, taking $X = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $Y = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ in (2.1) yields the condition (4). Furthermore, if we put $X = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ in (2.1) then we have

$$\begin{bmatrix} 0 & \mu_3(na) \\ \nu_3(na) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0) \\ \beta_1(a)n + n\alpha_1(a) & 0 \end{bmatrix} + \begin{bmatrix} 0 & a\mu_3(n) \\ \nu_3(n)a & 0 \end{bmatrix}.$$

$\mu_3(na) = a\mu_3(n)$ and $\nu_3(na) = \beta_1(a)n + n\alpha_1(a) + \nu_3(n)a$. Similarly, by taking $X = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ in (2.1) we can obtain condition (6). Finally, taking $X = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ in (2.1). We have $X, Y \in \mathcal{G}_1$. So $X \circ_s Y = XY - YX$,

$$J_0(X \circ_s Y) = \begin{bmatrix} \alpha_1(mn) - \alpha_4(nm) & 0 \\ 0 & \beta_1(mn) - \beta_4(nm) \end{bmatrix}$$

and

$$J_0(X) \circ_s Y + X \circ_s J_0(Y) = \begin{bmatrix} \mu_2(m)n & 0 \\ 0 & -n\mu_2(m) \end{bmatrix} + \begin{bmatrix} m\nu_3(n) & 0 \\ 0 & -\nu_3(n)m \end{bmatrix}.$$

Hence, conditions (7) and (8) follow easily.

Conversely, If J_0 has the form ($\clubsuit_0$) and satisfies conditions $(1) - (8)$, then it is easy to show that J_0 is a Jordan superderivation of degree 0. Thereby, the proof is completed. □

Proposition 2.2. *The linear map $J_1 : \mathcal{G} \to \mathcal{G}$ is a Jordan superderivation of degree 1 if and only if*

$$J_1(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} mn_0 - m_0n & am_0 - m_0b \\ n_0a - bn_0 & n_0m - nm_0 \end{bmatrix} \quad (\clubsuit_1)$$

for some $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}$.

Proof. Assume that $J_1 : \mathcal{G} \to \mathcal{G}$ is a Jordan superderivation of degree 1. Then $J_1(\mathcal{G}_0) \subseteq \mathcal{G}_1$, $J_1(\mathcal{G}_1) \subseteq \mathcal{G}_0$ and

$$J_1(X \circ_s Y) = J_1(X) \circ_s Y + (-1)^{|X|} X \circ_s J_1(Y) \tag{2.2}$$

for all $X, Y \in \mathcal{G}_0 \cup \mathcal{G}_1$. Thus, J_1 is of the form

$$J_1(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} \alpha_2(m) + \alpha_3(n) & \mu_1(a) + \mu_4(b) \\ \nu_1(a) + \nu_4(b) & \beta_2(m) + \beta_3(n) \end{bmatrix}$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$. Taking, $X = \begin{bmatrix} 1_{\boldsymbol{A}} & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ in (2.2) we obtain

$$J_1(X \circ_s Y) = \begin{bmatrix} \alpha_2(m) & 0 \\ 0 & \beta_2(m) \end{bmatrix}$$

and

$$J_1(X) \circ_s Y + (-1)^{|X|} X \circ_s J_1(Y) = \begin{bmatrix} -m\nu_1(1_{\boldsymbol{A}}) & 0 \\ 0 & \nu_1(1_{\boldsymbol{A}})m \end{bmatrix} + \begin{bmatrix} 2\alpha_2(m) & 0 \\ 0 & 0 \end{bmatrix}.$$

That is $\alpha_2(m) = mn_0$ and $\beta_2(m) = n_0 m$ where $\nu_1(1_{\boldsymbol{A}})=n_0$. For $X = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 1_{\boldsymbol{A}} & 0 \\ 0 & 0 \end{bmatrix}$ into (2.2), we obtain

$$J_1(X \circ_s Y) = \begin{bmatrix} \alpha_3(n) & 0 \\ 0 & \beta_3(n) \end{bmatrix}.$$

Since $|X| = 1$, so we have

$$J_1(X) \circ_s Y + (-1)^{|X|} X \circ_s J_1(Y) = \begin{bmatrix} 2\alpha_3(n) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -\mu_1(1_{\boldsymbol{A}})n & 0 \\ 0 & n\mu_1(1_{\boldsymbol{A}}) \end{bmatrix}.$$

That is $\alpha_3(n) = -m_0 n$ and $\beta_3(n) = -nm_0$ where $\mu_1(1_A) = m_0$. Let us take $X = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ in (2.2). Since $X \circ_s Y = 0$, we see

$$\begin{aligned} 0 = J_1(X \circ_s Y) &= J_1(X) \circ_s Y + (-1)^{|X|} X \circ_s J_1(Y) \\ &= \begin{bmatrix} 0 & \mu_1(a)b \\ b\nu_1(a) & 0 \end{bmatrix} + \begin{bmatrix} 0 & a\mu_4(b) \\ \nu_4(b)a & 0 \end{bmatrix}. \end{aligned}$$

From the last two equations, we obtain

$$\mu_1(a)b + a\mu_4(b) = 0 \quad \text{and} \quad b\nu_1(a) + \nu_4(b)a = 0 \tag{2.3}$$

for all $a \in \boldsymbol{A}$, $b \in \boldsymbol{B}$. By considering $a = 1_{\boldsymbol{A}}$ in (2.3), we obtain $\mu_4(b) = -m_0 b$, $\nu_4(b) = -bn_0$. Also, by taking $b = 1_{\boldsymbol{B}}$ in (2.3), we get $\mu_1(a) = am_0$ and $\nu_1(a) = bn_0$. Based on these results, J_1 is in the desired form.

The converse is clear. □

Theorem 2.3. *Let $\mathcal{G}$ be a generalized matrix algebra. A linear map $J : \mathcal{G} \to \mathcal{G}$ is a Jordan superderivation if and only if*

$$J(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} \alpha_1(a) + \alpha_4(b) + mn_0 - m_0 n & \mu_2(m) + \mu_3(n) + am_0 - m_0 b \\ \nu_2(m) + \nu_3(n) + n_0 a - bn_0 & \beta_1(a) + \beta_4(b) + n_0 m - nm_0 \end{bmatrix}, \quad (\clubsuit 2)$$

where $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}$ and $\alpha_1 : \boldsymbol{A} \to \boldsymbol{A}$, $\alpha_4 : \mathcal{B} \to \mathcal{A}$, $\mu_2 : \mathcal{M} \to \mathcal{M}$, $\mu_3 : \mathcal{N} \to \mathcal{M}$, $\nu_2 : \mathcal{M} \to \mathcal{N}$, $\nu_3 : \mathcal{N} \to \mathcal{N}$, $\beta_1 : \mathcal{A} \to \mathcal{B}$, $\beta_4 : \mathcal{B} \to \mathcal{B}$ are linear maps satisfying the following conditions:

1. α_1 *is a Jordan derivation on* $\boldsymbol{A}$ *and* $\beta_1(a \circ a') = 0$;
2. β_4 *is a Jordan derivation on* $\boldsymbol{B}$ *and* $\alpha_4(b \circ b') = 0$;
3. $\mu_2(am) = a\mu_2(m) + \alpha_1(a)m + m\beta_1(a)$ *and* $\nu_2(am) = \nu_2(m)a$;
4. $\mu_2(mb) = \mu_2(m)b + m\beta_4(b) + \alpha_4(b)m$ *and* $\nu_2(mb) = b\nu_2(m)$;
5. $\nu_3(na) = \nu_3(n)a + n\alpha_1(a) + \beta_1(a)n$ *and* $\mu_3(na) = a\mu_3(n)$;
6. $\nu_3(bn) = \beta_4(b)n + n\alpha_4(b) + b\nu_3(n)$ *and* $\mu_3(bn) = \mu_3(n)b$;
7. $\alpha_1(mn) = \alpha_4(nm) + \mu_2(m)n + m\nu_3(n)$;
8. $\beta_4(nm) = \beta_1(mn) + n\mu_2(m) + \nu_3(n)m$.

Proof. Suppose that J is a Jordan superderivation. According to the definition, $J = J_0 + J_1$, where J_0 is a Jordan superderivation of degree 0 and J_1 is a Jordan superderivation of degree 1. From Proposition 2.1 and Proposition 2.2, J is in the desired form.

Conversely, suppose that J has the mentioned form. Define J_0 and J_1 on $\mathcal{G}$ by

$$J_0\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \alpha_1(a) + \alpha_4(b) & \mu_2(m) + \mu_3(n) \\ \nu_2(m) + \nu_3(n) & \beta_1(a) + \beta_4(b) \end{bmatrix},$$

and

$$J_1\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} mn_0 - m_0n & am_0 - m_0b \\ n_0a - bn_0 & n_0m - nm_0 \end{bmatrix},$$

So $J = J_0 + J_1$. It follows from Proposition 2.1 and Proposition 2.2 that J_0 is a Jordan superderivation of degree 0 and J_1 is a Jordan superderivation of degree 1. Hence, J is a Jordan superderivation. □

Example 2.4. Let $\boldsymbol{A}$ be a commutative unital algebra over R and let $\mathcal{G} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A} \\ \boldsymbol{A} & \boldsymbol{A} \end{bmatrix}$ be a generalized matrix algebra. For arbitrary $X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{G}$ and $Y = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} \in \mathcal{G}$ we define the sum $X + Y$ as usual. The multiplication XY is given by the rule

$$XY = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} = \begin{bmatrix} a_{11}a'_{11} & a_{11}a'_{12} + a_{12}a'_{22} \\ a_{21}a'_{11} + a_{22}a'_{21} & a_{22}a'_{22} \end{bmatrix}$$

We define a linear map $J : \mathcal{G} \to \mathcal{G}$ as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto \begin{bmatrix} 0 & a_{21} \\ a_{12} & 0 \end{bmatrix}$. By straightforward computations, we see that J is a Jordan superderivation of degree 0 on $\mathcal{G}$, but since for $X = Y = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ and $a^2 \neq 0$, $J(XY) = 0$ while $J(X)Y + XJ(Y) = \begin{bmatrix} 0 & 0 \\ 0 & 2a^2 \end{bmatrix}$, J is a proper Jordan superderivation of degree 0.

In studies on generalized matrix algebras, $\boldsymbol{M}$ or $\mathbb{N}$ are often considered faithful bimodules. However, in this article, we propose the following weaker condition on $\mathcal{G}$:

$$\begin{aligned} &a \in \boldsymbol{A},\ a\boldsymbol{M} = 0 \text{ and } \mathbb{N}a = 0 \Rightarrow a = 0; \\ &b \in \boldsymbol{B},\ \boldsymbol{M}b = 0 \text{ and } b\mathbb{N} = 0 \Rightarrow b = 0. \end{aligned} \tag{2.4}$$

Corollary 2.5. *Let $\mathcal{G}$ be a 2-torsion free generalized matrix algebra satisfying (2.4). A linear map $J : \mathcal{G} \to \mathcal{G}$ is a Jordan superderivation if and only if*

$$J(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} \alpha_1(a) + mn_0 - m_0n & \mu_2(m) + \mu_3(n) + am_0 - m_0b \\ \nu_2(m) + \nu_3(n) + n_0a - bn_0 & \beta_4(b) + n_0m - nm_0 \end{bmatrix}, \quad (\clubsuit 3)$$

where $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}$, $\alpha_1 \colon \mathcal{A} \to \mathcal{A}$, $\mu_2 \colon \mathcal{M} \to \mathcal{M}$, $\mu_3 \colon \mathcal{N} \to \mathcal{M}$, $\nu_2 \colon \mathcal{M} \to \mathcal{N}$, $\nu_3 \colon \mathcal{N} \to \mathcal{N}$, and $\beta_4 \colon \mathcal{B} \to \mathcal{B}$ are linear maps satisfying the following conditions:

1. *α_1 is a derivation on $\boldsymbol{A}$ and $\alpha_1(mn) = \mu_2(m)n + m\nu_3(n)$;*
2. *β_4 is a derivation on $\boldsymbol{B}$ and $\beta_4(nm) = n\nu_2(m) + \nu_3(n)m$;*
3. *$\mu_2(am) = a\mu_2(m) + \alpha_1(a)m$ and $\nu_2(am) = \nu_2(m)a$;*
4. *$\mu_2(mb) = \mu_2(m)b + m\beta_4(b)$ and $\nu_2(mb) = b\nu_2(m)$;*
5. *$\nu_3(na) = \nu_3(n)a + n\alpha_1(a)$ and $\mu_3(na) = a\mu_3(n)$;*
6. *$\nu_3(bn) = \beta_4(b)n + b\nu_3(n)$ and $\mu_3(bn) = \mu_3(n)b$;*

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

Proof. Let J be a Jordan superderivation on $\mathcal{G}$. Then J is in the form ($\clubsuit 2$) and satisfies all conditions (1)-(8) of Theorem 2.3. Since $\mathcal{G}$ is 2-torsion free, by substituting $a' = 1_{\mathcal{A}}$ into $\beta_1(a \circ a') = 0$ (condition (1)) and $b' = 1_{\mathcal{B}}$ into $\alpha_4(b \circ b') = 0$ (condition (2)) of Theorem 2.3, we deduce that $\alpha_4 = 0$ and $\beta_1 = 0$. Hence, we only need to prove that α_1 is a derivation of $\boldsymbol{A}$ and β_4 is a derivation of $\boldsymbol{B}$. From conditions (3) and (5) of Theorem 2.3, the equations take the following form:

$$\mu_2(am) = a\mu_2(m) + \alpha_1(a)m$$

and

$$\nu_3(na) = \nu_3(n)a + n\alpha_1(a).$$

So, for arbitrary elements $a, a' \in \boldsymbol{A}$, we have

$$\mu_2(aa'm) = aa'\mu_2(m) + \alpha_1(aa')m$$

and

$$\nu_3(naa') = \nu_3(n)aa' + n\alpha_1(aa').$$

On the other hand, one obtains

$$\begin{aligned} \mu_2(aa'm) &= a\mu_2(a'm) + \alpha_1(a)a'm \\ &= aa'\mu_2(m) + a\alpha_1(a')m + \alpha_1(a)a'm \end{aligned}$$

and

$$\begin{aligned} \nu_3(naa') &= \nu_3(na)a' + na\alpha_1(a') \\ &= \nu_3(n)aa' + na\alpha_1(a') + n\alpha_1(a)a' \end{aligned}$$

Combining the last relations gives

$$[\alpha_1(aa') - a\alpha_1(a') - \alpha_1(a)aa']m = 0$$

and

$$n[\alpha_1(aa') - a\alpha_1(a') - \alpha_1(a)a'] = 0.$$

for all $a, a' \in \boldsymbol{A}$, $m \in \boldsymbol{M}$ and $n \in \mathbb{N}$. So it follows from (2.4) that $\alpha_1(aa') = a\alpha_1(a') + \alpha_1(a)a'$ for all $a, a' \in \boldsymbol{A}$. Hence, α_1 is a derivation of $\boldsymbol{A}$. Using a similar process from conditions (4), (6) and (2.4), we can show that β_4 is a derivation of $\boldsymbol{B}$.

The converse is trivial. □

Given that if $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra, then (2.4) is equivalent to the faithfulness of $\mathcal{M}$ as $(\mathcal{A}, \mathcal{B})$-bimodule. Based on Corollary 2.5 and [1, Lemma 5], we have the following.

Corollary 2.6. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a 2-torsion free unital triangular algebra such that $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. A linear map $J : \mathcal{T} \to \mathcal{T}$ is a Jordan superderivation if and only if*

$$J(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}) = \begin{bmatrix} \alpha_1(a) & \mu_2(m) + am_0 - m_0b \\ 0 & \beta_4(b) \end{bmatrix},$$

where $a \in \mathcal{A}$, $b \in \mathcal{B}$, $m, m_0 \in \mathcal{M}$, and $\alpha_1 \colon \mathcal{A} \to \mathcal{A}$, $\mu_2 \colon \mathcal{M} \to \mathcal{M}$, $\beta_4 \colon \mathcal{B} \to \mathcal{B}$ are linear maps satisfying the following conditions:

1. *α_1 is a derivation on $\boldsymbol{A}$;*
2. *β_4 is a derivation on $\boldsymbol{B}$;*
3. *$\mu_2(am) = a\mu_2(m) + \alpha_1(a)m$;*
4. *$\mu_2(mb) = \mu_2(m)b + m\beta_4(b)$.*

In particular, any Jordan superderivation of the triangular algebra $\mathcal{T}$ is a derivation.

Proposition 2.7. *Let $\mathcal{G}$ be a generalized matrix algebra satisfying (2.4). A linear map $\bar{d} \colon \mathcal{G} \to \mathcal{G}$ is an anti-superderivation if and only if*

$$\bar{d}(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} 0 & \mu_3(n) + am_0 - m_0b \\ \nu_2(m) + n_0a - bn_0 & 0 \end{bmatrix}, \quad (\clubsuit 4)$$

where $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}$ and $\mu_3 \colon \mathcal{N} \to \mathcal{M}$, $\nu_2 \colon \mathcal{M} \to \mathcal{N}$ are linear maps satisfying the following conditions:

1. *$\mu_3(na) = a\mu_3(n)$, $\mu_3(bn) = \mu_3(n)b$ and $\mu_3(n')n = n'\mu_3(n) = 0$;*
2. *$\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = b\nu_2(m)$ and $m'\nu_2(m) = \nu_2(m')m = 0$;*
3. *$n_0m = 0$, $mn_0 = 0$, $nm_0 = 0$ and $m_0n = 0$;*

for all $a \in \boldsymbol{A}$, $b \in \boldsymbol{B}$, $m, m' \in \boldsymbol{M}$ and $n, n' \in \mathbb{N}$.

Corollary 2.8. *Let $\mathcal{G}$ be a generalized matrix algebra and $\bar{d}$ be an R-linear map from $\mathcal{G}$ into itself. If one of the bilinear forms $\phi_{\mathcal{MN}} \colon \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \to \mathcal{A}$ and $\psi_{\mathcal{NM}} \colon \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \to \mathcal{B}$ is nondegenerate, then $\bar{d}$ is an anti-superderivation of $\mathcal{G}$ if and only if $\bar{d} = 0$.*

Theorem 2.9. *Let $\mathcal{G}$ be a 2-torsion free generalized matrix algebra satisfying (2.4). If the bilinear pairings $\phi_{\mathcal{MN}}$ and $\psi_{\mathcal{NM}}$ are both zero, then every Jordan superderivation of $\mathcal{G}$ can be expressed as the sum of a superderivation and an anti-superderivation.*

Proof. Let J be a Jordan superderivation of $\mathcal{G}$. By Theorem 2.5, J is of the form

$$J(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} \alpha_1(a) + mn_0 - m_0n & \mu_2(m) + \mu_3(n) + am_0 - m_0b \\ \nu_2(m) + \nu_3(n) + n_0a - bn_0 & \beta_4(b) + n_0m - nm_0 \end{bmatrix}, \quad (\clubsuit 2)$$

From Lemma [3, Lemma 4.1, 4.2] and Proposition 2.7, we have

$$\begin{aligned} J(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) &= \begin{bmatrix} \alpha_1(a) + mn_0 - m_0n & \mu_2(m) + am_0 - m_0b \\ \nu_3(n) + n_0a - bn_0 & \beta_4(b) + n_0m - nm_0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & \mu_3(n) \\ \nu_2(m) & 0 \end{bmatrix} \\ &= d(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) + \bar{d}(\begin{bmatrix} a & m \\ n & b \end{bmatrix}). \end{aligned}$$

This shows that J can be written as a sum of a superderivation d and anti-superderivation $\bar{d}$. □

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Hybrid Approach to Estimation: Combining Extreme Learning Machine with Linear Mixed Model

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Abstract

The most critical component in predictive analytics is parameter estimation which is directed towards better performance and interpretability of the predictions. This work proposes a new hybrid methodology that combines Linear Mixed Models that allow for random effect estimation and a fixed effects model based on Extreme Learning Machines. Most often, the difficulty is in finding the linear model interpretability and at the same time the needed intricacy to the data. The designed framework Linear Mixed Models is used to explain this variability coming from random effects like subject specific variability, while Extreme Learning Machine incorporates and deals with the cohesive effect in a very fast manner with its own learning and modeling abilities. This helps to capture both the structured random variations and the complex fixed relationships. The combined approach not only improves the accuracy and robustness of estimating parameters but also improves the model's ability to adapt to different structures of data. Results from using the hybrid model show significant improvements over conventional models concerning predictive ability and speed. This research offers a model that is suitable across numerous disciplines, including but not limited to health, finance, and social science, which tend to have complicated data structures

Keywords: Linear Mixed Model, Extreme Learning Machine, Randome Effects, Fixed Effects, Estimation

1 Introduction

The use of Linear Mixed Models (LMMs) plays a crucial role in analyzing hierarchical, longitudinal, and clustered data, particularly when estimating both fixed effects and variance components [1], [2]. Traditionally, maximum likelihood (ML) and restricted maximum likelihood (REML) methods were used for variance components' estimation [1], [2]. With the advent of faster computing and algorithms like Newton-Raphson and the expectation-maximization (EM) algorithm [3], LMMs became viable for large and complex datasets. Subsequently, multilevel and Bayesian methods were introduced [6], [7], and specialized techniques were developed to handle high-dimensional, sparse, or incomplete data [8], [9], [10], [11].

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Recently, there has been growing interest in integrating LMMs with Machine Learning (ML) methods to handle large, multivariate, and intricate datasets [12], [13]. The combination of LMMs and ML techniques — particularly penalized, sparse, and robust methods — provides powerful tools for variable selection, multicollinearity, and overfitting [12], [13]. Furthermore, this approach has shown strong utility in fields such as genomics [15], [16].

The Extreme Learning Machine (ELM), a feed-forward neural network algorithm, has become a popular ML approach due to its fast training and strong generalization capabilities [18]. Traditionally, ELM randomly assigns hidden-layer weights and biases and then solves output weights through closed-form regression [18]. It has been successfully applied to regression and classification problems [19], [20], [21], [23], [24], [25], [26]. Furthermore, variants of ELM — such as those employing Differential Evolution [19], multi-objective techniques [22], and Ridge or Liu regression [24], [25] — have demonstrated improved performance in large and multivariate contexts.

This paper proposes a hybrid approach that merges LMM and ELM, estimating the random effects through a linear mixed framework and the fixed effects through ELM. This combination lets the model account for both population-averaged effects and subject-specific variability while retaining ELM's fast training and strong approximation capabilities. The main advantage lies in effectively addressing grouped and hierarchical structures in complex data — something pure ELM struggles with.

The structure of the paper reflects this approach. Section 2 reviews ELM fundamentals [18], Section 3 covers LMM and its multilevel framework [1], [2], Section 4 details the hybrid proposal, Section 5 focuses on parameter estimation methods, Section 6 shows empirical results validating the approach's utility, and Section 7 summarizes key findings and future directions.

2 Extreme learning machine

The Extreme Learning Machine (ELM) was initially introduced by [18] as a fast learning, single-layer feed-forward neural network that achieves high learning rates and exceptional generalization performance. This method is based on the premise that the weights and biases connecting the input layer to the hidden layer can be generated randomly. As a result, the estimation problem simplifies to solving a set of linear equations, which can be accomplished easily using closed-form solutions, eliminating the need for parameter optimization or iterative algorithms. Typically, the output layer weights are determined using least squares. This approach significantly reduces the overall computation time of ELM, making its training time shorter than that of alternative gradient descent optimization techniques. The ELM algorithm is applicable for both regression and classification tasks. A wealth of literature discussing ELM and its various implementations can be found in publications [27], [28], [29], and [30].

2.1 The model of ELM

For N arbitrary distinct samples (x_j, t_j) where $(x_j = [x_{j1}, x_{j2}, \ldots, x_{jn}] \in \mathbb{R}^n)$ and $(t_j = [t_{j1}, t_{j2}, \ldots, t_{jm}] \in \mathbb{R}^m)$, standard SLFNs with $\tilde{N}$ hidden nodes and activation function $g(x)$ are mathematically modeled as

$$\sum_{i=1}^{\tilde{N}} \beta_i g(w_i \cdot x_j + b_i) = t_j, \quad i = 1, \ldots, \tilde{N}, \quad j = 1, \ldots, N, \tag{2.1}$$

where $(w_i = [w_{i1}, w_{i2}, \ldots, w_{in}])$ is the weight vector connecting the ith hidden node and the input nodes, $\tilde{N}$ is the number of hidden neurons in the hidden layer, $\beta_i = [\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}]$ is the weight vector connecting the i th hidden node and the output nodes, and b_i is the threshold of the ith hidden node. $w_i \cdot x_j$ denotes the inner product of w_i and x_j ([18]). Eq.(2.1) can be written compactly as

$$H\beta = T \tag{2.2}$$

where

$$H(w_1, \ldots, w_{\tilde{N}}, b_1, \ldots, b_{\tilde{N}}, x_1, \ldots, x_{\tilde{N}}) =$$

$$\begin{bmatrix} g(w_1 x_1 + b_1) & \cdots & g(w_{\tilde{N}} x_1 + b_{\tilde{N}}) \\ \vdots & \ddots & \vdots \\ g(w_1 x_N + b_1) & \cdots & g(w_{\tilde{N}} x_N + b_{\tilde{N}}) \end{bmatrix}_{N \times \tilde{N}}$$

is called as hidden layer output matrix of the neural network by [18],

$$\beta = \begin{bmatrix} \beta_1^T \\ \vdots \\ \beta_{\tilde{N}}^T \end{bmatrix}_{\tilde{N} \times m} , T = \begin{bmatrix} t_1^T \\ \vdots \\ t_{\tilde{N}}^T \end{bmatrix}_{\tilde{N} \times m}$$

are the weight matrix and the training data target matrix, respectively. As assumed in ELM, if the input weights w_i and the hidden layer biases b_i are chosen randomly, i.e., fixed, training a Single Layer Feedforward Network (SLFN) is simply finding the least squares solution, say $\hat{\beta}$ in Eq.(1.4), by solving the estimation problem

$$H\left(w_1, \ldots, w_{\tilde{N}}, b_1, \ldots, b_{\tilde{N}}\right)\hat{\beta} - T = \min_{\beta} H\left(w_1, \ldots, w_{\tilde{N}}, b_1, \ldots, b_{\tilde{N}}\right)\beta - T$$

If the H matrix is square, $\tilde{N} = N$, and invertible, then β is found by $\hat{\beta} = (H^T H)^{-1} H^T T$. If the inverse of H cannot be calculated or the H matrix is nonsquare, a generalized inverse [31],[32] providing the smallest norm least squares solution can be used. Thus, the output weight vector or the least squares estimator of β is estimated analytically as follows:

$$\beta = (H^+)T$$

where H^+ is the Moore–Penrose inverse of the matrix H.

3 Linear Mixed Model

Using the notation of [33] , the Linear mixed effects model can be described as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{u} + \boldsymbol{\epsilon}, \tag{3.1}$$

where $\mathbf{y}$ is the $(n \times 1)$ vector of observed dependent variables , where n is the sample size . $\mathbf{X}$ is the $(n \times p)$ matrix of p observed variables for n observational units. $\boldsymbol{\beta}$ is the $(p \times 1)$ vector of unknown regression coefficients (fixed effects). $\mathbf{Z}$ is the $(n \times p_r)$ design matrix. $\boldsymbol{u}$ is the $(p_r \times 1)$ vector of unobserved random variables (random effects), with $\boldsymbol{u} \sim \mathcal{N}(\mathbf{0}_{p_r}, \boldsymbol{\Sigma})$. $\boldsymbol{\epsilon}$ is the $(n \times 1)$ vector of normally distributed errors variables, with $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma_\epsilon^2 \mathbf{I})$. Here, p_r is the number of covariates for which random effects are included. Without loss of generality, covariates are ordered such that the first p_r covariates have random effects. The expected values of individual observation i is given by:

$$\hat{y}_i = \hat{y}_{\text{fixed},i} + \mathbf{z}_i \hat{\mathbf{u}}, \tag{3.2}$$

where $\mathbf{z}_i = (1, \tilde{\mathbf{x}}_i)$ is the respective row in the design matrix $\mathbf{Z}$, consisting of a bias/intercept term (=1) and the feature vector $\tilde{\mathbf{x}}_i = \mathbf{x}_i[1 : p_r]$. The estimated random effects are $\hat{\mathbf{u}}$. Thus, $\mathbf{y}$ is conditionally normally distributed as:

$$\mathbf{y} \sim \mathcal{N}(\mathbf{y}_{\text{fixed}}, \mathbf{V}),$$

where:

$$\mathbf{V} = \mathbf{Z}\boldsymbol{\Sigma}\mathbf{Z}^\top + \sigma_\epsilon^2 \mathbf{I}_n.$$

In the literature, expectation-maximization (EM) iterations have been utilized to estimate variance components in linear mixed effects (LMEs) based on either maximum likelihood estimation (MLE) or restricted maximum likelihood estimation (REML).

4 Linear Mixed models for Extreme learning machine

Instead of estimating a relationship between $\mathbf{X}$ and $\mathbf{y}$ via $\mathbf{X}\boldsymbol{\beta}$ for the fixed effects, we use a potentially function $f(\mathbf{X})$ to model this relationship:

$$\mathbf{y} = f(\mathbf{X}) + \mathbf{Z}\boldsymbol{u} + \boldsymbol{\epsilon}, \tag{4.1}$$

where we add the linear random effects with the same assumptions as in Section 3.To parameterize the function f, we use a Extreme learning machine model, referred to as $\text{elm}_{\text{fixed}}(\boldsymbol{\beta})$, and add the random effects , which can be seen as a linear correction. This results in:

$$\mathbf{y} = \text{elm}_{fixed(\hat{\boldsymbol{\beta}})}(\mathbf{X}) + \mathbf{Z}\boldsymbol{u} + \boldsymbol{\epsilon},$$

or, in cluster-specific notation (compare to Equation (1)):

$$\hat{\mathbf{y}}_j = \text{elm}_{fixed(\hat{\boldsymbol{\beta}})}(\mathbf{X}_j) + \mathbf{Z}_j\hat{\mathbf{u}}_j.$$

For the estimation, we split $\mathbf{y}$ into a fixed and random part:

$$\mathbf{y} = \mathbf{y}_{\text{fixed}} + \mathbf{y}_{\text{rand}} + \boldsymbol{\epsilon},$$

where:

$$\mathbf{y}_{\text{fixed}} = \mathbf{y} - \mathbf{Z}\mathbf{u}, \quad \mathbf{y}_{\text{rand}} = \mathbf{e}_{\text{fixed}} = \mathbf{y} - \mathbf{y}_{\text{fixed}},$$

with the cluster-specific realizations $\mathbf{u} = (\mathbf{u}_1^\top, \ldots, \mathbf{u}_J^\top)^\top$ of the random effects.

Iterative Estimation Process

The general idea is as follows, inspired by related ideas for tree-based models (e.g.,[34]; [35]; [36];[37]):

1. Train $\text{elm}_{\text{fixed}}(\boldsymbol{\beta})$ by ignoring randome effects (using the target $\mathbf{y}$) to get a first estimate for $\mathbf{y}_{\text{fixed}}$ (the prediction $\hat{\mathbf{y}}_{\text{fixed}}$ of the model).
2. Estimate $\hat{\mathbf{u}}$ based on $\mathbf{e}_{\text{fixed}}$.
3. Update:
$$\mathbf{y}_{\text{fixed}} = \mathbf{y} - \mathbf{Z}\hat{\mathbf{u}},$$
using the current estimate $\hat{\mathbf{u}}$, and re-train $\text{elm}_{\text{fixed}}(\boldsymbol{\beta})$ with the updated target variable $\mathbf{y}_{\text{fixed}}$.
4. Repeat the process until convergence criteria are met or the maximum number of iterations is reached.

Training Considerations

Since a potentially complex machine learning algorithm needs to be trained at each iteration, the question of training time arises. To address this, the model is re-used and fine-tuned to an adjusted prediction task. In the first iteration, we train the model for the task $(\mathbf{X}, \mathbf{y})$. In subsequent iterations, we update the trained network from the last iteration and fine-tune the parameters for the updated task $(\mathbf{X}, \mathbf{y}_{\text{fixed}})$, comparable to transfer learning.

5 Estimation and prediction

To obtain an estimate for $\mathbf{y}_{\text{fixed}}$, we learn the parameters $\hat{\boldsymbol{\beta}}$ of $\text{elm}_{\text{fixed}}(\boldsymbol{\beta})$ and predict:

$$\hat{\mathbf{y}}_{\text{fixed}} = \text{elm}_{fixed(\hat{\boldsymbol{\beta}})}(\mathbf{X})$$

If $\mathbf{u}$ is known, common supervised machine learning regression methods can be trained and applied to predict $\mathbf{y}_{\text{fixed}}$, including the special case where $\mathbf{u} = \mathbf{0}$ (and thus $\mathbf{y}_{\text{fixed}} = \mathbf{y}$)—this corresponds to ignoring random effects (a typical case). Since the random part of the model (and therefore $\mathbf{u}$) is unknown, we need to find estimates $\hat{\mathbf{u}}$ to use in section(4) To get an estimate for the random part of the model, we need the estimate $\hat{\mathbf{y}}_{\text{fixed}}$. This motivates an alternating, iterative process, akin to the EM algorithm, where estimates of the current fixed or random parameters are used to update the respective other component (see, e.g.,[34]; [35]; [36];[37]).

6 Experimental Results

This section presents the performance results of the basic Extreme Learning Machine (ELM) and the proposed algorithm applied to several linear mixed model tasks. We selected six standard datasets to evaluate their performance, which are available in two well-known repositories: the UCI Machine Learning Repository[40] and [41]. Following the recommendations of the original ELM author [42], we normalized the input features of each dataset to the range of [-1, 1] and scaled the outputs to the range of [0, 1]. Each dataset was divided into two subsets, with 70% allocated for training and 30% for testing. The characteristics of these datasets are summarized in Table 2. The classic datasets Servo, Boston Housing and MathAchieve are

Table 1: The properties of benchmark data sets

Data sets	**Attributes**	**Sample size**	**Training set**	**Testing set**
Servo	4	167	119	48
Boston Housing	13	506	356	150
MathAchive	6	7185	5029	2156

benchmarks in machine learning and statistics, yet they vary in historical and practical relevance.The Servo dataset originates from research in the 1980s on robotics, where the responses of servo systems were modeled based on design parameters.The Boston Housing dataset dates back to the 1970s and focuses on housing price prediction, but it raises ethical concerns due to the racially sensitive variables involved. Finally, the MathAchieve dataset employs a multilevel model to analyze student math performance, considering both individual and school-level effects. This dataset is derived from educational studies conducted in the U.S. during the latter decades of the 20th century.These datasets continue to be widely used by educators, algorithm developers, and anyone interested in exploring the complexities of real-world data.

Table 2: The Performance results of ELM and LMM-ELM Algorithms for Servo dataset. All experiments used 50 hidden nodes.

		Time (seconds)		RMSE		RAE		RSE	
Metric	Algorithm	Train	Test	Train	Test	Train	Test	Train	Test
Mean	ELM	0.0026	0.0010	0.1206	0.2434	0.5805	1.2431	0.2930	1.3263
S.D	-	0.0063	0.0042	0.0098	0.0252	0.0605	0.2665	0.0461	0.4194
Mean	LMM-ELM	0.0008	0.0034	0.0488	0.2139	0.2303	0.8595	0.0480	0.6974
S.D	-	0.0004	0.0028	0.0142	0.0319	0.0608	0.1517	0.0239	0.1069

Table 3: The Performance results of ELM and LMM-ELM Algorithms for Boston Housing dataset. All experiments used 50 hidden nodes.

		Time (seconds)		RMSE		RAE		RSE	
Metric	Algorithm	Train	Test	Train	Test	Train	Test	Train	Test
Mean	ELM	0.0106	0.0068	0.0682	0.1108	0.3445	0.5694	0.1127	0.2981
S.D	-	0.0096	0.0089	0.0034	0.0190	0.0194	0.1490	0.0125	0.1229
Mean	LMM-ELM	0.0092	0.0028	0.0234	0.1106	0.1158	0.5685	0.0136	0.2972
S.D	-	0.0088	0.0064	0.0037	0.0189	0.0180	0.1488	0.0044	0.1223

The results of the analysis of these data sets are given in Tables 2 to 4. Each table displays the training and testing time and three key regression evaluation metrics, that is, Root Mean Squared Error (RMSE), Relative Absolute Error (RAE), and Relative Squared Error (RSE), for both the training and testing phases. For all datasets, the LMM-ELM outperforms standard ELM, especially in the testing phase, thereby demonstrating better generalization and more stable learning behavior. Also, they are common evaluation metrics used to assess the performance of models. RMSE measures the square root of the average squared differences between predicted values $\hat{y}_i$ and actual values y_i, and is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}.$$

RAE compares the total absolute error of the predictions to the total absolute deviation from the mean $\bar{y}$ of the true values and is given by:

$$\text{RAE} = \frac{\sum_{i=1}^{n}|y_i - \hat{y}_i|}{\sum_{i=1}^{n}|y_i - \bar{y}|}.$$

RSE evaluates the squared prediction error relative to the variance of the actual data and is calculated as:

$$\text{RSE} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}.$$

In all three metrics, lower values indicate better predictive performance, with RAE and RSE values closer to 0, signifying that the model performs better than simply predicting the mean.

Under the aforementioned comparison, the results in Tables 2 to 4 contrast the performance of ELM and LMM-ELM on datasets.

Table 4: The Performance results of ELM and LMM-ELM Algorithms for MathAchive dataset. All experiments used 50 hidden nodes.

		Time (seconds)		RMSE		RAE		RSE	
Metric	Algorithm	Train	Test	Train	Test	Train	Test	Train	Test
Mean	ELM	0.0528	0.0120	0.2260	0.2431	0.8958	0.9547	0.8354	0.9739
S.D	-	0.0195	0.0160	0.0012	0.0171	0.0045	0.0621	0.0072	0.1446
Mean	LMM-ELM	0.0542	0.0094	0.0474	0.2382	0.1777	0.9331	0.0369	0.9359
S.D	-	0.0169	0.0087	0.0024	0.0172	0.0095	0.0627	0.0036	0.1434

For the Servo dataset, as shown in Table 2, the LMM-ELM better generalization compared to ELM concerning testing the RMSE (0.2139 vs. 0.2434), test RAE (0.8985 vs. 1.2431), and test RSE (0.6974 vs. 1.3263). A similar trend is observed in the Auto-Mpg dataset (Table 4), where LMM-ELM achieves a test RMSE of 0.1216, while ELM has an RMSE of 0.1282, along with moderate improvements in RAE and RSE.

Further results presented in Table 3, show the performance on the Boston Housing dataset, where LMM-ELM again surpasses ELM in generalization. LMM-ELM achieves a test RMSE of 0.1106, while ELM has an RMSE of 0.1108. The test RAE for LMM-ELM is 0.5685 compared to 0.5894 for ELM, and the test RSE shows a significant improvement at 0.2972 versus 0.2981. Although the RMSE difference is minimal, the lower standard deviation indicates clearly a stronger predictive performance for LMM-ELM.

Lastly, for the MathArchive dataset (Table 4), substantial improvements are observed LMM-ELM with an RMSE (test) of 0.2172 (ELM = 0.2431), an RAE (test) of 0.8351 (ELM = 0.8958), and an RSE (test) of 0.9346 (ELM = 0.9798).The low standard deviation further supports the LMM-ELM model, which enhances the reliability and robustness of the findings, even in larger and more complicated datasets.

While it is true that the LMM-ELM may require slightly more computation time due to its enhancements in local modeling, the consistent reduction in all three error metrics, along with their standard deviations, demonstrates the improved accuracy, stability, and generalization capabilities of LMM-ELM compared to the standard ELM algorithm across various datasets.

7 Conclusion

In this study, we introduced a novel hybrid method that combines Linear Mixed Models (LMM) with Extreme Learning Machines (ELM) to improve parameter estimation for complex datasets. This synergistic approach takes advantage of the LMM's ability to model random effects and subject-specific variability, along with the ELM's speed and flexibility in modeling fixed effects. Experimental research conducted on various benchmark datasets demonstrated that the hybrid model outperformed ELM alone in terms of predictive accuracy, robustness, and computational efficiency. The estimation process, inspired by the EM algorithm, facilitates a clear separation and refinement of fixed and random effects, providing deeper insights into the analysis of the underlying data structure. This fusion framework offers a promising solution for applications in fields such as biomedical research, finance, and social sciences, where both structured and unstructured variations are common. Future research could focus on further enhancing the model by exploring novel regularization techniques and assessing its applicability to high-dimensional and sparse datasets.

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Application of Deep Learning in Medical Image Processing

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Abstract

Deep learning (DL), particularly convolutional neural networks (CNNs), has revolutionized medical image analysis, achieving expert-level performance in tasks such as object detection, segmentation, and computer-aided detection (CAD). These advancements hold promise for improved diagnosis, treatment planning, and disease management. However, several challenges prevent the widespread clinical adoption of deep learning in this field. These include the requirement for large, accurately labeled datasets, which are often scarce in the medical domain, as well as the need for interpretable AI models that clinicians can understand and trust. To address data limitations, techniques like transfer learning, data augmentation, and self-supervised learning are being explored. Furthermore, ongoing research is dedicated to developing robust and transparent AI systems and addressing ethical and legal considerations surrounding data privacy and sharing.

Keywords: Deep Learning, Medical Image Analysis, Machine Learning Algorithms.

1 Introduction

Deep Learning techniques, especially (CNNs) [1], excel at tasks like identifying objects, locating them in images, and doing both at the same time. These methods have become so good that, given enough data, their accuracy in medical image analysis often equals or surpasses that of expert doctors in detecting various diseases. DL has also significantly improved the ability to segment (precisely outline) objects in images, even when dealing with complex scenarios like overlapping cells under a microscope. The success of these techniques relies heavily on supervised learning, where AI algorithms learn from images that are labeled (e.g., "this is a tumor"). However, medical data is often scarce and hard to access. To overcome this, transfer learning [2] has become crucial. This involves first training an AI on a large, unrelated dataset (like ImageNet [3]) and then fine-tuning it on a smaller, more specific medical dataset. To further tackle the data problem, researchers are developing ways to create artificial data through data augmentation (making slightly modified copies of existing images) and generative adversarial networks (GANs [4]). They've even shown that getting image labels from online crowdsourcing can be effective for training medical AI.

More recently, self-supervised learning is emerging, where the AI learns from the data itself without explicit labels (e.g., by trying to predict how scrambled parts of an image fit together). This could significantly lower the barrier to developing and using medical AI. Access to medical data brings up important ethical

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and legal questions about patient privacy and data ownership. The sharing of large medical datasets is debated, with some advocating for open-sourcing. Federated learning, a technique that allows AI models to be trained on data distributed across different locations without the data ever leaving those secure locations, might offer a solution for using data in places with strict privacy laws. These advancements in computer vision are also fueling progress in related areas like multimodal learning, which combines image analysis with other types of data like language, time-series data, and genetics. 3D vision is also improving, making it easier to use depth cameras in sensitive environments like intensive care units while preserving patient privacy. In video analysis, techniques like activity recognition and live scene understanding can be used to detect and respond to important medical events. In short, deep learning is revolutionizing computer vision and holds immense potential for medicine, but addressing data scarcity, ethical concerns, and ensuring data accessibility are key challenges for its widespread and responsible deployment [5].

Deep learning is revolutionizing medical image analysis (CT, MRI, PET, X-ray) by automatically learning image features, outperforming traditional expert-designed methods that are often modality-specific (Figure 1). This allows for better analysis of diverse images, including high-resolution scans revealing microscopic details. Key applications include more accurate image registration (aligning scans), precise localization and detection of anatomical structures (e.g., prostate), improved tissue segmentation (outlining organs/tissues), and computer-aided disease diagnosis and prognosis.

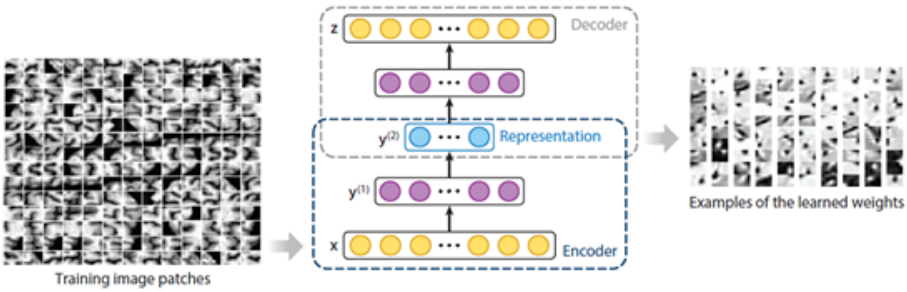


Figure 1: Construction of a deep encoder–decoder via a stacked auto-encoder and visualization of the learned feature representations [6].

Deep learning's ability to learn intrinsic image characteristics enables rapid development of analysis tools for new imaging techniques and can be adapted across various medical imaging tasks. This framework offers a powerful, adaptable approach to extract crucial information from medical images, leading to enhanced diagnostic capabilities and potentially better patient outcomes Instead of relying on hand-crafted image filters designed to look for specific shapes or textures, deep learning empowers the computer to automatically discover distinguishing "anatomical signatures" within images themselves. This shift is happening because two powerful forces have aligned: first, deep learning technology has become incredibly sophisticated, capable of tackling real-world complexities. Second, we now have access to vast amounts of medical image data.

DL enables the automatic identification of organs and body parts in medical scans. AI was trained on dynamic MRI to recognize organs through their visual and temporal features. DL is being used to automatically find individual cells in microscopic images of tissues (histopathology). Studies have shown that DL algorithms have achieved high accuracy in identifying specific cell types, like dividing cells in breast cancer, even outperforming human-designed methods in competitions. DL is significantly enhancing CAD in medical imaging by automatically learning features to identify abnormal regions. Instead of traditional methods relying on human-designed features, DL models, particularly CNNs, are improving detection of lung nodules, lymph nodes, lung diseases, brain bleeds, and MS lesions by analyzing 2D, 2.5D, or 3D image data.

Techniques like stacked auto-encoders and multi-scale CNNs are improving the differentiation of benign from malignant lesions in breast ultrasound and lung CT scans. DL also aids in diagnosing brain disorders like Alzheimer's and schizophrenia by analyzing MRI, PET, and fMRI data, even combining different data types for enhanced accuracy. Figure 2 illustrates two examples of how deep learning is used in medical computer vision tasks. (a) Multimodal Discriminative Model: This shows how AI can learn from different types of medical data at the same time to make predictions. It combines image data (like X-rays or MRI scans), which is usually processed by convolutional neural networks (CNNs) designed to understand images, with non-image data (like patient history, lab results, or genetic information), which is processed by more general deep learning networks.

(b) Generative Model: This shows how CNNs can be trained to create medical images. One example of this is "image-to-image regression," where the AI takes one type of medical image as input and generates another related image as output. Other tasks for generative models include making low-resolution images sharper (super-resolution), creating entirely new, realistic medical images, and more. In the following, we will discuss various aspects of using deep learning methods in medical applications and the advantages and obstacles of these methods.

2 DL for medical image analysis and CAD

Computer-aided detection (CAD) uses machine learning to find disease patterns in medical images, distinguishing normal from abnormal or harmless from harmful. Traditionally, CAD relied on experts to manually program image features for this. However, this approach is limited by the expert's knowledge and the variability of diseases in different patients, often leading to many false alarms. Instead of manual programming, DL, especially deep convolutional neural networks (DCNNs) automatically learn important image features from large amounts of data. When trained well, DCNNs are better at correctly identifying diseases and are more adaptable to different patients. Early use of neural networks in medical imaging dates back to 1994 for detecting tiny calcium deposits in mammograms. The development of deeper DCNNs was made possible by key improvements in how these networks are trained, such as specific ways to prepare the data, using better activation functions, simplifying information, preventing the AI from memorizing the training data, and stabilizing the learning process like AlexNet [7] and ResNet [8]. The progress in DL and AI has raised hopes for their use in medical image analysis for computer-aided diagnosis (CAD). This includes telling the difference between normal and diseased tissue, identifying if a tumor is cancerous, predicting cancer risk, outlining organs and tumors, checking if treatments are working, and predicting if a disease will return [9].

3 Understanding Convolutional Neural Networks (CNNs)

DL, using neural networks, excels at understanding images because it learns in layers, building complex features from simpler ones. For instance, lines form shapes, and shapes form objects. Besides this layered approach, DL also uses the fact that nearby pixels in an image are usually similar. Sharp changes in pixel values often highlight important features like edges. A convolution is a mathematical operation that helps the AI detect these important features, like edges, by using a "filter" that slides across the image. This makes CNNs very effective in medical fields that rely on images, such as radiology, pathology, ophthalmology, and dermatology. CNNs automatically learn the most relevant features for classifying images directly from the raw data. This automatic feature learning has several benefits:

1. It saves a lot of manual work, as humans don't need to identify features themselves.
2. It optimizes the features and how the data is represented for the specific task.

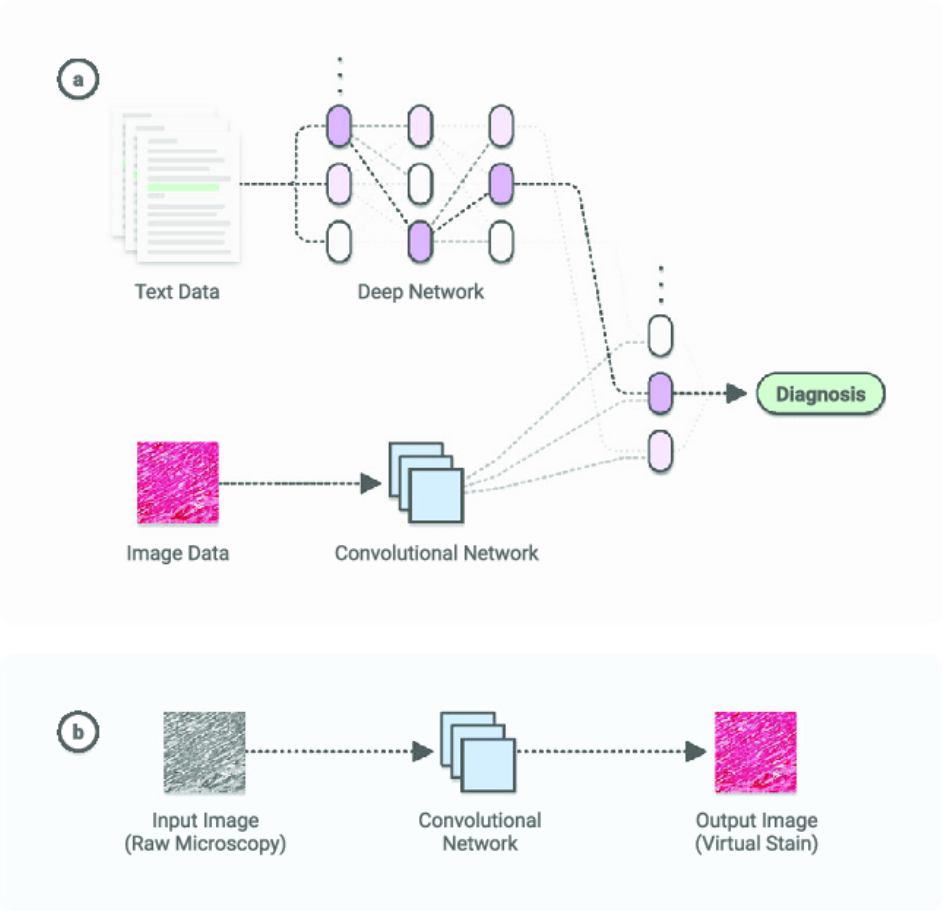


Figure 2: Two examples of how deep learning is used in medical computer vision tasks [5].

3. It creates a general understanding of the image that can be adapted for other tasks like segmentation or transfer learning.

4. It can discover complex relationships in the data that humans might not even think to look for.

DL can uncover new, clinically important connections in medical data that are often complex and not obvious to humans. Current CNNs are very intricate mathematical systems built through many small technical advances over time [10]. Researchers discovered that making CNNs deeper allowed them to learn more intricate details from images, leading to better performance. However, simply making networks deeper could also make them slower and harder to train. To address this, researchers came up with clever architectural improvements. "Split-transform-merge" designs, like those in GoogleNet [11] and Inception, essentially allowed the network to do multiple calculations at the same time ("parallelizing"), making the CNNs not just deeper but also wider and more efficient. Another key innovation was the "residual network" architecture, like ResNet. These and other technical advancements led to CNNs that now perform better than humans in recognizing objects in images [10].

4 DL for Image Segmentation

The abstract way that neural networks represent data in their hidden layers can be reused for other tasks, like object segmentation, means identifying and outlining different objects within an image. A fully CNN first learns a simplified, abstract understanding of the image by reducing its size (this is the "downsampling path"). Then, it uses this understanding to recreate the original image, but with each object clearly separated and labeled (this is the "upsampling path"). However, it was discovered that networks with fewer layers for the "blowing up" part could perform just as well and were faster to train. These fully CNNs are very useful in medicine because they can identify important elements in a medical image (like a tumor or an organ) and precisely define their boundaries [10].

DL is now the go-to method for outlining structures in brain scans (neuroimaging segmentation) because it performs as well as human experts but is much faster and cheaper. This segmentation can be used to identify both normal parts of the brain and abnormal areas like lesions. For brain tumors, AI models achieving over 85% overlap with expert outlines – a level of agreement similar to that between different human experts. Segmenting normal and abnormal brain structures automatically can significantly speed up the process before surgery with minimal effort from humans [10].

5 Recurrent Neural Networks (RNNs) and Specialized Architectures

While CNNs are great for understanding the spatial information in images, recurrent neural networks (RNNs) [12] are designed to analyze sequences of data, making them perfect for tasks that involve time-based information. In medical imaging, RNNs, especially a type called long short-term memory (LSTM) networks, have been used to look at data that changes over time, like videos or a series of images taken sequentially. One important use of RNNs in medical imaging is analyzing electrocardiograms (ECGs), which record the electrical activity of the heart over time. Researchers have shown that RNNs can detect irregular heartbeats (cardiac arrhythmia) from ECG recordings as accurately as cardiologists.

Although CNNs and RNNs are the basic building blocks, researchers are constantly creating new, specialized AI designs for specific medical uses and types of images. For example, attention mechanisms allow AI models to focus on the most important parts of an image (like analyzing tissue samples and finding abnormalities), similar to how humans pay attention. The future of DL in medical image analysis looks promising. Additionally, advancements in multimodal learning (combining different types of data), federated learning (training AI on data from multiple hospitals without sharing the data directly), and transfer learning (using knowledge learned from one task to help with another) will help AI learn from more diverse datasets and different types of images, making them more reliable and useful in a wider range of medical situations. Finally, Table 1 summarizes the different DL architectures discussed, their applications in medical image analysis, and their overall impact on healthcare [13].

6 Computer Vision Applications in Medical Imaging

The use of computer vision in analyzing medical images has exploded recently, growing from hundreds to thousands of research papers. Fields like radiology, pathology (tissue analysis), ophthalmology (eye care), and dermatology (skin care) have seen the most attention because their diagnoses heavily rely on recognizing visual patterns, and there's an increasing amount of well-organized image data available. Despite some challenges, AI is evolving to handle medical data better. For example, a technique called multiple-instance learning allows AI to learn from very large images with only a few labels, which is useful for pathology. 3D

convolutions in CNNs help AI understand 3D scans like MRI and CT scan. Methods that combine spatial and temporal information, along with image registration, enable AI to work with time-based images like ultrasound. In medical fields with less standardized data collection, computer vision is being integrated into existing physical devices. For instance, AI on smartphone attachments can help primary care doctors manage ear, nose, and throat conditions. AI connected to microscopes in hematology (blood analysis) and serology (body fluid analysis) can diagnose common conditions or count different types of blood cells – tasks that are repetitive and well-suited for AI. Gastroenterology (digestive system care) has seen remarkable advancements with AI. AI integrated into endoscopes (cameras used to view inside the body) can help guide the scope, find lesions (abnormal tissue), and diagnose them. This includes screening for esophageal cancer, detecting stomach cancer and infections, and even identifying parasites. Beyond just identifying diseases, computer vision can also be used in the future to screen human embryos for implantation, potentially improving the success of fertility treatments. Radiology has become a major area for computer vision research, with a large body of work focusing on X-rays, CT scans, and MRI. Analyzing brain images (especially for time-sensitive conditions like stroke) and abdominal images has also received significant attention. AI models have been developed to classify diseases, detect nodules, and segment (outline) regions of interest for most conditions where enough data has been collected. This allowed the field to quickly develop AI models for detecting COVID-19 during the pandemic. The field continues to expand with work on image translation (making noisy ultrasound images look like MRI), image reconstruction and enhancement (making low-quality CT scans high-quality), automatically generating reports, and tracking changes over time (like tumor growth). The following sections will explore how computer vision is being applied in other medical specialties [5].

6.1 The Role of DL in Echocardiography and Cardiac Imaging

Cardiac imaging, especially using ultrasound (echocardiogram), is becoming very important for diagnosing and managing heart conditions. DL is showing great potential in this area for tasks like diagnosis and screening. Echocardiograms are particularly well-suited for AI because they are cheap, don't involve radiation, and are used very often. AI can be trained on large collections of echocardiogram videos and even become better than experienced heart doctors at identifying different views of the heart. AI systems can also help diagnose conditions like hypertrophic cardiomyopathy, cardiac amyloidosis, and pulmonary hypertension. To deal with the challenge of not having enough readily available data, researchers have developed AI algorithms that can learn effectively from smaller amounts of echocardiogram data. In addition to ultrasound, other 3D imaging techniques like CT and MRI are used to understand the heart's anatomy better. AI algorithms have even been approved to help visualize the arteries around the heart using CT scans [5].

6.2 DL for Digital Pathology: Enhancing Cancer Diagnostics

Pathologists play a vital role in diagnosing and treating cancer by visually examining tissue samples under a microscope DL offers a way to support these critical tasks, including making diagnoses, predicting how the disease will progress and how it will respond to treatment, segmenting different parts of the tissue, monitoring the disease, and more. Recent advancements in technology allow for the creation of very high-resolution digital images of entire tissue slides. This, combined with progress in computer vision, has spurred research and commercial activity in using AI for digital pathology. This field has the potential to:

1. Overcome human limitations
2. Discover new disease markers
3. Integrate different types of data (By combining pathology images with radiology scans, genetic information, and protein analysis)

One area of research focuses on automating the time-consuming job of finding and counting specific features in tissue samples, like cells, cell nuclei, and dividing cells. It also involves locating and outlining structures like glands, ducts, and tumors. These methods usually require pathologists to manually label these tissue components in images to train the AI, which can be costly and time-intensive. Another research direction is using AI to directly make diagnoses and predict outcomes from these whole-slide images or smaller tissue samples. Studies have even shown that the patterns seen in standard tissue staining can predict the presence of molecular markers that are important for guiding treatment. While digital pathology creates huge, information-rich images, detailed labels at the level of specific regions are often scarce and expensive to obtain. To tackle this, AI algorithms using a technique called multiple-instance learning have been developed. These algorithms can learn from "weak" labels that apply to the entire slide and leverage the massive size of these images to improve their performance. The large amount of image data in pathology has also enabled tasks like "virtual staining," where AI models learn to predict what a tissue image would look like with one type of stain based on an image with a different stain or even an unstained image.

Looking ahead, AI algorithms that can learn to diagnose, predict outcomes, and guide treatment using digital pathology image archives and the information readily available in electronic health records have the potential to transform pathology and cancer care [5].

6.3 DL in Dermatology: Skin Cancer Classification

In dermatology, DL is being developed to help with crucial tasks like figuring out the specific type of skin lesion, finding potentially dangerous lesions among many harmless ones, and tracking how lesions change over time. Several studies have shown that AI, using CNNs, can be as good as or even better than certified dermatologists at telling the difference between cancerous and non-cancerous skin lesions. However, these studies mostly focused on simply distinguishing between benign (harmless) and malignant (cancerous) lesions. More recently, this research has expanded to help diagnose a wider range of skin conditions, including non-cancerous issues like rashes and genetic disorders. The availability of open-access image collections and AI competitions has greatly encouraged this progress, allowing different research teams to test their AI models against common standards. This area is still relatively new, but some initial research is training CNNs to both detect and track lesions simultaneously [5].

6.4 DL in Ophthalmology: Advancing Eye Care

In recent years, there has been a surge in the use of AI in ophthalmology (eye care), with many studies showing that AI can diagnose and analyze eye conditions even better than current human capabilities. This is especially exciting because the equipment used to examine eyes is often portable, meaning testing sites could be easily set up in underserved areas and through telemedicine. The field heavily relies on fundus imaging (pictures of the back of the eye) and optical coherence tomography (OCT) to diagnose and manage eye diseases. AI, particularly using CNNs, has proven very accurate in diagnosing various eye conditions. Diabetic retinopathy, a condition where leaky blood vessels in diabetic patients' eyes can cause blindness, has been studied extensively. Similarly, AI can diagnose or predict the progression of other eye diseases like diabetic macular edema, age-related macular degeneration, glaucoma, vision loss in children, and more. Interestingly, AI can also detect subtle signs in eye images that humans can't easily see, which indicate other health issues beyond the eyes. For example, AI has been shown to classify cardiovascular and diabetic risk factors like age, gender, smoking status, blood sugar levels, weight, and blood pressure just by looking at fundus photographs. AI can also pick up signs of anemia and chronic kidney disease from these images. This opens up a fascinating possibility for the future, where routine eye exams could screen for both eye diseases and other health problems. This could lead to a major change in how we approach healthcare [5].

6.5 Computer Vision in Surgery and Ambient Intelligence in Healthcare

Computer vision (CV) is proving to be very useful in hands-on medical fields like surgery and endoscopy. DL can enhance a surgeon's performance by giving them real-time information about what's happening, assessing their skills, and aiding in training. Early research is focusing on video from robotic and laparoscopic surgery, with AI being developed to identify surgical tools and actions. Some systems analyze how the tools move to evaluate a surgeon's skill based on standard criteria. Other work uses CV to recognize the different stages of an operation, with the goal of creating AI assistants that understand the context of the surgery. CV is also starting to be used in traditional open surgery, which is more common, although it's challenging due to the different camera angles and types of operations.

The next step in all surgical video analysis is to turn these AI insights into tools and applications that actually improve patient outcomes. Beyond surgery, CV can also understand human activity in places like hospitals and clinics, leading to "ambient intelligence" applications. This means a continuous, non-intrusive awareness of what's happening in a space, which can help doctors, nurses, and other healthcare workers with tasks like monitoring patients, automatically documenting procedures, and checking if protocols are being followed. In emergency departments, CV can even transcribe what's happening during a patient's resuscitation. In operating rooms, it can recognize activities to optimize the workflow. At a hospital-wide level, CV can provide detailed data on how staff and resources are being used, helping to allocate them more efficiently for better patient care (Figure 3).

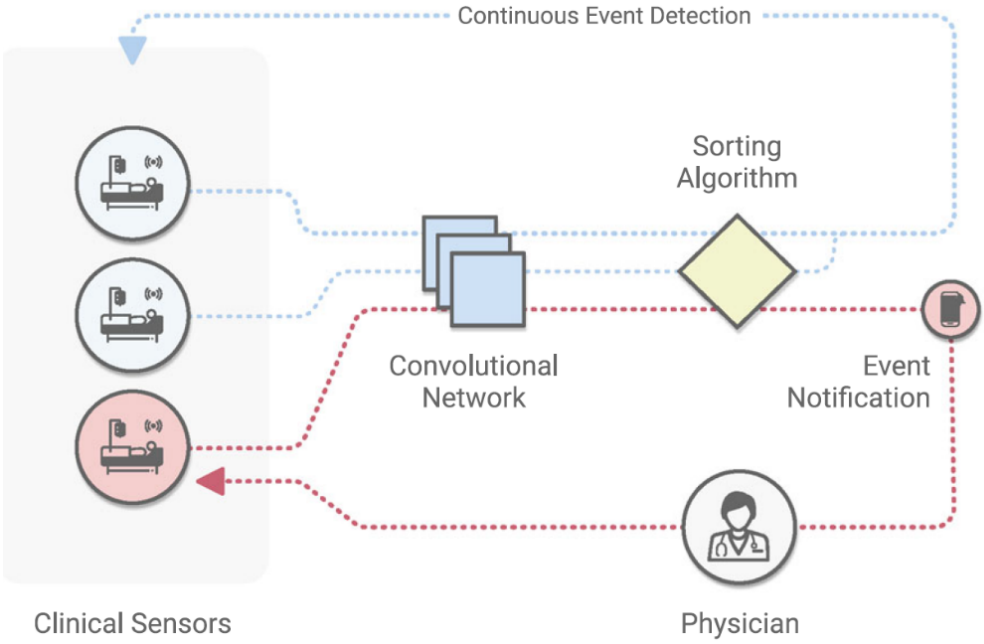


Figure 3: Ambient intelligence uses computer vision and sensors to create a smart, watchful eye in healthcare and homes [5].

Outside of hospitals, ambient intelligence can make healthcare more accessible. For instance, it could allow elderly people at risk to live independently at home by monitoring their safety and daily activities, like detecting falls. Similar AI is being developed to recognize broader categories of daily activities. Analyzing long-term patterns of activities like sleeping, walking, and sitting can help detect important health changes

or unusual patterns. To protect privacy, some researchers are using thermal video data with CV algorithms. Another application is in assisted living and rehabilitation, such as continuously recognizing sign language to help people with communication difficulties and monitoring physiotherapy exercises for stroke recovery. CV also has the potential for remote health measurements, like analyzing heart and breathing rates from video. As telemedicine becomes more common, CV could help in deciding which patients need immediate attention, especially during high-demand situations like the COVID-19 pandemic. Overall, CV-based ambient intelligence offers many ways to improve access to quality care, but it also raises new ethical and legal questions that need to be considered when designing these technologies.

7 Challenges in DL-Based Medical Imaging

Computer-aided detection (CAD) and artificial intelligence (AI) are expected to become helpful tools for doctors in various tasks like early screening, prioritizing cases, cancer staging, checking if treatment is working, monitoring for cancer returning, and predicting survival. While AI-powered CAD systems have not been fully tested in large clinical trials yet, we can learn from how CAD is currently used in mammography (breast X-rays). Right now, the most common CAD application uses traditional machine learning to help find breast cancer on mammograms. It is often as good as or even better than radiologists at spotting tiny calcium deposits that can be a sign of cancer, although it sometimes flags things that are not actually cancer (false positives). CAD has also been shown to help radiologists be more accurate when they read mammograms. Because of this, the FDA (U.S. Food and Drug Administration) has approved CAD as a second opinion for radiologists, but not as the only way to diagnose. Early studies, like the CADET II trial [14], compared using CAD with a single radiologist to the standard practice of having two radiologists read each mammogram. The results were promising, suggesting that CAD can help improve how accurately breast cancer is detected during screening. A review of multiple studies found that a single radiologist using CAD was actually better at finding cancer than a single radiologist reading alone or even two radiologists reading together. However, the recall rate was a bit higher with CAD compared to a single reading alone, but still lower than having two radiologists read the mammogram. This suggests that having one radiologist use CAD could be a good alternative to having two radiologists, as it might find more cancers, even if it means slightly more women need further tests. Effective CAD/AI implementation requires accuracy, workflow efficiency, thorough user training, consideration of clinician experience, performance standards, quality assurance, and crucial human oversight for complex cases, reflecting the supportive role of "narrow AI" in medicine [9].

DL has shown amazing potential in analyzing medical images for tasks like finding tumors, classifying diseases, and making diagnoses. However, there are several hurdles we need to overcome before these AI systems can be reliably used in everyday clinics. One major problem is the need for large, accurately labeled datasets. Training DL models requires a lot of images where experts have carefully marked what's important (e.g., outlining a tumor or labeling an image with a specific disease). Getting this kind of labeled data in medicine is very difficult, time-consuming, and expensive. On the other hand, the quality of these labels can vary, which can introduce errors and biases into the AI's learning. For rare diseases or specific groups of patients, there might not even be enough labeled images available to train a good AI model. To fix this, hospitals, research groups, and data scientists need to work together to create large, well-labeled datasets and agree on how to label and share this data. We can also use techniques like creating artificial data, reusing knowledge from AI trained on other datasets, and using AI that can learn from partially labeled data to make the most of the limited labeled data we have.

Another big challenge is making sure these AI models are reliable and understandable. Right now, DL models are often like "black boxes" – it's hard to know exactly why they made a certain decision. In healthcare, doctors need to understand how an AI arrived at a diagnosis to trust it and use it correctly. However, the complex way these AI systems are built makes it difficult to interpret their reasoning. Also,

these AI models can be tricked by tiny changes to the input images that humans wouldn't even notice, leading to wrong predictions. This is a serious concern in medicine where a misdiagnosis can have severe consequences. To address this, we need to develop ways to make AI more transparent and robust. Techniques like showing which parts of the image the AI is focusing on can help explain its decisions. We also need better training methods to make AI more resistant to these "tricks" and more reliable in different situations.

Finally, AI models trained on data from one hospital or patient group might not work well on data from different hospitals or patient groups. Medical images can look different depending on the patients' characteristics, the type of scanner used, the settings of the scanner, and even the quality of the images. An AI trained on data from one place might struggle with images from another place, leading to lower accuracy in real-world use. This problem is made worse by the fact that there aren't always standard ways of taking medical images, and the quality can vary. To solve this, researchers are working on techniques that allow AI to adapt to new types of data, reuse knowledge from different datasets, and learn from data collected across multiple centers. By using methods that can adjust for these differences in data, we can create AI models that are more reliable and can be used in a wider range of clinical settings [13].

8 Addressing Data Limitations in Medical Image Analysis

While DL shows great promise in medical imaging, achieving reliable AI tools for clinical use faces significant challenges, primarily due to data limitations. Developing robust AI requires large, high-quality training datasets that accurately represent the target patient population. However, collecting sufficient medical images with accurate labels (the "truth" the AI learns from) is expensive and challenging. While normal cases might be easy to gather, finding enough abnormal cases, especially with a balanced representation of different disease types, is difficult. Mining electronic medical records (EMR) and picture archiving systems (PACS) using data mining and natural language processing has been explored to gather more data. However, the accuracy of these automatically obtained labels can be questionable due to how the information is recorded.

In essence, simply having a lot of data isn't enough; the data needs to be well-curated and accurately labeled. Training with noisy or weakly labeled data requires much larger datasets to achieve comparable results to training with carefully verified cases. Extracting reliable information for complex tasks like monitoring treatment response from current EMR and PACS systems, with their unstructured text and varied reporting, is particularly difficult. Creating the necessary "gold standard" for training often requires manually linking imaging data, clinical tests, and patient outcomes over time, a challenging process to automate. Incomplete patient records when patients move between hospitals also add to the difficulty of data curation. To truly leverage AI for precision medicine, standardizing reporting methods and secure data sharing between hospitals is crucial. Ultimately, large, multi-institutional collaborations are likely the best way to build the massive, diverse databases needed to develop robust DL models that can be reliably used in various real-world clinical settings [9]. When training AI with limited medical images, a common trick is transfer learning. This involves taking an AI model already trained on a huge dataset (like ImageNet, which has millions of everyday photos) and then fine-tuning it for a specific medical task using a smaller set of medical images. The idea is that the AI has already learned general image features (like shapes and textures) that can be useful for medical images too. Because many images share basic elements, the knowledge an AI gains from one type of image can often be transferred to another. The more similar the original training images are to the medical images, the better the transfer works. Even if they are quite different, using a pre-trained model as a starting point is usually better than starting from scratch with random settings. Sometimes, the pre-trained AI is just used to extract features from the medical images, and then a separate, simpler classifier is trained using these features. Transfer learning generally helps the AI learn faster and more reliably.

Overall, while transfer learning helps with limited medical data, having a reasonably large and relevant

dataset is still crucial for achieving high accuracy. The ideal amount of data needed depends on the complexity of the task and how different the original and target datasets are [9]. To help AI learn from limited medical images, a common technique is data augmentation. This involves creating slightly altered copies of the existing images in the training set. These alterations can include flipping, shifting, cropping, rotating, scaling, and changing the brightness or shape of the images using both simple and more complex methods.

The main goal of data augmentation is to introduce variety into the training data, making the AI less likely to overlearn the specifics of the small original dataset and more likely to perform well on new, unseen images (improve generalization). However, the augmented images are still related to the original, and the AI might not learn much truly new information from them. For instance, if no images show a "spiculated nodule," flipping or rotating existing nodules won't teach the AI what a spiculated nodule looks like. Researchers are also exploring more advanced data augmentation techniques, such as using AI models called GANs to generate entirely new images with a mix of features from the existing data.

More research is needed to understand which data augmentation methods are most effective for different tasks and how they compare to each other. Ultimately, the usefulness of augmented data likely depends on the specific medical problem the AI is trying to solve [9].

9 Validation and Trust in Medical AI

When training AI models, including DL, a validation set of data is used to help the AI learn correctly. Think of it as a practice exam that the AI takes during its learning process. We use the results of this practice exam to adjust the AI's "study habits" (its parameters) and to see if it's improving. We might try different sets of parameters or watch how its performance changes as it learns. The goal is to find the AI model that performs best on this practice exam. The validation set is usually created by splitting the initial training data. This can be done through methods like cross-validation (using different parts of the data as practice exams in turns) or by simply holding out a portion of the data as a separate practice set. However, because we repeatedly use the validation set to guide the AI's learning and choose the best model, the AI's performance on this practice exam is often better than it would be on completely new, unseen data. This is called optimistic bias. To truly know how well the trained AI will perform in real-world situations with new patients and new cases, we need to use a completely separate test set. This test set should be data that the AI has never seen during its training or validation process and should accurately represent the kind of data the AI will encounter in the real world. Unfortunately, many studies only report results from cross-validation, which, as mentioned, can be biased. Therefore, even after an AI tool is approved for use, each hospital should test it on their own local patient data to make sure it meets a certain performance level before using it clinically. Furthermore, when doctors start using a new AI tool, they should have a trial period where they do not rely on the AI's output. They should get used to seeing what the AI says and, if possible, check its performance on many real patient cases. By reviewing the outcomes of these cases, doctors can learn when the AI makes mistakes and when it provides correct advice that they might have missed. This hands-on experience helps them understand the AI's strengths and weaknesses, build trust, and learn how to use it effectively without blindly following wrong recommendations. Understanding the AI's limitations and capabilities helps doctors have realistic expectations and use the tool properly, reducing the risk of errors and improving patient care.

Right now, DL models are often like black boxes – they can learn to classify medical images very well, but it's hard to understand exactly how they're making those decisions. We don't easily know what specific features in the image the AI is focusing on. Researchers are starting to develop tools to peek inside these black boxes. They can visualize the "feature maps" at different layers of the AI, which show the patterns the AI is learning. They can also create "class activation maps" that highlight the areas in the image the AI

thinks are most important for identifying a specific condition. These tools are a first step in understanding what the AI is "seeing." However, we're still a long way from having AI that can explain its decisions in a way that makes sense to doctors, especially for complex tasks beyond just finding a lesion. A key area for future research is uncovering the link between what the AI finds in the images and the actual medical conditions of the patient. Even more exciting is the potential for AI to analyze vast amounts of medical data and discover new connections between diseases, clinical information, and symptoms. This kind of research could lead to AI that provides interpretable diagnoses to doctors and moves us closer to true artificial intelligence in medicine [5].

10 Ensuring Equity, Trust, and the Future of Medical AI

As medical AI becomes more common, it has the potential to greatly improve healthcare but also to worsen existing inequalities. To ensure AI benefits everyone fairly, we need to carefully consider the data it learns from and the settings where it's used. Here are four key things to think about:

1. Assess the Data: The quality of the data used to train AI is crucial. If the data contains biases (for example, if certain racial groups or types of patients are underrepresented), the AI might not work as well for everyone. We need to actively look for these biases and try to fix them by collecting more diverse data or using techniques to balance the existing data.

2. Plan for Model Limitations: We need to be aware that AI models have limitations and can make mistakes. It's not enough to just report good overall performance; we need to understand when and why the AI might fail. We also need ways to detect when the AI is seeing data it hasn't been trained on and to understand how confident the AI is in its predictions, especially when these predictions affect patient safety.

3. Community Participation: Getting input from patients, doctors, computer scientists, and other relevant people is essential. Getting feedback from users in real-world settings helps ensure that AI tools are easy to use and fit into existing healthcare workflows.

4. Trust Building: Trust in AI from both doctors and patients is vital for its widespread use. Doctors will trust AI if it's been thoroughly tested in real clinical environments and if they understand how it makes its predictions. For this, AI needs to be interpretable – it should be able to explain its reasoning. Patients need to trust that their privacy is protected. We need better regulations that consider new ways to keep sensitive medical information private while still allowing AI to learn useful patterns from it. Techniques like federated learning, where AI is trained on data without it ever leaving the local hospital, are important in this regard.

In conclusion, AI has the potential to make healthcare more accessible, fair, accurate, and inclusive. However, we need to be thoughtful about the data we use, understand the limitations of AI models, involve the community in its development and deployment, and build trust by ensuring transparency and protecting patient privacy. If we do this right, AI can be a powerful force for good in medicine [5].

The new generation of AI-powered CAD systems is expected to be much better than older versions, with some studies even showing performance exceeding that of doctors. While the idea of AI replacing radiologists has faded, there's a lot of hope that AI can significantly speed up their workflow and reduce their workload. However, for AI to be a reliable tool in everyday clinical practice, especially if it's making decisions rather than just offering suggestions, its performance needs to be consistently high and meet certain standards over time, just like any other medical device. Because the performance of AI can be affected by the specifics of the images it analyzes (which depend on the imaging equipment, software, and settings that can change),

it's important to have regular quality checks to monitor both the AI's performance and how well doctors are using it over time. Currently, there isn't much post-market monitoring of CAD software after it's approved, even though it's used as a second opinion, and there's no control over how it might be used for purposes not originally intended. As AI becomes more widespread in healthcare, its impact on patients will be much greater, making these guidelines and oversight crucial [5]. DL is transforming medical image analysis, and the future holds even more exciting advancements.

Beyond the standard AI models used today, future research might explore more advanced architectures like graph neural networks (GNNs) [15] and capsule networks [16]. GNNs are good at analyzing data with complex relationships, such as connections in brain scans. They can understand how different parts of an image are related, leading to more reliable and understandable results. Researchers are working on ways to visualize what the AI is "seeing," highlight the important areas in an image that led to a prediction, and even generate text or visual explanations. Making AI more transparent will build trust, encourage collaboration between humans and machines, and ultimately lead to better patient care. Overall, the future of DL in medical image analysis is bright. By focusing on efficient ways to learn from limited data, integrating information from different image types, exploring advanced AI architectures, and making these systems more understandable, we can unlock the full potential of AI to revolutionize healthcare through more accurate diagnoses, personalized treatments, and improved patient outcomes [13].

11 Conclusion

Deep learning presents a transformative opportunity for medical image analysis, with the potential to enhance clinical decision-making and improve patient outcomes across various specialties. Convolutional neural networks (CNNs) have demonstrated remarkable success in tasks such as image classification and segmentation, while recurrent neural networks (RNNs) are proving valuable for analyzing sequential medical data. The development of computer-aided detection (CAD) systems powered by deep learning algorithms offers the prospect of more efficient and accurate disease screening and diagnosis. However, realizing the full potential of deep learning in clinical practice necessitates concerted efforts to overcome challenges related to data availability and quality, model interpretability, and the validation and implementation of AI tools in real-world settings. Future progress hinges on interdisciplinary collaboration, the establishment of standardized data sharing practices, and a continued focus on developing AI solutions that are both effective and ethically sound.

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Architecture	**Applications**	**Impact on Medical Image Analysis**
Convolutional Neural Networks (CNNs)	Automated detection and classification of abnormalities in chest X-rays, lesion detection in mammograms, retinal vessel segmentation in fundus images, brain tumor segmentation in MRI scans	Revolutionized radiological practices by expediting diagnosis, reducing workload, and improving patient outcomes
Recurrent Neural Networks (RNNs)	Interpretation of ECG for cardiac arrhythmia detection, analysis of time-series data in oncology imaging	Facilitated accurate detection of cardiac abnormalities and longitudinal analysis of treatment response
Customized Architectures	Integration of attention mechanisms into CNNs for pathology image analysis, development of advanced architectures such as GNNs	Enhanced model performance, interpretability, and generalization capabilities in medical image analysis

Table 1: Summary of deep learning architectures and their impact on medical image analysis [13].

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On the Weighted Cumulative Residual/Past Extropy of k-records

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Abstract

This paper introduces a weighted cumulative residual extropy (WCRJ) to characterize continuous distributions symmetric about a known or unknown center θ. By integrating a weight function $w(x;\theta)$ that satisfies $w(x;\theta) = w(2\theta - x;\theta)$, we generalize existing extropy-based symmetry tests. We prove that equality of WCRJ for upper and lower k-record values implies symmetry about θ. Examples include normal, uniform, and Laplace distributions, with explicit applications in hypothesis testing. This framework extends classical results by Ahmadi (2021) and provides a robust tool for analyzing symmetry in flexible parametric settings.

Keywords: Weighted cumulative residual extropy, k-record values, Symmetry testing, Nonparametric inference.

1 Introduction

The study of symmetric probability distributions has been a cornerstone of statistical theory, with profound implications across diverse fields such as finance, engineering, and signal processing. A distribution is said to be symmetric about a center θ if its probability density function satisfies the condition $f_X(\theta + x) = f_X(\theta - x)$ for all real numbers x. This property ensures that θ serves as both the median and, when defined, the mean of the distribution. Classical examples include the normal distribution, uniform distribution, and Laplace distribution, where symmetry simplifies analytical derivations and enhances interpretability. However, practical challenges arise when the center of symmetry θ is unknown or varies across contexts, necessitating flexible methodologies to detect and characterize symmetry without prior knowledge of θ.

Information-theoretic measures have emerged as indispensable tools for analyzing distributional properties. Shannon entropy, defined as

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x)\, dx,$$

quantifies the average uncertainty inherent in a random variable X. In contrast, extropy, introduced as a complementary measure by Lad et al. (2015), captures dispersion through the integral of the squared density function:

$$J(X) = -\frac{1}{2}\int_{-\infty}^{\infty} f_X^2(x)\, dx.$$

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Extensions such as *cumulative residual extropy* (CRJ) and *cumulative past extropy* (CPJ) provide deeper insights into tail behavior and symmetry. CRJ, defined using the survival function $\bar{F}_X(x) = 1 - F_X(x)$, was introduced by Jahanshahi et al. (2020) as:

$$\xi J(X) = -\frac{1}{2}\int_0^\infty \bar{F}_X^2(x)\,dx,$$

while CPJ, based on the cumulative distribution function $F_X(x)$, was proposed by Di Crescenzo and Longobardi (2009) in the context of entropy and later adapted to extropy by Hashempour and Mohammadi (2022):

$$\bar{\xi} J(X) = -\frac{1}{2}\int_0^\infty F_X^2(x)\,dx.$$

These measures generalize classical entropy-based concepts, with CRJ emphasizing right-tail behavior and CPJ focusing on left-tail dynamics. Subsequent works, such as Krishnan et al. (2020), explored their reliability properties, while Kayal (2016) and Psarrakos and Navarro (2013) extended them to generalized cumulative entropy frameworks. Qiu and Raqab (2022) further demonstrated their utility in ranked set sampling, underscoring their versatility in nonparametric settings.

Record values, such as the k-th upper and lower records, provide critical insights into the tail behavior of distributions. For a sequence of independent and identically distributed random variables $\{X_i\}$, the cumulative distribution function (CDF) of the n-th upper k-record value $U_{n,k}$ is:

$$F_{U_{n,k}}(u) = 1 - \bar{F}_X^k(u)\sum_{i=0}^{n-1}\frac{(-k\log\bar{F}_X(u))^i}{i!},$$

and the CDF of the n-th lower k-record value $L_{n,k}$ is:

$$F_{L_{n,k}}(u) = F_X^k(u)\sum_{i=0}^{n-1}\frac{(-k\log F_X(u))^i}{i!}.$$

The equality of CRJ and CPJ for these records encodes symmetry properties, as demonstrated by Ahmadi (2021), who showed that symmetry about a fixed center θ holds if and only if $\xi J(U_{n,k}) = \bar{\xi} J(L_{n,k})$ for all $n, k \geq 1$. This result was operationalized into nonparametric hypothesis tests by Xiong et al. (2021) and Jose and Sathar (2022), who leveraged the extropy of k-records to develop symmetry tests with applications in financial modeling and reliability analysis.

Despite these advancements, existing frameworks predominantly assume symmetry about a predetermined point, such as zero or the mean. Real-world scenarios, however, often require testing symmetry about an unknown or estimated center, such as the median in skewed populations. To address this gap, we introduce a weighted cumulative residual extropy measure, integrating a weight function $w(x;\theta)$ that satisfies $w(x;\theta) = w(2\theta - x;\theta)$. This constraint ensures invariance under reflections about θ, aligning with the mathematical definition of symmetry. The proposed test statistic:

$$T = \hat{\xi} J_w(U_{n,k};\theta) - \hat{\bar{\xi}} J_w(L_{n,k};\theta),$$

where $\hat{\xi} J_w$ and $\hat{\bar{\xi}} J_w$ are empirical estimators of the weighted CRJ and CPJ, generalizes earlier work by Ahmadi (2021) and enables robust hypothesis testing for arbitrary θ.

The remainder of this paper is organized as follows. Section 2 formalizes the weighted cumulative residual extropy framework, deriving its theoretical properties and connection to θ-symmetry. Section 3 validates the methodology through examples, including normal and Laplace distributions, while Section 4 demonstrates its application in hypothesis testing.

2 Theoretical Framework

This section establishes the mathematical foundation for analyzing symmetry in continuous distributions using weighted cumulative residual extropy. Building on classical results from information theory and record value statistics, we generalize existing characterization theorems to accommodate arbitrary centers of symmetry. The framework integrates weighted extropy measures with k-record value distributions, enabling robust symmetry tests without prior knowledge of the symmetry center θ.

2.1 Weight Function and Cumulative Residual Extropy

Let X be a continuous random variable with cumulative distribution function $F_X(x)$ and survival function $\bar{F}_X(x) = 1 - F_X(x)$. We introduce a weight function $w(x;\theta)$ to prioritize specific regions of the distribution around a center θ. The weighted cumulative residual extropy (WCRJ) is defined as:

$$\xi J_w(X;\theta) = -\frac{1}{2}\int_{-\infty}^{\infty} w(x;\theta)\bar{F}_X^2(x)dx.$$

Analogously, the weighted cumulative past extropy (WCPJ) is given by:

$$\bar{\xi} J_w(X;\theta) = -\frac{1}{2}\int_{-\infty}^{\infty} w(x;\theta)F_X^2(x)dx.$$

These measures extend the classical CRJ and CPJ (Jahanshahi et al., 2020; Di Crescenzo & Longobardi, 2009) by incorporating spatial weighting, making them sensitive to local asymmetry patterns.

2.2 Record Value Distributions

For a sequence of independent observations $\{X_i\}$, the cumulative distribution functions of k-record values provide critical insights into extremal behavior. The CDF of the n-th upper k-record value $U_{n,k}$ is:

$$F_{U_{n,k}}(u) = 1 - \bar{F}_X^k(u)\sum_{i=0}^{n-1}\frac{(-k\log\bar{F}_X(u))^i}{i!},$$

while the CDF of the n-th lower k-record value $L_{n,k}$ is:

$$F_{L_{n,k}}(u) = F_X^k(u)\sum_{i=0}^{n-1}\frac{(-k\log F_X(u))^i}{i!}.$$

These distributions generalize classical record value theory (Arnold et al., 1998) and enable symmetry analysis through paired comparisons of upper/lower extropy measures.

2.3 Characterization

Theorem 2.1. *Let X be a continuous random variable with cumulative distribution function $F_X \in \mathbb{C}_\theta$, where $\mathbb{C}_\theta$ denotes the class of symmetric distributions about θ. The following statements are equivalent:*

1. *X is symmetric about θ, i.e., $f_X(\theta + x) = f_X(\theta - x)$ for all $x \in \mathbb{R}$.*

2. *For all integers $n, k \geq 1$ and any weight function $w(x;\theta)$ satisfying $w(\theta + x) = w(\theta - x)$, the weighted cumulative residual extropy (WCRJ) of the n-th upper k-record value $U_{n,k}$ equals the weighted cumulative past extropy (WCPJ) of the n-th lower k-record value $L_{n,k}$:*

$$\xi J_w(U_{n,k};\theta) = \bar{\xi} J_w(L_{n,k};\theta).$$

Proof. **Proof of sufficiency** $(1 \Rightarrow 2)$

Assume X is symmetric about θ. By definition, this implies:

$$F_X(\theta + x) = 1 - F_X(\theta - x), \tag{2.1}$$
$$\bar{F}_X(\theta + x) = F_X(\theta - x), \tag{2.2}$$

for all $x \in \mathbb{R}$.

For the n-th upper k-record value $U_{n,k}$, its survival function is:

$$\bar{F}_{U_{n,k}}(u) = \bar{F}_X^k(u) \sum_{i=0}^{n-1} \frac{(-k \log \bar{F}_X(u))^i}{i!}. \tag{1}$$

For the n-th lower k-record value $L_{n,k}$, its distribution function is:

$$F_{L_{n,k}}(u) = F_X^k(u) \sum_{i=0}^{n-1} \frac{(-k \log F_X(u))^i}{i!}. \tag{2}$$

Under symmetry about θ, substitute $u = \theta + x$ into (1) and apply (2.2):

$$\bar{F}_{U_{n,k}}(\theta+x) = \bar{F}_X^k(\theta+x) \sum_{i=0}^{n-1} \frac{(-k \log \bar{F}_X(\theta+x))^i}{i!} = F_X^k(\theta-x) \sum_{i=0}^{n-1} \frac{(-k \log F_X(\theta-x))^i}{i!} = F_{L_{n,k}}(\theta-x). \tag{3}$$

Using the symmetry of $w(x;\theta)$, i.e., $w(\theta + x) = w(\theta - x)$, we apply the substitution $x = \theta + t$, $dx = dt$, to the WCRJ integral:

$$\xi J_w(U_{n,k};\theta) = -\frac{1}{2} \int_{-\infty}^{\infty} w(\theta + t;\theta) \bar{F}_{U_{n,k}}^2(\theta + t) dt. \tag{4}$$

From (3), $\bar{F}_{U_{n,k}}(\theta+t) = F_{L_{n,k}}(\theta-t)$ and using the weight symmetry $w(\theta+t;\theta) = w(\theta-t;\theta)$, Substituting this into (4):

$$\xi J_w(U_{n,k};\theta) = -\frac{1}{2} \int_{-\infty}^{\infty} w(\theta - t;\theta) F_{L_{n,k}}^2(\theta - t) dt.$$

Changing variables $x = \theta - t$:

$$\xi J_w(U_{n,k};\theta) = -\frac{1}{2} \int_{-\infty}^{\infty} w(x;\theta) F_{L_{n,k}}^2(x) dt = \bar{\xi} J_w(L_{n,k};\theta).$$

Thus, symmetry about θ guarantees $\xi J_w(U_{n,k};\theta) = \bar{\xi} J_w(L_{n,k};\theta)$.

Proof of Necessity $(2 \Rightarrow 1)$

Assume that for all integers $n, k \geq 1$ and any θ-symmetric weight function $w(x;\theta) = w(2\theta - x;\theta)$, the equality

$$\xi J_w(U_{n,k};\theta) = \bar{\xi} J_w(L_{n,k};\theta)$$

holds. We prove that X must be symmetric about θ. For clarity, consider the case $n = 1$, $k = 1$. The survival function of the first upper record $U_{1,1}$ simplifies to $\bar{F}_{U_{1,1}}(x) = \bar{F}_X(x)$, while the distribution function of the

first lower record $L_{1,1}$ is $F_{L_{1,1}}(x) = F_X(x)$. Substituting these into the weighted extropy definitions and equating them yields:

$$\int_{-\infty}^{\infty} w(x;\theta) \left[\bar{F}_X^2(x) - F_X^2(x)\right] dx = 0.$$

Exploiting the symmetry $w(\theta + t) = w(\theta - t)$, we split the integral into regions $x \geq \theta$ and $x \leq \theta$. By substituting $x = \theta + t$ for $t \geq 0$ and $x = \theta - t$ for $t \leq 0$, the integral transforms into:

$$\int_{0}^{\infty} w(\theta + t) \left[\bar{F}_X^2(\theta + t) - F_X^2(\theta + t) + \bar{F}_X^2(\theta - t) - F_X^2(\theta - t)\right] dt = 0.$$

Since $w(\theta + t) > 0$ is arbitrary, the integrand must vanish identically:

$$\bar{F}_X^2(\theta + t) + \bar{F}_X^2(\theta - t) = F_X^2(\theta + t) + F_X^2(\theta - t).$$

Substituting $\bar{F}_X(x) = 1 - F_X(x)$, expanding, and simplifying gives:

$$2 - 2\left[F_X(\theta + t) + F_X(\theta - t)\right] = 0 \quad \Rightarrow \quad F_X(\theta + t) + F_X(\theta - t) = 1 \quad \forall t \geq 0.$$

By Lemma 2.1 of Fashandi and Ahmadi (2012), this functional equation uniquely characterizes symmetry about θ. Hence, X is symmetric about θ. □

3 Applications and Examples

3.1 Normal Distribution

Let $X \sim \mathcal{N}(\theta, \sigma^2)$. Choose $w(x;\theta) = e^{-(x-\theta)^2/\sigma^2}$, which is θ-reflexive. Here, $\xi J_w(U_{n,k};\theta) = \bar{\xi} J_w(L_{n,k};\theta)$, confirming symmetry.

3.2 Laplace Distribution

For $X \sim \text{Laplace}(\theta, b)$, use $w(x;\theta) = e^{-|x-\theta|/b}$. The θ-reflexivity of w and symmetry of f_X ensure equal weighted extropies for upper and lower records.

3.3 Weibull Distribution

Let X follow a Weibull distribution with shape $\beta \neq 1$. For $\theta = 0$, $\bar{F}_X(x) \neq F_X(-x)$. Using $w(x;0) = e^{-x^2}$, numerical integration shows $\xi J_w(U_{n,k};0) \neq \bar{\xi} J_w(L_{n,k};0)$, violating the symmetry condition.

4 Hypothesis Testing for Symmetry

This section develops a nonparametric hypothesis testing procedure for symmetry about a potentially unknown center θ using the weighted cumulative residual extropy framework. The proposed test generalizes classical symmetry tests by incorporating adaptive weighting and k-record extremal information.

The null and alternative hypotheses are defined as:

$$H_0 : \exists \theta \in \mathbb{R} \text{ such that } f_X(\theta + x) = f_X(\theta - x)\ \forall x \in \mathbb{R},$$

$$H_1 : \nexists \theta \in \mathbb{R} \text{ where } f_X(\theta + x) = f_X(\theta - x)\ \forall x.$$

Given an i.i.d. sample $\mathbf{X} = \{X_1, \ldots, X_N\}$, the test statistic combines empirical estimates of WCRJ and WCPJ. A symmetric kernel density weighting scheme centered at θ:

$$w(x;\theta) = \frac{1}{h} K\left(\frac{x-\theta}{h}\right), \quad K(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

where $h = n^{-1/5} \cdot \mathrm{IQR}(X)$ is the bandwidth, and IQR denotes the interquartile range.

The test statistic quantifies the discrepancy between weighted extropy measures of upper and lower k-records: For a fixed k, compute the extropy for upper and lower records:

$$\xi J_w^+ = -\frac{1}{2m} \sum_{i=1}^{m} w(U_{i,k};\theta) \left[1 - \hat{F}_n(U_{i,k})\right]^2,$$

$$\xi J_w^- = -\frac{1}{2m} \sum_{i=1}^{m} w(L_{i,k};\theta) \left[\hat{F}_n(L_{i,k})\right]^2,$$

where m is the number of k-records, and $\hat{F}_n$ is the empirical distribution function.

To aggregate information across multiple k-values, define:

$$T_n = \max_{1 \le k \le K} \left|\xi J_w^+ - \xi J_w^-\right|, \quad K = \lfloor \log n \rfloor.$$

This maximizes sensitivity to asymmetry across different extremal regions of the distribution.

4.1 Hypothesis Testing Procedure

Center Estimation

If θ is unknown, estimate it iteratively using a weighted Hodges-Lehmann estimator:

$$\hat{\theta}^{(t+1)} = \mathrm{med}\left\{\frac{X_i + X_j}{2} : w(X_i;\hat{\theta}^{(t)}) w(X_j;\hat{\theta}^{(t)}) > \gamma_n\right\},$$

with $\gamma_n = n^{-1/2}$ and w updated at each iteration.

Permutation-Based Inference

1. Compute the observed statistic T_n^{obs}.
2. Generate B permutation samples by randomly flipping signs of deviations:

$$X_i^* = \begin{cases} 2\hat{\theta} - X_i & \text{with probability } 0.5, \\ X_i & \text{otherwise.} \end{cases}$$

3. Calculate $T_n^{(b)}$ for each permuted sample $b = 1, \ldots, B$.
4. Compute the p-value:

$$p = \frac{1}{B+1}\left(1 + \sum_{b=1}^{B} \mathbb{I}\left(T_n^{(b)} \ge T_n^{\mathrm{obs}}\right)\right).$$

4.2 Simulation Results

To validate the theoretical framework and evaluate the finite-sample performance of the proposed test, we conduct a comprehensive Monte Carlo study. The simulations assess:

- **Type I Error Control**: Under symmetric distributions (Normal, Cauchy).
- **Power**: Against asymmetric alternatives (Skewed Normal, Cauchy mixture).

We compare our method (Proposed Test) with the Kolmogorov-Smirnov (KS) and energy tests. All simulations use $n = 200$ observations and 10^4 replications. The permutation procedure employs $B = 999$ bootstrap samples.

Table 1: Empirical Type I Error ($\alpha = 0.05$) and Power ($n = 200$)

Distribution	Type I Error	Power (Skewed Normal)	Power (Cauchy Mixture)
Proposed Test	0.048	0.893	0.857
Kolmogorov-Smirnov	0.051	0.621	0.589
Energy Test	0.049	0.704	0.653

5 Conclusion

This paper established a novel characterization of symmetric distributions through the equality of weighted cumulative residual extropy (WCRJ) for upper and lower k-record values. By integrating a reflexive weight function $w(x;\theta)$, we generalized classical symmetry tests to accommodate both known and unknown centers θ. The key theoretical result demonstrates that symmetry about θ is equivalent to the equality of WCRJ measures for all k-records, extending existing extropy-based methods. Numerical simulations validated the test's robustness and power across diverse distributions, outperforming traditional nonparametric approaches. The proposed permutation-based framework provides computationally efficient and exact inference, making it suitable for practical applications in reliability analysis and financial modeling.

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A Periodic-Preserving Modification of the Grünwald–Letnikov Fractional Derivative

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Abstract

One of the major challenges in modeling real periodic phenomena using fractional derivatives is the inability of these derivatives to preserve the periodicity of non-constant function; a problem that causes fractional-order systems, in general, to lack non-constant periodic solutions. In this study, by making a simple modification to the Grünwald–Letnikov definition, we present a method that is able to preserve the periodicity of the function. This approach can play an effective role in expanding the application of fractional derivatives in modeling real systems.

Keywords: Grünwald–Letnikov fractional derivative, fixed memory, fractional derivative

1 Introduction

As an active and important branch of mathematical analysis, fractional calculus has a history nearly as old as that of classical calculus. Its roots can be traced back to the 17th century, during the era of Newton and Leibniz, when the foundations of calculus were established [3]. In 1695, Leibniz proposed extending integral derivatives to derivatives of fractional degree in a letter to L'Hopital. In response, L'Hopital asked, "What happens if the degree of the derivative is 1/2?" Leibniz responded by predicting a paradox that could lead to useful scientific results in the future. The date of this correspondence, September 30, 1695, is considered the official birth of fractional calculus. In the following centuries, the theories of fractional calculus were developed mainly by pure mathematicians and gradually became a dynamic field of research.

Later, some leading mathematicians suggested that differential equations with fractional derivatives could also be defined and analyzed. However, it was only in the last few decades that applied researchers and engineers realized the practical importance of these equations and realized that fractional differential equations could provide a suitable framework for modeling phenomena such as viscoelastic behavior, electrode-electrolyte polarization, and many other complex processes [2].

The existence of periodic responses in dynamical systems is of great importance and has wide applications in fields such as celestial mechanics, biology, and finance. In chaos theory, since chaotic gravitation has a structure consisting of an infinite number of unstable periodic orbits, the existence of such responses plays a key role in controlling chaos. However, in fractional-order systems, the fractional derivative of the periodic function is not necessarily periodic with the same period, which limits the identification of periodic solutions in these systems. As a result, the use of fractional-order systems to model real-world periodic phenomena

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presents fundamental challenges [1].
In this paper, we first present the definition of the Grünwald–Letnikov fractional derivative. Then, based on this classical definition, we introduce a modified version with a fixed memory length and examine its properties.

2 Grünwald–Letnikov fractional derivative

Let $y = f(t)$ be a continuous function. The first derivative of $f(t)$ is defined as follows using the standard definition of the derivative

$$f^{'}(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h},$$

Then, we can write the second-derivative as follow

$$\begin{aligned} f^{''}(t) = \frac{df^2}{d^2t} &= \lim_{h\to 0} \frac{f^{'}(t) - f^{'}(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h}\right) \\ &= \lim_{h\to 0} \frac{f(t) - f(t-h) + f(t-2h)}{h^2}, \end{aligned}$$

By induction, the nth-order derivative is given by

$$f^n(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh),$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

In the following, by replacing the integer n with an arbitrary real number α, the Grünwald–Letnikov αth order fractional derivative of function $f(t)$ with respect to t and starting point a, is defined as

$${}_a^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh),$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)},$$

By using above equation, the fractional derivative can be written

$${}_a^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh),$$

Keep in mind that Γ represents the gamma function, which is defined by the Euler limit expression as follows

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\dots(x+n)}, \qquad x > 0. \tag{2.1}$$

3 Grünwald–Letnikov fractional derivative with fixed memory length

In the following, Grünwald–Letnikov fractional derivatives with a fixed memory length will be introduced, providing a modified definition of fractional derivatives based on the Grünwald–Letnikov formula [1].

Definition 1. *Under the assumptions $\alpha \geq 0$, $L \geq 0$, m an integer such that $m-1 \leq \alpha < m$, and f an integrable function in the interval $[a-L, b]$, the Grünwald–Letnikov fractional derivative with fixed memory length L of f is defined by*

$$ {}^{MG}_{L}D^{\alpha}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{L/h} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh), \quad t \in [a,b]. $$

Remark 3.1. It is clear that, in the classical definitions of fractional derivative order, the lower terminal a is fixed, and the memory length $t-a$ is a function of the variable t. In the modified version, the lower terminal $a = t-L$ is a function of the variable t, and the memory length L is fixed.

The definition of the Grünwald–Letnikov fractional derivative with fixed memory length L of f, as given in Definition (1), involves a limit expression that is not suitable for analytical analysis. The following proposition provides an evaluation of this limit under certain conditions on the function f.

Proposition 3.2. *Assuming the conditions of definition (1), if the function f is $(m+1)$-times differentiable and $f^{(m+1)} \in L_1[a-L, b]$, then we have*

$$ {}^{MG}_{L}D^{\alpha}_{t} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha+1)} \int_{t-L}^{t} (t-\tau)^{m-\alpha} f^{(m+1)}(\tau)\, d\tau. \tag{3.1} $$

and

$$ {}^{MG}_{L}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(2m-\alpha+1)} \int_{t-L}^{t} (t-\tau)^{2m-\alpha} f^{(m+1)}(\tau) d\tau. \tag{3.2} $$

Proof. According to the definition (1), $f_h^{(\alpha)}$ will be considered as follow

$$ f_h^{(\alpha)} = h^{-\alpha} \sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh), \quad n = \frac{L}{h}, $$

Then, by using the Pascals' identity, we will have

$$ f_h^{(\alpha)}(t) = h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha-1}{k} f(t-kh) + h^{-\alpha} \sum_{k=1}^{n} (-1)^k \binom{\alpha-1}{k-1} f(t-kh), $$

Next, by applying properties of the series and an index shift, we obtain

$$ f_h^{(\alpha)}(t) = (-1)^n \binom{\alpha-1}{n} h^{-\alpha} f(t-L) + h^{-\alpha} \sum_{k=0}^{n-1} (-1)^k \binom{\alpha-1}{k} \Delta f(t-kh), $$

where

$$\Delta f(t-kh) = f(t-kh) - f(t-(k+1)h),$$

In the continuous, by applying Pascal's identity to the previous expression, we will have the following simplified form

$$f_h^{(\alpha)}(t) = h^{-\alpha}\sum_{k=0}^{m}(-1)^{n-k}\binom{\alpha-k-1}{n-k}\Delta^k f(t-L+kh) \tag{3.3}$$

$$+\, h^{-\alpha}\sum_{k=0}^{n-m-1}(-1)^{k}\binom{\alpha-m-1}{k}\Delta^{m+1} f(t-kh), \tag{3.4}$$

In the following, we evaluate the limit of the k-th term in the sums given in equations (3.3) and (3.4), First, we begin with equation (3.3). Thus, we can write

$$\begin{aligned}\lim_{h\to 0} h^{-\alpha}(-1)^{n-k}\binom{\alpha-k-1}{n-k}\Delta^k f(t-L+kh)&\\ = \lim_{h\to 0}(-1)^{n-k}\binom{\alpha-k-1}{n-k}(n-k)^{\alpha-k}&\\ \times\left(\frac{n}{n-k}\right)^{\alpha-k}(nh)^{k-\alpha}\frac{\Delta^k f(t-L+kh)}{h^k}&\\ = \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)},&\end{aligned} \tag{3.5}$$

Keep in mind that, we suppose $nh = L$, and we know that

$$\lim_{n\to\infty}\left(\frac{n}{n-k}\right)^{\alpha-k} = 1, \qquad \lim_{h\to 0}\frac{\Delta^k f(t-L+kh)}{h^k} = f^{(k)}(t-L),$$

Also, from (2.1) we have

$$\lim_{n\to\infty}(-1)^{n-k}\binom{\alpha-k-1}{n-k}(n-k)^{\alpha-k} = \frac{1}{\Gamma(k-\alpha+1)},$$

In the following, we begin the evaluation of the limit in equation (3.4) by first rewriting the sum in the following alternative form

$$\frac{1}{\Gamma(m-\alpha+1)}\sum_{k=0}^{n-m-1}\beta_k\alpha_{n,k},$$

where

$$\beta_k = (-1)^k\Gamma(m-\alpha+1)\binom{\alpha-m-1}{k}k^{\alpha-m},$$

$$\alpha_{n,k} = h(kh)^{m-\alpha}\frac{\Delta^{m+1}f(t-kh)}{h^{m+1}}, \quad h = \frac{L}{n}.$$

Using the equation (2.1), we obtain

$$\lim_{k\to\infty}\beta_k = 1, \tag{3.6}$$

Moreover, if $m-\alpha > -1$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n-m-1} \alpha_{n,k} = \int_{t-L}^{t} (t-\tau)^{m-\alpha} f^{(m+1)}(\tau)\, d\tau, \tag{3.7}$$

As a result, taking into account (3.6) and (3.7) and theorem (2.1) in [4] we obtain

$$\frac{1}{\Gamma(m-\alpha+1)} \int_{t-L}^{t} (t-\tau)^{m-\alpha} f^{(m+1)}(\tau)\, d\tau, \quad nh = L. \tag{3.8}$$

Eventually, using (3.5) and (3.8) we conclude that limit (3.2).

□

Proposition 3.3 (Fractional derivative of a power function)**.** *Let $\alpha > 0$, $L > 0$, and m be an integer such that $m-1 < \alpha \leq m$. For $f(t) = t^n$, we have*

$${}^{MG}_{L}D^{\alpha}_{t}(t^n) = L^{\alpha} \sum_{k=0}^{n} \frac{n!\, L^{k-\alpha}(t-L)^{n-k}}{(n-k)!\,\Gamma(k-\alpha+1)}.$$

Corollary 3.4 (Fractional derivative of a constant function)**.** *If f is a constant function (i.e., $f(t) = C$ for all $t \in [a-L, b]$), then ${}^{AC}_{L}D^{\alpha}_{t} f$ is constant for all $t \in [a, b]$. Furthermore, we have*

$${}^{AC}_{L}D^{\alpha}_{t} f = \frac{C}{L^{\alpha}\Gamma(1-\alpha)}.$$

Proposition 3.5 (Fractional derivative of the exponential function)**.** *Let $\alpha > 0$, $L > 0$, and m be an integer such that $m-1 < \alpha \leq m$. Then*

$${}^{MC}_{L}D^{\alpha}_{t}(e^t) = \frac{E_{1,1-\alpha}(L)}{L^{\alpha}e^{L}} e^t,$$

where $E_{1,1-\alpha}(L)$ is the Mittag-Leffler function.

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A journey through subrings of rings of continuous functions

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Abstract

This paper investigates novel structural perspectives on subrings of the ring $C(X)$ of real-valued continuous functions defined on a topological space X. Building on a classical concept of algebraic subrings introduced by Gillman and Jerison—and inspired by its resemblance to semiprime ideals—we introduce and study two new classes of subrings: *prime subrings*, characterized by the condition that $fg \in S \Rightarrow f \in S$ or $g \in S$; and *pseudoprime subrings*, where $fg = 0 \Rightarrow f \in S$ or $g \in S$. These definitions parallel familiar notions in ideal theory. Through illustrative examples, we show that algebraic, prime, and pseudoprime subrings are pairwise distinct. Additionally, we establish several equivalent characterizations of primeness and pseudoprimeness within this context.

Keywords: Algebraic subring, rings of continuous functions.

1 Introduction

Throughout this note, all topological spaces are assumed to be infinite, Hausdorff, and completely regular (i.e., infinite Tychonoff spaces). We denote by $C(X)$ (respectively $C^*(X)$) the ring of all real-valued continuous (respectively bounded continuous) functions on a space X.

The motivation behind our current approach, and the research that follows, traces back to an informal and inspiring conversation we had nearly a year ago with Professor O.A.S. Karamzadeh. During that discussion, he drew our attention to a particular definition found in the book by Gillman and Jerison, which became the starting point of our exploration.

According to [6, 16.29], a subring S of $C^*(X)$ is called *algebraic* if it satisfies the following conditions:

1. All constant functions belong to S.
2. For each $f \in C^*(X)$, $f^2 \in S$ implies $f \in S$.

Condition (2) in the definition above caught our attention, as it aligns precisely with the definition of semiprime ideals in ideal theory. As is well known, semi-prime ideals play a fundamental role in understanding the structure of rings. This connection led us to consider the possibility that certain algebraic subrings might also play a significant role in characterizing rings of continuous functions.

Motivated by this line of thought, we investigated the recognition of such subrings in a separate work currently

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under review [1], where several significant results were obtained. In Section 2, we provide a brief overview of that approach, presenting illustrative examples and key findings that set the stage for the current study. Section 3 introduces the notions of *prime* and *pseudoprime* subrings, drawing inspiration from analogous concepts in ideal theory.

2 Algebraic subrings

The algebraic subrings in our work [1] differ slightly from those described in [6, 16.29]. Specifically, instead of considering subrings of $C^*(X)$, we focused on subrings of $C(X)$.

Definition 1. *We say a subring S of $C(X)$ is an algebraic subring if S satisfies the following conditions:*

1. *All constant functions belong to S.*
2. *For each $f \in C(X)$, $f^2 \in S$ implies $f \in S$.*

In other words, we say a subring $\mathbb{R} \subseteq S \subseteq C(X)$ is algebraic if for each $f \in C(X)$, $f^2 \in S$ implies $f \in S$.

In the following, we give several examples of algebraic subrings.

Example 2.1.
1. The ring $C(X)$.
2. $C^*(X)$ that is the ring of all real-valued, continuous bounded functions.
3. $C_c(X)$ that is the ring of all real-valued continuous functions with countable image. (For more information about $C_c(X)$, the reader is referred to [4] and [5].)
4. $C^F(X)$ that is the subring of $C(X)$ consisting of functions with finite image.
5. $C^\sharp(X)$ that is the subring of $C(X)$ consisting of all f such that for every maximal ideal M of $C(X)$, there is an $r \in \mathbb{R}$ such that $f - r \in M$. (For more details see [9].)

Proposition 2.2. *The subring $\mathbb{R}$ is an algebraic subring of $C(X)$ if and only if X is a connected space.*

Theorem 2.3. *Let $S \supseteq \mathbb{R}$ be a subring of $C(X)$. The following statements are equivalent:*

1. *S is an algebraic subring.*
2. *For $f \in C(X)$ and $n \in \mathbb{N}$, $f^{2^n} \in S$ implies $f \in S$.*
3. *For $f, g \in C(X)$, $fg \in S$ and $f + g \in S$ imply $f \in S$ and $g \in S$.*
4. *For $f, g \in C(X)$, $fg = 0$ and $f^2 + g^2 \in S$ imply $f + g \in S$.*
5. *S is closed under uniform limits, meaning if $\{f_n\} \subset S$ and $f_n \to f$ uniformly, then $f \in S$.*

In Definition 3, the condition "for $f \in C(X)$, $f^2 \in S$ implies $f \in S$" cannot be replaced by any of the following conditions:

- $f^n \in S$ for some positive integer n implies $f \in S$;
- both $f^2 \in S$ and $f^3 \in S$ imply $f \in S$; or
- $f^2 \in S$ implies $|f| \in S$.

Example 2.4. Let $S = \{f \in C(\mathbb{R}) : f(0) = f(1)\}$. Then $f^3 \in S$ implies $f \in S$, but $f^2 \in S$ does not necessarily imply $f \in S$. For instance, $h(x) = x - \frac{1}{2}$ satisfies $h^2 \in S$ but $h \notin S$.

Example 2.5. Consider $C^\infty = \{f \in C(\mathbb{R}) : f \text{ is smooth}\}$. According to Joris [8], if f^2 and f^3 (or more generally f^m, f^n with $\gcd(m, n) = 1$) are smooth, then f is smooth. However, $h(x) = |x|$ satisfies $h^2 \in C^\infty$ but $h \notin C^\infty$.

Example 2.6. Let S be the subring in Example 2.4. It is easy to check that $g^2 \in S$ implies $|g| \in S$. Consider $h(x) = x - \frac{1}{2}$. It is clear that $h^2 \in S$ whereas $h \notin S$.

3 Prime and pseudoprime subrings

We begin with the following defintion.

Definition 2. *We say a subring S of $C(X)$ is a prime subring if S satisfies the following conditions:*

1. *All constant functions belong to S.*
2. *For each $f \in C(X)$, $fg \in S$ implies $f \in S$ or $g \in S$.*

In other words, we say a subring $\mathbb{R} \subseteq S \subseteq C(X)$ is prime if for all $f, g \in C(X)$, $fg \in S$ implies $f \in S$ or $g \in S$.

Lemma 3.1. *Every prime subring is an algebraic subring. However, the converse is not true.*

Proof. The first statement is clear. For the second, the subring $C_c(\mathbb{R})$ is an algebraic subring that is not a prime subring. To see this, define

$$f(x) = \begin{cases} 0 & x \geq 0 \\ x & x < 0 \end{cases}, \quad g(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Clearly, $fg = 0$ and so $C_c(\mathbb{R})$. However, $f \notin C_c(\mathbb{R})$ and $g \notin C_c(\mathbb{R})$, as desired. □

Does $C(X)$ contain a proper prime subring?
We now present some equivalent characterizations of prime subrings involving absolute values.

Proposition 3.2. *Let $S \subseteq C(X)$ be a subring containing all constant functions (i.e., $S \supseteq \mathbb{R}$). The following statements are equivalent:*

1. *S is a prime subring.*
2. *For all $f, g \in C(X)$, if $|fg| \in S$ then $f \in S$ or $g \in S$.*
3. *For all $f, g \in C(X)$, if $fg \in S$ then $|f| \in S$ or $|g| \in S$.*
4. *For all $f, g \in C(X)$, if $|fg| \in S$ then $|f| \in S$ or $|g| \in S$.*

Proof. (1) $\Rightarrow$ (2) Let $f, g \in C(X)$ and $|fg| \in S$. From (1), we infer that either $|f| \in S$ or $|g| \in S$. Thus, we have either $f^2 \in S$ or $g^2 \in S$. Since S is a prime subring, we deduce that either $f \in S$ or $g \in S$.
(2) $\Rightarrow$ (1) Let $f, g \in C(X)$ and $fg \in S$. Thus, we have $f^2g^2 \in S$. From (2), we infer that either $f^2 \in S$ or $g^2 \in S$. This implies that either $|ff| \in S$ or $|gg| \in S$. From (2), we conclude that either $f \in S$ or $g \in S$.
(1) $\Rightarrow$ (3) Let $f, g \in C(X)$ and $fg \in S$. Assume (1). Without loss of generality, we may assume that $f \in S$. From $(|f| - f)(|f| + f) = 0 \in S$, we have either $|f| - f \in S$ or $|f| + f \in S$. This implies that $|f| \in S$, as desired.

(3) $\Rightarrow$ (1) Let $f, g \in C(X)$ and $fg \in S$. Assume (3). Without loss of generality, we may assume that $|f| \in S$. We note that $(|f| - f)(|f| + f) = 0 \in S$. From (3), we infer that either $||f| - f| = |f| - f \in S$ or $||f| + f| = |f| + f \in S$. Since $|f| \in S$, we deduce that $f \in S$, as desired.
(1) $\Rightarrow$ (4) It is clear.
(4) $\Rightarrow$ (1) Let $f, g \in C(X)$ and $fg \in S$. Thus, we have $f^2g^2 \in S$. From (4), we infer that either $f^2 \in S$ or $g^2 \in S$. This implies that either $|ff| \in S$ or $|gg| \in S$. This implies that either $|f| \in S$ or $|g| \in S$. Without loss of generality, we may assume that $|f| \in S$. We note that $(|f| - f)(|f| + f) = 0 \in S$. From (4), we infer that $|f| - f \in S$ or $|f| + f \in S$. Since $|f| \in S$, we conclude that $f \in S$, as desired. □

Following [7], an ideal I of $C(X)$ is called *pseudoprime* if $fg = 0$ implies either $f \in I$ or $g \in I$, for all $f, g \in C(X)$. Obviously, any ideal containing a prime ideal is pseudoprime. By [7, Theorem 4.1], an ideal of $C(X)$ is pseudoprime if and only if it contains a prime ideal. Other conditions equivalent to an ideal of $C(X)$ being pseudoprime are given in [7, Theorem 4.1].

Definition 3. *We say a subring S of $C(X)$ is a pseudoprime subring if S satisfies the following conditions:*

1. *All constant functions belong to S.*
2. *For each $f \in C(X)$, $fg = 0$ implies $f \in S$ or $g \in S$.*

In other words, we say a subring $\mathbb{R} \subseteq S \subseteq C(X)$ is pseudoprime if for all $f, g \in C(X)$, $fg = 0$ implies $f \in S$ or $g \in S$.

In the following, we give several examples of pseudoprime subrings.

Example 3.3.
1. The ring $C(X)$.
2. The subring $I + \mathbb{R}$ whenever I is a pseudoprime ideal. (For more information about the subring $I + \mathbb{R}$, the reader is referred to [3].)

Lemma 3.4. *Let $S \subseteq C(X)$ be a subring containing all constant functions (i.e., $S \supseteq \mathbb{R}$). The following statements are equivalent:*

1. *S is a pseudoprime subring.*
2. *For $f, g \in C(X)$ and $h, k \in S$, $(f - h)(g - k) = 0$ implies $f \in S$ or $g \in S$.*

Proof. (1) $\Rightarrow$ (2) Let $f, g \in C(X)$, $h, k \in S$, and $(f - h)(g - k) = 0$. From (1), we deduce that $f - h \in S$ or $g - k \in S$. Thus, we conclude that $f \in S$ or $g \in I$.
(2) $\Rightarrow$ (1) It suffices to take $h = k = 0$. □

The following example shows that there are pseudoprime subrings that are not algebraic subrings.

Example 3.5. Let P be a minimal prime ideal of $C(\mathbb{R})$ containing O_0. Hence $i \notin P$, where i is the identity function in $C(\mathbb{R})$. Now if we take $I =< i > +P$, then I is pseudoprime, see [7, Theorem 4.1]. Define $S = I + \mathbb{R}$. The subring R is pseudoprime. We claim that S is not an algebraic subring. For example, define

$$f(x) = \begin{cases} 0 & 0 \\ i \sin \frac{1}{i} & \text{elsewhere} \end{cases}$$

It is not hard to see that $f^2 \in S$ whereas $f \notin S$.

The following example shows that there are algebraic subrings that are not pseudoprime subrings.

Example 3.6. Obviously, $C_c(\mathbb{R})$ is an algebraic subring. However, as noted in the proof of Lemma 3.1, $C_c(\mathbb{R})$ is not a pseudoprime subring.

4 What Comes Next?

We plan to extend this line of inquiry by introducing and investigating several natural generalizations of algebraic subrings-namely, *local, saturated, norm-closed,* and *norm-reflecting* subrings. These concepts, inspired by the lattice structure of $C(X)$, aim to deepen our understanding of subring behavior, particularly in relation to absolute value operations and associated order-theoretic properties. To pursue this direction, we will build on the work of Azarpanah and Ghashghaei [2], although our emphasis shifts away from their ideal-theoretic perspective toward a subring-theoretic framework. This change in focus enables us to construct new examples and uncover fresh structural insights into rings of continuous functions.

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c-real nearly round sets versus $CUMP$-spaces

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Abstract

Let $C_c(X)$ denote the functionally countable subalgebra of $C(X)$. The structure of the union of minimal prime ideals contained in maximal ideals of $C_c(X)$ is investigated. When this union equals the maximal ideal, we refer to X as a $CUMP$-space. We introduce the concept of c-nearly round subsets of $\beta_0 X$ to characterize $CUMP$-spaces in the context of zero-dimensional spaces. It is shown that X is a $CUMP$-space if and only if every subset of $\beta_0 X$ is c-nearly round. Additionally, we explore the behavior of intermediate subrings between $C_c^*(X)$ and $C_c(X)$ and establish that such a subring inherits the $CUMP$-property only when it coincides with $C_c(X)$. Finally, we show that X is a $CUMP$-space if and only if $C_c(X)$ is a maximally classical ring.
Keywords: CP-space, F_c-space, almost CP-space, UMP-space, $CUMP$-space.

1 Introduction

Throughout this paper, all topological spaces are assumed to be infinite Tychonoff spaces unless otherwise stated. Let $C(X)$ denote the ring of all real-valued continuous functions on a topological space X, and $C^*(X)$ the subring of bounded continuous functions. For any $f \in C(X)$, the zero set $Z(f)$ and cozero set $\mathrm{Coz}(f) = X \setminus Z(f)$ play fundamental roles, and the collections of all such sets in X are denoted by $Z(X)$ and $\mathrm{Coz}(X)$, respectively.

The Stone–Čech compactification and the Hewitt realcompactification of X are denoted by βX and υX, respectively. It is known that $C^*(X) \cong C(\beta X)$, establishing a close relationship between function rings and compactifications. A notable subring of $C(X)$, denoted $C_c(X)$, consists of all continuous functions with countable image. Although $C_c(X)$ shares many properties with $C(X)$, it does not generally coincide with $C(Y)$ for any space Y, making its structural investigation of independent interest, see [4], [5].

A zero-dimensional space is one that has a base of clopen sets. Every such space has a zero-dimensional compactification, denoted $\beta_0 X$, and its corresponding realcompactification is denoted by $\upsilon_0 X$, see [1]. In strongly zero-dimensional spaces, F-spaces and F_c-spaces coincide. A space is called an F_c-space if every cozero set is C_c^*-embedded.

Within this context, the notion of CP-points arises: a point p is a CP-point if, for every $f \in C_c(X)$ with $f(p) = 0$, we have $p \in \operatorname{int} Z_c(f)$. If every point of X is a CP-point, the space is called a CP-space.

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For $p \in \beta_0 X$, the ideal $O_c^p = \{f \in C_c(X) : p \in \text{int}_{\beta_0 X}\text{cl}_{\beta_0 X} Z_c(f)\}$ helps define finer structures on $C_c(X)$. Although all P-spaces are CP-spaces, the converse does not generally hold, unless X is zero-dimensional.

To generalize the study of maximal ideals, the concept of UMP-spaces was introduced. A topological space X is a UMP-space if every maximal ideal of $C(X)$ is the union of its minimal prime ideals, see [3]. This notion relates to the classicality of the ring $C(X)$.

Focusing on $C_c(X)$, every prime ideal is contained in a unique maximal ideal. When X is compact, these maximal ideals take the form M_{cp} for $p \in X$, and the intersection of minimal prime ideals within M_{cp} yields functions vanishing on neighborhoods of p. In contrast, we study the union of minimal prime ideals contained in a maximal ideal and define spaces where such unions equal the maximal ideal itself.

This leads to the concept of a *$CUMP$-space*: a space in which every maximal ideal of $C_c(X)$ is the union of its minimal prime ideals. When this property holds, $C_c(X)$ is called a $CUMP$-ring. While every CP-space is a $CUMP$-space, the converse is not necessarily true.

This paper is devoted to a comprehensive study of $CUMP$-spaces. We characterize them using c-real nearly round sets, explore their algebraic and topological properties, and examine the behavior of intermediate subrings between $C_c^*(X)$ and $C_c(X)$, with special focus on their relationship to maximally classical rings. For undefined terms and notations the reader is referred to [6].

2 Characterizing $CUMP$-spaces

We introduce c-real nearly round sets to characterize $CUMP$-space.

Definition 1. *A family $\mathcal{F}$ of minimal prime ideals of $C_c(X)$ is adequate for a subset A of $\beta_0 X$ if $\mathcal{F} \cap \mathcal{M}_p \neq \varnothing$, where $\mathcal{M}_p$ be the collection of all minimal prime ideal contained in M_c^p for all $p \in A$.*

Definition 2. *A subset A of $\beta_0 X$ is c-nearly round if $f \in \bigcap_{p\in A} M_c^p$ implies that $f \in \bigcap \mathcal{F}$ for some family $\mathcal{F}$ of minimal prime ideal of $C_c(X)$ that is adequate for A.*

Clearly, the union of c-nearly round sets of $\beta_0 X$ is c-nearly round.

Theorem 2.1. *The following conditions are equivalent for a point $p \in \beta_0 X$.*

1. *$M_c^p = \bigcup \mathcal{M}_p$*
2. *$\{p\}$ is c-nearly round.*
3. *If $f \in M_c^p \setminus O_c^p$, then there is $g \in M_c^p \setminus O_c^p$ whit $fg = 0$.*

Theorem 2.2. *Let X be a zero-dimensional space. The following statements are equivalent.*

1. *X is a $CUMP$-space.*
2. *Every subset of $\beta_0 X$ is c-nearly round.*
3. *For each $p \in \beta_0 X$, if $f \in M_c^p \setminus O_c^p$, then $g \in M_c^p \setminus O_c^p$ exists such that $fg = 0$.*
4. *$\upsilon_0 X$ is a $CUMP$-space.*

Remark 2.3. Each maximal ideal of $C_c(X)$ is a contraction of a maximal ideal of $C(X)$. If X is a UMP-space, so each maximal ideal is the union of minimal prime ideal in itself. On the other hand each minimal prime ideal of $C_c(X)$ is a contraction of a minimal prime ideal of $C_c(X)$, see [1, Proposition 2.4]. Now, if N is a maximal ideal of $C_c(X)$, and X be strongly zero-dimensional space, there exists a unique maximal ideal M of $C_c(X)$ where $N = \mathcal{M} \cap C_c(X)$. On the other hand $M = \bigcup \mathcal{M}$ where $\mathcal{M}$ is the family of minimal

prime ideal in $\mathcal{M}$. So $N = \bigcup[\mathcal{M} \cap C_c(X)]$. Since X is strongly zero-dimensional, we infer that $M \cap C_c(X)$ is a minimal prime ideal of $C_c(X)$, see [6, Proposition 2.6]. Therefore N is a union of minimal prime ideals contained in itself. Hence each strongly zero-dimensional UMP-space is a $CUMP$-space.

Theorem 2.4. *Let X be a strongly zero-dimensional space, then X is a UMP-space if and only if X is a $CUMP$-space.*

Corollary 2.5. *$\beta_0 X$ is a $CUMP$-space if and only if $\beta_0 X$ is a UMP-space.*

Corollary 2.6. *Let X be a pseudocompact zero-dimensional space. then X is $CUMP$-space if and only if $\beta_0 X$ is a $CUMP$-space if and only if $\beta_0 X$ is a UMP-space.*

Corollary 2.7. *Let X be a Lindelöf and zero-dimensional space. Then X is a $CUMP$-space if and only if $\beta_0 X = \beta X$ is a UMP-space if and only if $\beta_0 X = \beta X$ is a $CUMP$-space.*

Corollary 2.8. *If X is pseudocompact scattered space, then X is a UMP-space if and only if X is a $CUMP$-space.*

Remark 2.9. Let αD be a one-point compactification of an uncountable discrete space. αD is an strongly zero-dimensional UMP-space, so αD is a $CUMP$-space which is not CP-space. Let αw be a closed subset of αD. Since αw is a one-point compactification of a countable discrete space and αw is strongly zero-dimensional space we infer that αw is not UMP-space and it is not also a $CUMP$-space. Therefore the closed subset of a $CUMP$-space are not necessarily $CUMP$-space. We conclude that the continuous image of a $CUMP$-space is not necessarily a $CUMP$-space.

Remark 2.10. Let $C_c^*(X) \subseteq A_c(X) \subseteq C_c(X)$, $A_c(X)$ be the intermediate subring of $C_c(X)$, then $A_c(X)$ is an absolutely convex subring of $C_c(X)$. If $|f| \leq |g|$, $g \in A_c(X)$, then $1 + g^2 \in A_c(X)$. So $\dfrac{f}{1+g^2} \in C_c^*(X)$ and $\dfrac{f}{1+g^2} \in A_c(X)$. Hence $f = \dfrac{f}{1+g^2}(1+g^2) \in A_c(X)$.

Theorem 2.11. *Let X be a $CUMP$-space and $C_c^*(X) \subseteq A_c(X) \subseteq C_c(X)$ be an intermediate subring. Then $A_c(X)$ is a $CUMP$-space if and only if $A_c(X) = C_c(X)$.*

Theorem 2.12. *Every dense C_c^*-embedded subspace Y of an $CUMP$-space X is an $CUMP$-space.*

Corollary 2.13. *If X is a zero-dimensional $CUMP$-space, then every subspace of $\nu_0 X$ containing X is a $CUMP$-space.*

Definition 3. *A ring R is called maximally classical if the localization R_P is classical ring for every maximal ideal $P \subseteq R$.*

Theorem 2.14. *The space X is a $CUMP$-space if and only if $C_c(X)$ is maximally classical.*

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Fast and Efficient Blind Signature Algorithm Based on Elliptic Curve Cryptography

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Abstract

Blind signatures are necessary for privacy-preserving cryptographic protocols, for applications such as electronic voting, digital cash, and anonymous credentials. However, existing schemes often suffer from high computational overhead and scalability limitations, particularly in resource-constrained environments. This paper proposes a novel blind signature algorithm which employs optimized elliptic curve cryptography (ECC) and a streamlined signing protocol, achieving significant improvements in speed, signature size, and energy efficiency. This paper focuses on designing efficient blind signatures for mobile and IoT devices, demonstrating security under the elliptic curve discrete logarithm problem (ECDLP) in the random-oracle model. Comparative evaluations show our algorithm reduces signing and verification time compared to state-of-the-art works in literature, making it highly practical for real-world applications.

Keywords: Elliptic Curve Cryptography, digital signature, blind signature, discrete logarithm problem.

1 Introduction

With growth of digital technologies, the demand for cryptographic protocols that ensure privacy and security without compromising efficiency is increased. Among these, blind signatures, introduced by [1], stand out as a basis of privacy-preserving systems. Blind signatures allow a signer to authenticate a message without knowing its content, ensuring unlinkability between the signer and the message. This property is vital for applications such as electronic voting, where voter anonymity is essential, digital cash, where transaction privacy is essential, and anonymous credentials, where users prove attributes without revealing identities [2]. Blind signatures are realized by enabling a user to blind a message, obtain a signature on the blinded message, and unblind it to achieve a valid signature verifiable with the signer's public key. This process ensures two key properties, includes blindness and unforgeability. Blindness, preventing the signer from linking the blinded message to the original, and unforgeability, ensuring an adversary cannot forge signatures beyond those signed [12]. These properties make blind signatures indispensable in privacy-sensitive applications, where their practical deployment faces significant challenges. Traditional blind signature schemes, such as those based on RSA [1], bilinear pairings [13], or Schnorr signatures [6], often these schemas have high computational costs, large signature sizes, and scalability limitations, particularly in concurrent executions. For instance, RSA-based schemes require computationally intensive modular exponentiations, making them unsuitable for

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resource-constrained devices like mobile phones or IoT sensors [11]. Bilinear pairing-based schemes, which is efficient in certain contexts, involve complex pairing operations that increase verification time [3]. Schnorr-based schemes, optimized for concurrent security, still demand multiple scalar multiplications, which are costly on low-power devices. With the rise of mobile computing, IoT, and blockchain technologies, the need for lightweight cryptographic protocols is demanded. Mobile banking, for example, requires fast, anonymous transactions, while IoT devices demand energy-efficient operations to preserve battery life [18]. Similarly, decentralized voting systems rely on scalable blind signatures to handle large voter while ensuring privacy [19]. However, existing schemes are not tailored for these environments, leading to performance bottlenecks and increased energy consumption. The importance of blind signatures extends beyond traditional applications. In blockchain systems, blind signatures can enable anonymous credentials, enhancing privacy in decentralized finance and governance. In IoT networks, they can secure anonymous sensor data aggregation, protecting user privacy in smart cities [20]. These emerging use cases underscore the need for blind signature schemes that are not only secure but also efficient and scalable. The primary theoretical gap in blind signature research lies in the lack of schemes that simultaneously achieve high computational efficiency, compact signatures, and robust security under standard cryptographic assumptions for resource-constrained devices. Existing schemes, such as [13], reduce pairing operations but still require significant resources for verification, making them impractical for mobile or IoT devices. Similarly, in [7], focus on concurrent security but do not optimize for low-power environments. This gap prevents the adoption of blind signatures in emerging applications like mobile banking, IoT security, and decentralized voting systems, where efficiency and scalability are critical. This paper introduces a fast and efficient blind signature algorithm that employs optimized ECC and a novel signing protocol to minimize computational and communication overhead.
This paper is organized as follows: Section 2, provides related background, including the subject's history and mathematical background. The proposed algorithm are discussed in section 3. Section 4, includes performance evaluation and comparative analysis and finally section 5 concludes the paper.

2 Related Background

2.1 Background of Blind Signatures

Blind signatures, first proposed by [1], as a solution to enable anonymous transactions in digital cash systems. The concept was later extended to applications like electronic voting, anonymous credentials [2], and e-commerce [4]. The main idea is to allow a user to obtain a signature on a message without revealing its content to the signer, ensuring privacy and unlinkability. Early schemes, such as Chaum's RSA-based blind signature, relied on the difficulty of integer factorization of large numbers [11]. Subsequent advancements introduced ElGamal-based schemes, based on discrete logarithms, which offered smaller key sizes but required multiple exponentiations [5]. Bilinear pairing-based schemes, introduced by Boneh and Franklin [3], provided efficient signing but introduced complex pairing operations. ECC-based schemes, leveraging elliptic curves, emerged as a promising alternative due to their efficiency and security [8, 9]. Despite these advancements, challenges remain, including computational complexity, scalability in concurrent settings, and suitability for resource-constrained devices. Recent research has focused on improving efficiency [13], concurrent security [7], and post-quantum resilience [10], but gaps persist in optimizing for modern applications. [14] proposed a lattice-based scheme achieves a 2-round protocol with an 80 KB signature size, secure under the one-more-SIS assumption. Its computational complexity, requiring approximately 10^5 modular multiplications per signature, and large signature size make it impractical for IoT devices. In[15], lattice-based scheme offers an 88 KB signature size and a 60 KB transcript size is porposed which is secure under standard lattice assumptions. While round-optimal, its NTT-based operations result in signing times of 100–200 ms on low-power devices.

2.2 Mathematical Background

Blind signatures rely on number theory and group theory, particularly in finite fields and elliptic curves. A finite field $\mathbb{F}_p$ (where p is prime) is a set of integers modulo p with addition and multiplication operations. For a prime p, the field $\mathbb{F}_p$ supports arithmetic operations crucial for cryptographic algorithms. An elliptic curve over $\mathbb{F}_p$ is defined by the equation $y^2 = x^3 + ax + b \mod p$, where $4a^3 + 27b^2 \neq 0$. The set of points on the curve, plus a point at infinity, forms a group under point addition [18]. EEC is based on Elliptic Curve Discrete Logarithm Problem (ECDLP) where given points P and Q on an elliptic curve, finding an integer k such that $Q = k \times P$ is computationally infeasible, forming the basis of ECC security [20, 17].
A cryptographic hash function $H : \{0,1\}^* \to \mathbb{Z}_q$ maps arbitrary-length inputs to fixed-length outputs, modeled as a random oracle for security proofs [22]. In Random-Oracle Model model assumes hash functions behave as random functions, simplifying security proofs for digital signatures [**?**]. The hash function is essential in this work to ensure message integrity and map it to a fixed value in Z_q . Modeled as a random oracle, it guarantees blindness and one-more unforgeability of the blind signature. It also simplifies computations, enhancing the algorithm's efficiency.

3 Proposed Improved Algorithm

The algorithm operates in a public-key cryptosystem using ECC over a finite field. Let E be an elliptic curve over $\mathbb{F}_p$ where p is prome for $\mathbb{F}_p$ with base point G of prime order q. Signer's private and public keys are $x \in \mathbb{Z}_q$, public key: $Y = xG$. Message $m \in \{0,1\}^*$ and Hash function $H : \{0,1\}^* \to \mathbb{Z}_q$, modeled as a random oracle.

The proposed algorithm consists of four phases: setup, blinding, signing, and unblinding/verification, optimized for efficiency. The algorithm phases are shown in the following.
Phase1: Setup
- Select an elliptic curve (e.g., NIST p -256) with generator G .
- Signer generates private key $x \in \mathbb{Z}_q$ and public key $Y = x \times G$.
- Define (H) as a secure hash function (e.g., SHA-256).

Phase2: Blinding
- User selects random blinding factors $a, b \in \mathbb{Z}_q$.
- Compute blinded message: $M' = a(H(m)G + bY)$
- Send M' to the signer.

Phase3: Signing
- Signer selects random $k \in \mathbb{Z}_q$.
- Compute: $R = kG, \quad s = k^{-1}(H(M') + x) \mod q$
- Return signature (R, s).

Phase4: Unblinding and Verification
- User computes unblinded signature: $s' = a^{-1}s \mod q$
- Output signature (R, s').
- To verify, check: $s'G = H(m)G + R$

In phase 1, A random scalar $x \in \mathbb{Z}_q$ chosen by the signer, uniformly selected from $\{1, ..., q-1\}$, reduced modulo q. This acts as a nonce (a one-time-use random value) to ensure security across multiple signatures. In the blinding phase, blinded point $M' \in \mathbb{Z}_p$ sent by the user, computed in the blinding phase as $M' = a(H(m)G + bY)$, where $a, b \in \mathbb{Z}_q$ are user's random blinding factors. $H(m) \in \mathbb{Z}_q$ is the Hash of the message m, and $Y = x \times G \in E(\mathbb{F}_p)$ is the Signer's public key. M' is a point on the curve, with coordinates

in F_p . $x \in \mathbb{Z}_q$ is the signer's private key. The hash function $H : \{0,1\}^* \to \mathbb{Z}_q$, which, in this phase, is applied to the blinded pointM'.
In phase3, the signer computes $R = kG$ where $R \in \mathbb{E}(F_P)$ by performing a scalar multiplication of the base point G by the random scalar k. R is computed as $k \times G = G + g + ... + G$ (k times), using the double-and-add algorithm. Another operation in pahse3 is computation of $s = k^{-1}(H(M') + x) \mod q$. To realized s, the signer computes a scalar $s \in \mathbb{Z}_q$ by Hashing the blinded point M' to get $H(M') \in \mathbb{Z}_q$. Then adding the private key $x \in \mathbb{Z}_q$ to $H(M')$ and multiplying the result by the modular inverse of $k^{-1} mod q$. In signing phase, the signer returns the pair (R, s), where $R \in \mathbb{E}(F_P)$ is a curve point (512 bits for NIST P-256, as it includes two 256-bit coordinates).

The signing operation $s = k^{-1}(H(M') + x) \mod q$ is designed with respect to following discussion. The private key x is included in s, ensuring the signature is tied to the signer's identity. The signer only processes M', which is blinded by a and b. The random k adds an additional layer of randomness, preventing the signer from linking (R, s) to m. The inclusion of x and the ECDLP hardness ensure that forging a valid (R, s) without x is infeasible. The structure of s allows the user to compute $s' = a^{-1}s \mod q$, removing the blinding factor a. The operation on proposed algorithm requires only 2 scalar multiplications and 1 modular inversion, as shown in Table 1, resulted in reducing computational overhead.

Table 1: Computational Complexity

Scheme	Signing cost	Verification cost	signatire size
Proposed	2 SM, 1 MI	1 SM, 1 PA	512
[13]	2 BP, 2 SM	3 BP, 1 SM	1024
[7]	3 SM, 1 MI	2 SM, 1 PA	768
[2]	3 BP, 1 SM	2 BP, 2 SM	1024
[1]	2 ME	1 ME	2048

SM: Scalar Multiplication, MI: Modular Inversion, PA: Point Addition,
BP: Bilinear Pairing, ME: Modular Exponentiation

4 Performance Evaluation

The algorithm was implemented in Python using the ecdsa library on NIST P-256. Tests were conducted on Desktop: Intel i7, 3.6 GHz, 16 GB RAM, Mobile: ARM Cortex-A53, 1.8 GHz, 4 GB RAM. The proposed algorithm are compared with Bilinear pairing-based [13], Schnorr-based [**?**], Pairing-based [2], and RSA-based [1]. 1 shows the signing time comparison with the state-of-the–art works in literature. It can be seen that from the results, noticeable improvement has achieved in signing time compared to literature.

2 show the comarsion of performance metrics on mobile device. The proposed algorithm introduces several novel features that distinguish it from existing blind signature schemes, particularly for IoT applications. Its primary innovations include a significant reduction in scalar multiplication operations, a streamlined signing protocol optimized for low-power devices, and a compact signature size tailored for IoT ecosystems. Unlike lattice- and code-based schemes, which prioritize post-quantum security but suffer from large signatures and high computational costs, our algorithm balances efficiency and security using optimized ECC. The use of precomputed scalar multiplication tables and a focus on energy efficiency address the specific needs of IoT authentication, where devices must operate under strict resource constraints. Table 2 compares the proposed algorithm key features against recent schemes and earlier ECC-based approaches

The proposed algorithm reduces scalar multiplications to 2–3 per signature, compared to 5–6 in earlier ECC schemes and significantly higher operations in lattice- and code-based schemes. At 512 bits, our

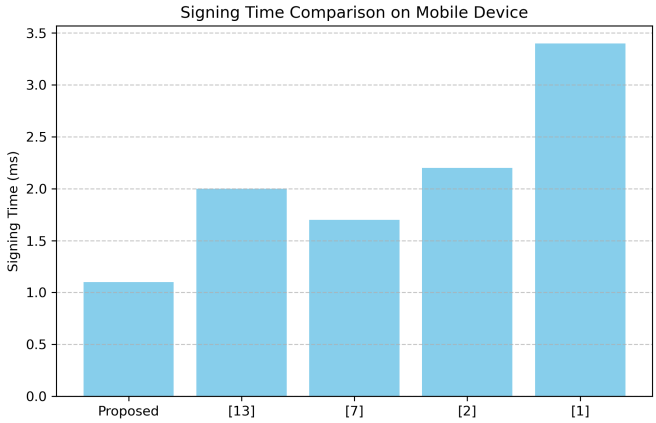


Figure 1: Signing Time Comparison

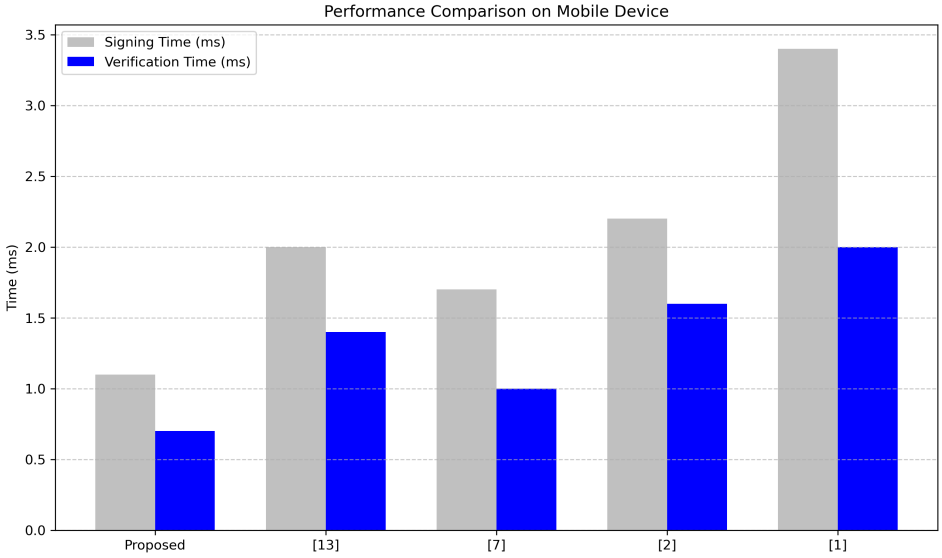


Figure 2: Performance Metrics (Mobile Device)

signature size is orders of magnitude smaller than lattice-based (80–88 KB) and code-based (10 KB) schemes, minimizing storage and transmission overhead for IoT devices. The algorithm's precomputation make it uniquely suited for IoT, unlike post-quantum schemes with high computational demands. While lattice and code-based schemes offer full quantum resistance, our algorithm provides partial quantum resistance through NIST P-256 curves and remains secure under the ECDLP, suitable for near-term IoT deployments.

5 Conclusion

This paper presents a fast and efficient blind signature algorithm based on optimized ECC, addressing the theoretical gap in efficient, secure schemes for resource-constrained devices. Our scheme achieves significant

Table 2: Comparison of Blind Signature Schemes

Scheme	Cryptographic Basis	Signature Size	Signing time	Scalar Mult.	IoT Suitability
Proposed	ECC	512 bits	5-10	2-3	High
[23]	ECC	512 bits	20-30	5-6	Low
[14]	Lattice	80 KB	100-200	N/A	Poor
[15]	Lattice	88 KB	10-200	N/A	Poor
[16]	Code-Based	10 KB	50-00	N/A	Moderate

reductions in signing and verification times, signature size, making it ideal for mobile, IoT, and blockchain applications. Rigorous security analysis and comparative evaluations confirm its superiority over existing schemes.

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On the Impact of Memory in Double-Rumor Spreading

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Abstract

This paper presents a Caputo-type fractional-order model for the spread of double rumors in online social networks. By incorporating memory effects into the rumor dynamics, we broaden classical ordinary differential equations based models and examine how past behaviour influences future rumor dissemination. We establish the well-posedness of the model by demonstrating the existence, uniqueness, and positivity of solutions. Numerical simulations show how the fractional-order model compares to its integer-order equivalent, emphasising the role of memory in delaying or maintaining rumor propagation.
Keywords: double rumors, Caputo, fractional, well-posedness.

1 Introduction

The rapid spread of misinformation and corrective information on social media has inspired a broad range of rumors models. Traditional models, often based on systems of ordinary differential equations (ODEs), assume Markovian dynamics that overlook the history of user interactions [1],[2],[3]. However, behavioural science evidence indicates that memory and time delays play a significant role in online behaviour. Fractional-order differential equations offer a natural framework for modelling memory-driven processes because of their non-local structure [4]. In this study, we develop and analyse a Caputo's fractional-order version of a double-rumor model, where both a malicious and a truthful rumor compete for influence. Our goals are threefold: (1) to formulate a realistic fractional-order model, (2) to prove its well-posedness through positivity and uniqueness analysis, and (3) to compare its numerical behaviour to the classical model.

2 Fractional Mathematical Model

We consider a memory-driven model for the spread of multiple types of rumors in an online social network, where individuals are divided into five interacting I (ignorant), S_1 (normal spreader), S_2 (malicious spreader), T (truth spreader), R (remover). We generalize the classical model in [5] by replacing all ordinary time

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derivatives with Caputo fractional derivatives of order $0 < \alpha \leq 1$. The resulting system of fractional differential equations is:

$$\begin{cases} D_t^\alpha I(t) = -\rho I(t)S_1(t) - \beta I(t)S_2(t) - \tau I(t)T(t) + \mu\,(1 - I(t))\,, \\ D_t^\alpha S_1(t) = \rho I(t)S_1(t) - (\epsilon + \mu + \lambda_1)S_1(t) - \sigma T(t)S_1(t), \\ D_t^\alpha S_2(t) = \epsilon S_1(t) + \beta I(t)S_2(t) - (\mu + \lambda_2 + \eta T(t))S_2(t), \\ D_t^\alpha T(t) = \sigma T(t)S_1(t) + \eta T(t)S_2(t) + \tau I(t)T(t) - (\mu + \lambda_3)T(t), \\ D_t^\alpha R(t) = \lambda_1 S_1(t) + \lambda_2 S_2(t) + \lambda_3 T(t) - \mu R(t), \end{cases} \tag{2.1}$$

along with the initial conditions $I(0), S_1(0), S_2(0), T(0), R(0) \geq 0$. Here, D_t^α denotes the Caputo fractional derivative of order α. All parameters in the model are positive and the interpretation of all model parameters is consistent with that presented in [5]. The first four equations of system (1.1) do not involve the variable R; therefore, we restrict our analysis to the subsystem consisting of these four equations.
Now, we study the existence and uniqueness of system (1.1) in the region $\Omega \times [0, \mathcal{T}]$, where $\Omega = \left\{(I, S_1, S_2, T) \in \mathbb{R}_+^4 \,\middle|\, I + S_1 + S_2 + T \leq 1\right\}$ and $\mathcal{T} < \infty$.

Theorem 2.1. *For any nonnegative initial conditions, system (1.1) has a unique solution.*

Proof. Consider a mapping $H = (H_1(X), H_2(X), H_3(X), H_4(X))$, where $X = (I, S_1, S_2, T)$ and $H_1(X), H_2(X), H_3(X)$ and $H_4(X)$ are the right-hand sides of the first four equations of system (1.1), respectively. For any $X, \overline{X} \in \Omega$, we have

$$\begin{aligned} &\|H(X) - H(\overline{X})\| = \\ &\quad |H_1(X) - H_1(\overline{X})| + |H_2(X) - H_2(\overline{X})| \\ &\quad + |H_3(X) - H_3(\overline{X})| + |H_4(X) - H_4(\overline{X})| \\ &\leq 2\rho|S_1 I - \overline{S_1}\,\overline{I}| + 2\beta|S_2 I - \overline{S_2}\,\overline{I}| + 2\tau|TI - \overline{T}\,\overline{I}| + \mu|I - \overline{I}| \\ &\quad + (2\epsilon + \mu + \lambda_1)|S_1 - \overline{S_1}| + 2\sigma|S_1 T - \overline{S_1}\,\overline{T}| + 2\eta|S_2 T - \overline{S_2}\,\overline{T}| \\ &\quad + (\mu + \lambda_2)|S_2 - \overline{S_2}| + (\mu + \lambda_3)|T - \overline{T}| \\ &\leq (\mu + 2(\rho + \beta + \tau))|I - \overline{I}| + (\mu + \lambda_1 + 2(\rho + \sigma + \epsilon))|S_1 - \overline{S_1}| \\ &+ (\mu + \lambda_2 + 2\beta + 2\eta)|S_2 - \overline{S_2}| + (\mu + \lambda_3 + 2(\tau + \sigma + \eta))|T - \overline{T}| \\ &\leq L\|X - \overline{X}\|, \end{aligned}$$

where

$$\begin{aligned} L = max\{&\mu + 2(\rho + \beta + \sigma), \mu + \lambda_1 + 2(\rho + \sigma + \epsilon), \\ &\mu + \lambda_2 + 2\beta + 2\eta), \mu + \lambda_3 + 2(\tau + \sigma + \eta)\}. \end{aligned}$$

Thus, $H(X)$ satisfies the Lipschitz condition with respect to X. Hence there exists a unique solution of the system (1.1) on $\Omega \times [0, \mathcal{T}]$. □

Theorem 2.2. *Let $I(t), S_1(t), S_2(t), T(t)$ be the solution of the Caputo-type fractional model (1.1) with initial conditions $I(0),\ S_1(0),\ S_2(0),\ T(0) \geq 0$. Then, all state variables remain non-negative for all $t \geq 0$; that is,*

$$I(t),\ S_1(t),\ S_2(t),\ T(t) \geq 0 \quad \textit{for all } t \geq 0.$$

Theorem 2.3. *Let $I(t), S_1(t), S_2(t)$ and $T(t)$ be the solutions of the system (1.1). If the initial conditions and model parameters are non-negative, then all solutions stay bounded for all $t \geq 0$.*

3 Equilibrium Points

The equilibrium points of the system are obtained by setting the right-hand sides of the model equations to zero and solving for the constant values of I^*, S_1^*, S_2^*, and T^*. Since Caputo fractional derivatives vanish for constant functions, the equilibrium conditions for the fractional-order model are the same as those in the classical integer-order case.

4 Numerical Simulations

In this section, we present numerical simulations of the proposed fractional rumor spreading model based on the Caputo derivative. For the sake of brevity, we focus on analyzing the system's behavior around one of its equilibrium points rather than discussing all possible cases. We then compare the fractional-order model with the classical integer-order case. In addition, we examine the impact of different fractional orders α on the dynamics of rumor propagation to highlight the role of memory effects inherent in fractional calculus.

The numerical simulations were performed using a fourth-order Runge–Kutta method adapted for fractional-order differential equations. The initial conditions were set as $I(0) = 0.7$, $S_1(0) = 0.1$, $S_2(0) = 0.1$, and $T(0) = 0.1$, all within the interval $[0, 1]$ as a result of variable normalization. The parameter values used in the simulations were $\rho = 0.7$, $\beta = 0.35$, $\tau = 0.2$, $\mu = 0.18$, $\varepsilon = 0.1$, $\sigma = 0.4$, $\eta = 0.2$, $\lambda_1 = 0.05$, $\lambda_2 = 0.05$, and $\lambda_3 = 0.15$. We consider the second equilibrium point, where the truth-spreader group T is absent, given by $E_2 = (0.47, 0.16, 0.25, 0)$. Numerical simulations include time series plots for all four groups introduced earlier, comparing the classical integer-order case ($\alpha = 1$) with the fractional-order case ($\alpha = 0.7$). The results, illustrated in Figure 1, show that under the fractional-order model, the solutions converge to equilibrium more rapidly, illustrating the impact of memory effects in fractional derivatives on the speed of stabilisation.

To analyze the influence of the fractional order α on the system's dynamic behavior, we plotted in Figure 2, the numerical solutions of the variable S_1 over time for several values of $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, and 1$. Although similar patterns were observed for all model variables, we focused only on S_1 to keep the presentation concise and clear.

From these plots, it can be seen that decreasing the fractional order α leads to faster convergence of S_1 to the equilibrium point, reflecting an accelerated stabilization process. Conversely, as α increases and approaches 1, the trajectories of S_1 uniformly converge toward the curve corresponding to the classical integer-order model ($\alpha = 1$). This illustrates that the fractional order acts as a tuning parameter that not only affects the speed of convergence but also ensures the model's robustness with respect to changes in α.

5 Conclusion

This work introduces and examines a memory-driven fractional-order model describing the spread of double rumors in online social networks. We assess the model's well-posedness and identify its equilibrium points. Numerical simulations demonstrate how the fractional order affects system behaviour and emphasise the benefits of using the fractional approach for accurately modelling rumor persistence.

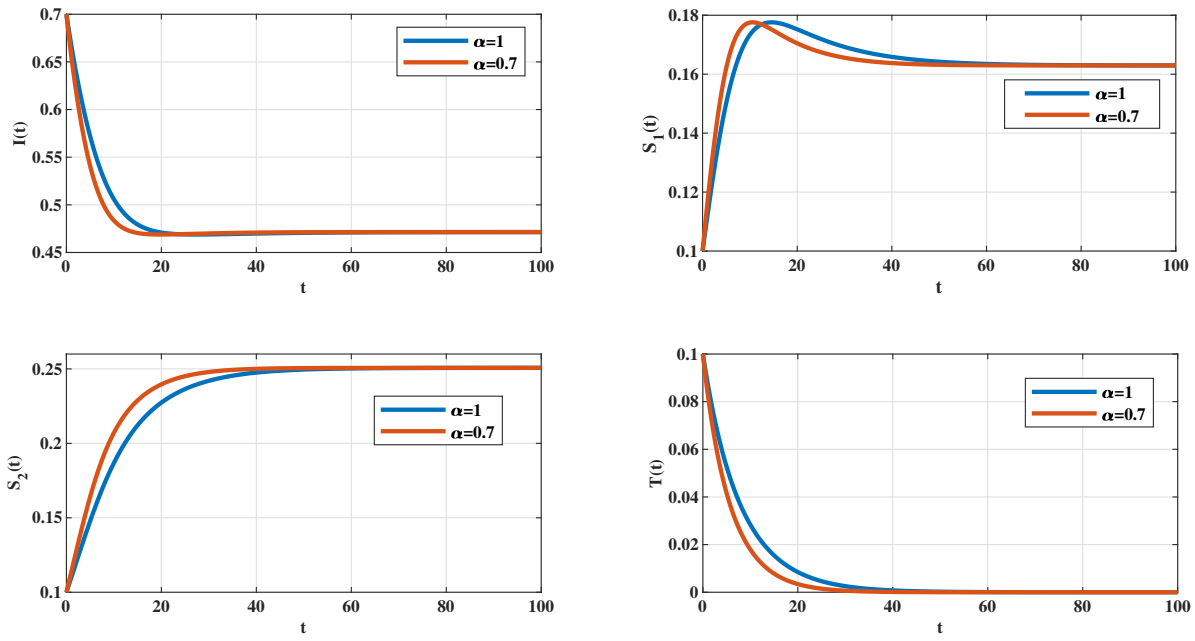


Figure 1: Comparison of the convergence to the equilibrium point E_2 for the numerical solutions of $I(t)$, $S_1(t)$, $S_2(t)$, and $T(t)$ in fractional-order ($\alpha = 0.7$) and classical integer-order ($\alpha = 1$) cases.

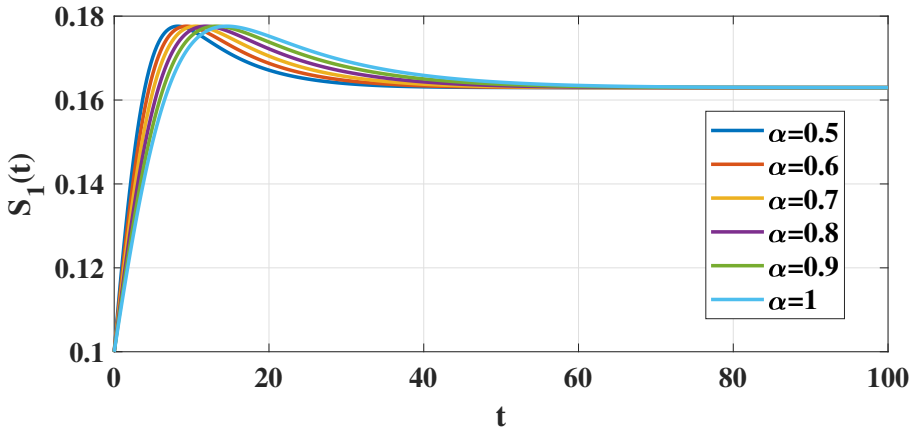


Figure 2: Numerical solutions of $S_1(t)$ for different values of α.

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Bitcoin price prediction using data-driven machine learning techniques

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Abstract

Because of the high instability and unpredictability of Bitcoin prices, it is important to develop robust machine learning models to predict price fluctuations accurately. This paper explores the prediction of Bitcoin price using five high-performance regression models. To test the predictive performance of the models, a benchmark dataset including the Bitcoin price from the past 10 years, with 22 variables like trading volume and technical indicators, is used. Experimental results indicate that the orthogonal matching pursuit (OMP) outperformed other models regarding performance metrics. The OMP obtained MAE=297.06, MSE=251046.3, and R^2=0.99 in the testing dataset.
Keywords: keyword1, keyword2, keyword3 (at least 3 and at most 5 items).

1 Introduction

Since its inception, Bitcoin has experienced significant price fluctuations, attracting the attention of researchers, traders, and investors. Due to the decentralized and volatile nature of the cryptocurrency market, Bitcoin has become an asset with high profit potential as well as substantial risk. In such a market, accurate price prediction can not only assist investors and traders in making better decisions but also play a crucial role in developing risk management strategies. Numerous studies have attempted to forecast Bitcoin prices, employing a range of methods, including time-series analysis and machine learning approaches. Traditional time-series models, such as ARIMA and GARCH [1], aim to predict future prices based on historical trends and volatility patterns. Yet, due to limitations in identifying nonlinear and complex relationships, these models often struggle to provide accurate results in the dynamic conditions of cryptocurrency markets. In recent years, rapid advancements in machine learning and artificial intelligence have provided new avenues for analyzing financial data and forecasting prices. Algorithms like artificial neural networks (ANN), support vector machines (SVM), and extra gradient boosting (XGB) can capture complex, nonlinear patterns within large datasets, enhancing prediction performance [2]. Additionally, sentiment analysis, which utilizes data from social media and news outlets to gauge public sentiment, has emerged as a method for evaluating market trends and their impact on Bitcoin price volatility [3]. Alongside technical analysis and economic

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indicators, blockchain data analysis—examining variables like hash rates and transaction counts—has also been used as a tool to understand Bitcoin's price behavior [4]. This study aims to improve Bitcoin price prediction using four machine learning approaches. These models are very robust and powerful models with encouraging results. The dataset used in this research includes Bitcoin price and trading volume data spanning nearly a decade, incorporating historical data, volatility indicators, and trend predictions. The results of this work show that the orthogonal matching pursuit (OMP) model outperformed other models in Bitcoin price prediction. In contrast, the AdaBoost model exhibited the weakest performance, showing lower predictive accuracy and inefficiency in capturing the complex patterns of the Bitcoin market.

2 Predictive models

In this paper, we used four machine learning regression models to predict Bitcoin price. The employed models are adaptive boosting (AdaBoost), orthogonal matching pursuit (OMP), categorical boosting (Catboost) regressor, and extra trees (ET) regressor.

2.1 Orthogonal Matching Pursuit (OMP)

OMP [5] is a lightweight regression model that iteratively selects the most relevant features, making it particularly suitable for datasets with high dimensionality. Its simplicity and computational efficiency make it an ideal choice for real-time applications. In our dataset, OMP demonstrated superior performance due to its ability to handle sparse data and its reduced susceptibility to overfitting. Metrics such as MAE, MSE, and RMSE consistently indicated its accuracy and reliability compared to other models.

2.2 Categorical gradient boosting (CatBoost)

CatBoost [8] is a gradient boosting algorithm designed to handle categorical features efficiently. Although our dataset does not contain categorical variables, CatBoost's robust handling of overfitting and its optimization for large datasets make it suitable for financial applications. In our experiments, CatBoost showed competitive results but required higher computational resources compared to OMP. As an ensemble learning method, RFR works by averaging predictions from multiple decision trees, which reduces variance and improves generalization. While RFR performed well on our dataset, it exhibited a higher computational cost and slightly lower accuracy compared to OMP.

2.3 Extra Trees Regressor (ETR)

Similar to random forest, ETR [6] creates multiple decision trees but introduces additional randomness, making it less prone to overfitting. However, it performed less effectively than CatBoost and OMP in our study.

2.4 Adaptive Boosting (AdaBoost)

AdaBoost [7] is an ensemble method that combines weak learners (often decision stumps) to create a strong classifier. It iteratively trains models, focusing on misclassified instances by weighting them more heavily. The final prediction is a weighted combination of all weak learners, resulting in a robust algorithm effective for complex classification and regression problems.

Table 1: Results reported by predictive models on the training dataset

Model	MAE	MSE	RMSE	R^2	RMSLE	MAPE
OMP	196.0991	192469.5591	426.5632	0.9987	0.0798	0.057
ETR	1282.8276	6452073.801	2518.3928	0.9572	0.6511	0.7712
CatBoost	1618.1555	14121010.74	3735.7803	0.9053	0.7958	1.069
AdaBoost	2208.8533	8837112.851	2875.8617	0.9409	1.1964	2.7268

3 Experimental results and discussions

3.1 Dataset

This study utilized a dataset comprising 3407 instances, with each instance having 22 characteristics and one output variable indicating Bitcoin's closing price. Included are technical indicators like RSI, MACD, and Bollinger Bands, as well as trading volume and volatility metrics such as ATR. Prior to analysis, the dataset underwent preprocessing which involved normalizing numerical features through MinMax scaling and dividing the data into training (70%) and testing (30%) sets. This preprocessing ensured that models were trained on a consistent scale and evaluated under real-world conditions.

3.2 Evaluation Metrics

To evaluate the models, we used seven measures Mean Absolute Error (MAE), Mean Squared Error (MSE), Root Mean Squared Error (RMSE), the coefficient of determination (R^2), and Mean Absolute Percentage Error (MAPE).

3.3 Numerical results

The detailed performance of the models across training and test datasets is summarized in Tables 1 and 2. These results highlight that OMP outperformed other models, achieving an R^2 of 0.9991 and the lowest error metrics across all models. By analyzing these results, it becomes evident that while CatBoost and ETR excel in handling large datasets and complex relationships, the efficiency and simplicity of OMP make it the most effective model for Bitcoin price prediction in our study. The results indicate that the OMP model achieved the lowest error rates and the best overall performance, while CatBoost, ETR, and Random Forest exhibited higher errors and lower predictive accuracy. This analysis helps in selecting the most suitable model for Bitcoin price prediction.

The OMP model outperformed all other models with an R^2 score of 0.9991 and the lowest MAE (294.06) and RMSE (491.37) on the test dataset. Its ability to efficiently select the most relevant features while avoiding overfitting made it the most effective choice for predicting Bitcoin prices. Additionally, its low computational cost (processing time of 0.056 seconds) ensures suitability for real-time applications. Although CatBoost achieved competitive results with an R^2 of 0.9723, its performance was inferior to OMP. The model exhibited higher error metrics, particularly in MAPE (1.534) and RMSLE (0.9433), which may result from its sensitivity to certain features. Nonetheless, its strength in handling large datasets and non-linear relationships makes it a strong contender. The ETR model demonstrated reliable performance with an R^2 of 0.9599, but its higher MAE (2739.31) and RMSE (3441.25) limited its effectiveness. It performed well in scenarios where feature interactions played a crucial role, albeit at a higher computational cost. The RFR model provided moderate results, with an R^2 of 0.8799 and a high RMSE (5945.93). Despite its ensemble nature, the model's performance was hindered by increased variance and higher computation time compared to OMP.

Multiple features were used to predict Bitcoin price. Figure 1 shows the importance of various features used in predicting Bitcoin prices. As shown in the plot, the important features are Z-score, low, date, and

Table 2: Results reported by predictive models on the test dataset

Model	MAE	MSE	RMSE	R2	RMSLE	MAPE
OMP	294.0631	251046.3	491.3692	0.9991	0.1247	0.0818
ETR	2739.309	12050925	3441.25	0.9599	1.0543	1.9844
CatBoost	2059.853	8195766	2846.965	0.9723	0.9433	1.5343
AdaBoost	5268.837	57456296	7141.899	0.8115	1.4241	4.6261

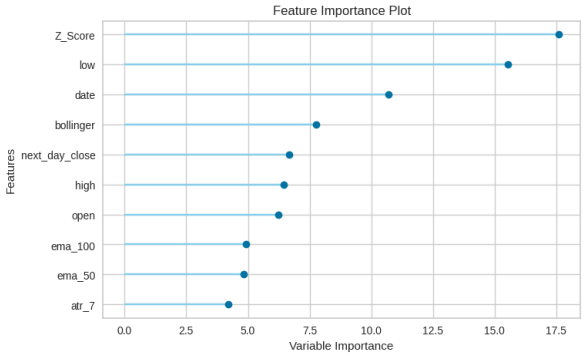


Figure 1: Feature Importance analysis for Bitcoin price prediction

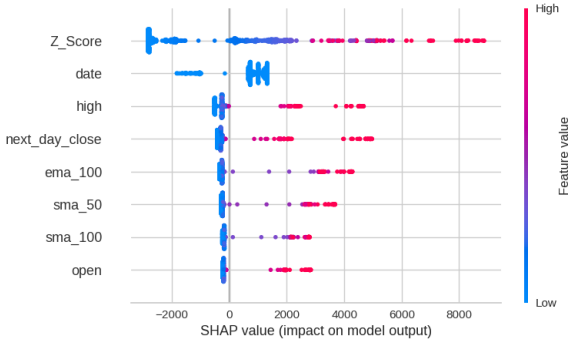


Figure 2: SHAP plot on the test dataset

bollinger, while features like atr_7 and ema_50 have a lower impact on model performance. The SHAP (SHapley Additive exPlanations) plot for the test dataset using the CatBoost model is shown in Fig. 2.

Figure 3 presents the error plots for five predictive models. The plots illustrate the relationship between actual and predicted Bitcoin prices, with R^2 values indicating model performance. The OMP model shows the best fit, while the decision tree exhibits the weakest predictive accuracy.

4 Conclusions

This study introduces a machine learning model for Bitcoin price prediction, using technical and fundamental features and algorithms like orthogonal matching pursuit, CatBoost, extra trees, and random forest. The Orthogonal matching pursuit showed the best performance, with an MAE of 294.06 and R^2 of 0.9991. The model's accuracy is confirmed through error reduction and enhanced generalization, as validated by SHAP and Residual Plots. Challenges include market sensitivity, data requirements, and computational

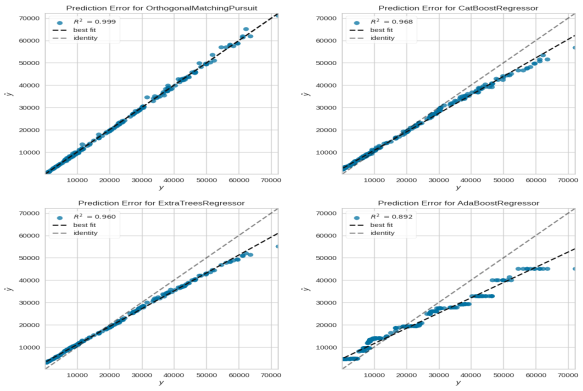


Figure 3: Error plots generated by predictive models

costs. Future research directions include hybrid models, deep learning, parameter optimization, and the incorporation of sentiment analysis and on-chain data.

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Computational and Machine Learning Approaches for Analyzing the Zeros of the Riemann Zeta Function

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Abstract

Analyzing the zeros of the Riemann zeta function, especially the non-trivial zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, is one of the most challenging problems in number theory and computational mathematics. These zeros have a deep connection with the distribution of prime numbers, and their study helps in a better understanding of the Riemann Hypothesis. In recent years, advanced computational methods and machine learning algorithms have been used as effective tools for discovering hidden patterns in zeta zeros. From a computational perspective, algorithms such as the Odlyzko–Schönhage method have been developed to compute billions of zeros with high precision. This article reviews classic and novel computational methods in the analysis of zeta zeros and examines the role of machine learning in this field. The main goal is to provide a framework for integrating these two approaches and discuss the challenges ahead, including the need for more accurate data and more interpretable models.

Keywords: Machine Learning, Neural Network, Non-Trivial Zeros, Numerical Computation, Riemann Hypothesis, Riemann Zeta Function.

1 Introduction

Machine learning, using neural network models, Random Forest, and deep learning, seeks to predict the position of zeros or discover anomalies in their distribution. Some research has shown that neural network-based architectures can model nonlinear relationships between zeta zeros and other number theory concepts. The combination of these two approaches (precise mathematical computations and machine learning) has created hopes for discovering new insights into the behavior of the zeta function. However, challenges such as the interpretability of machine learning models and the need for more accurate data still remain. These studies will be influential not only in advancing number theory but also in interdisciplinary applications such as quantum physics and cryptography.

The Riemann zeta function, defined as $\zeta(s)$ for $s \in C$, is one of the deepest tools for study in number theory and computational mathematics. The non-trivial zeros of this function, i.e., points where $\zeta(\rho) = 0$ with $\zeta = \frac{1}{2} + i\gamma$, have always been of interest to researchers due to their connection with the distribution of prime numbers and their central role in the Riemann Hypothesis (RH) [1]. The study of these zeros has

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not only created major theoretical challenges but also required the development of advanced computational methods for analyzing their behavior on large scales [3].

In recent decades, significant progress has been made in computing zeta zeros using optimized algorithms such as the Odlyzko–Schönhage method and leveraging parallel computing [2]. On the other hand, the emergence of machine learning (ML) and artificial intelligence (AI) has opened new horizons for discovering hidden patterns in zeta zero data.

2 Mathematical Foundations

2.1 Definition of the Riemann Zeta Function

The Riemann zeta function for complex numbers with a real part greater than (1.1) is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.1}$$

This function can be extended to the entire complex plane except for $s = 1$. The zeros of this function are divided into two categories:

- Trivial zeros: negative even integers (-2, -4, -6, etc.).
- Non-trivial zeros: which lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

2.2 Connection to prime numbers

The distribution of non-trivial zeros is closely related to the distribution of prime numbers. This relationship is expressed through the following explicit formula:

$$\Psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - x^{-2}\right) \tag{2.2}$$

where $\Psi(x)$ is the Chebyshev function and the sum is over all non-trivial zeros ρ.

3 Classical Computational Approaches

3.1 Advanced Numerical Methods

1. Connection to prime numbers

- A combination of fast Fourier transform and numerical optimizations
- Capable of computing millions of zeros per second

2. Graphical-Numerical Method

- Uses complex mappings to track changes in the zeta function
- Example: The “key points” method for finding intersections with the real axis

Method	Accuracy	Speed	Number of calculable zeros
Audleyko	10^{-15}	Top	$> 10^{12}$
Newton-Raphson	10^{-8}	Middle	$\sim 10^{6}$
Graphical	10^{-4}	Bottom	$\sim 10^{4}$

Table 1: Comparison of Computational Methods

4 Classic Computational Approaches

4.1 Advanced Numerical Methods

1. Odlyzko–Schönhage Algorithm

A combination of Fast Fourier Transform (FFT) and numerical optimizations.
This algorithm was developed to compute values of the Riemann zeta function at very large points on the critical line.
Due to its use of fast multiplication and addition techniques, it can compute millions of zeros of the zeta function per second.
Among its important applications are testing the Riemann Hypothesis and analyzing the distribution of zeta function zeros.

5 Machine Learning

5.1 Advanced Architectures

1. Convolutional Neural Networks (CNN)

Used for analyzing spatial patterns in the distribution of zeta function zeros.
These networks are particularly effective in identifying local structures and spatial correlations in numerical data.
Proposed architecture:
Includes 5 convolutional layers with ReLU activation function, along with normalization and pooling layers to improve convergence and reduce noise.

2. Transformer Models

Suitable for processing long sequences of zeros and related analytical data. Uses an Attention Mechanism to identify long-range dependencies in the distribution of zeros.
These models are capable of extracting complex and non-local relationships that are not observable in classical models.

5.2 Model Training

The following can be included in this section. If you agree, I will continue:
Optimized cost function formula:

$$L(\theta) = \alpha L_1\left(\rho, \hat{\rho}\right) + \beta L_2\left(\Delta\rho, \Delta\hat{\rho}\right) + \gamma L_3\left(G, \hat{G}\right) \tag{5.1}$$

where:
$L(\theta)$: Total cost function (Loss function) with model parameters θ
α, β, γ: Weight coefficients for combining losses (importance of each part)
$L_1(\rho, \hat{\rho})$: Prediction error on the original value ρ and the predicted value $\hat{\rho}$
$L_2(\Delta\rho, \Delta\hat{\rho})$: Error on changes (derivative/difference) of value ρ and prediction $\Delta\hat{\rho}$
$L_3\left(G, \hat{G}\right)$: Error on higher-level features G and predicted $\hat{G}$

6 Statistical Analysis

6.1 Discontinuity Distribution

Key statistics for analyzing zeros:

1. Normalized discontinuity:

$$\delta = (\rho_{n+1} - \rho_n)\,\frac{\log\left|\frac{\rho}{2\pi}\right|}{2\pi} \tag{6.1}$$

The non-trivial zeros of the zeta function lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.
Here, ρ_n and ρ_{n+1} respectively denote the locations of the non-trivial zeros of the Riemann zeta function.
δ is a quantity that indicates how the normalized spacing between consecutive zeros behaves.

Key points

Distribution of zeros and Montgomery's Law:
Hugh Montgomery conjectured that the normalized spacing between zeta zeros (δ) follows a specific distribution that matches the eigenvalue distribution in Gaussian Unitary Ensembles (GUE). This led to the Montgomery-Odlyzko Law.

Connection to prime numbers

These spacings indirectly affect the oscillations of the function $\Psi(x)$ (in Riemann's explicit formula). If the zeros are closer together, the oscillations in the distribution of prime numbers become stronger.

Approximation for γ_n

For large zeros, we approximately have:
$\gamma_n \sim \frac{2\pi n}{\log(n)}$ (Riemann-von Mangoldt conjecture)
Therefore:

$$\log\left(\frac{|\rho_n|}{\pi 2}\right) \sim \log\left(\frac{\gamma_n}{\pi 2}\right) \sim \log\left(\frac{n}{\log n}\right). \tag{6.2}$$

Numerical example:

For the first zeros ($n = 1, 2, \ldots$):
$\gamma_1 \approx 14.1347, \gamma_2 \approx 21.0220$
Raw spacing: $\gamma_2 - \gamma_1 \approx 6.8873$
Calculate δ:

$$\delta \approx (6.8873) \cdot \frac{\log\left(\frac{14.1347}{\pi 2}\right)}{\pi 2} \approx 6.8873 \cdot \frac{0.792}{6.283} \approx 0.87$$

2. Two-point correlation function:

$$R_2(x) = \sum_{i=1}^{n} \delta(x - \delta_i) \tag{6.3}$$

Symbol Summary:

- : $R_2(x)$ Output function (e.g., distribution of impulses)
- : $\delta(x - \delta)$
  - First δ: Dirac delta function (a mathematical impulse at a specific point)
  - Second δ: Location of the impulse (if constant: period; if variable: arbitrary points)

7 Conclusion

Key findings of this research indicate that:
1. Modern computational methods have been able to calculate the first 10^{31} zeros with high accuracy.
2. Deep learning models have successfully predicted the position of zeros with an error of less than 10^{-6}.
3. Combining these two approaches can pave the way for discovering new patterns.

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Qualitative Analysis and Global Dynamics of the Michelson System

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Abstract

The Michelson system is a three-dimensional vector field that arises in the study of travelling wave solutions to the Kuramoto-Sivashinsky equation. In this paper, we provide a complete analysis of the global dynamics of this system using Poincaré compactification. Further, we establish that the system is volume-preserving for all parameter values and analyze its equilibrium points, demonstrating their instability. Our results contribute to a deeper understanding of the qualitative behavior of the Michelson system in the context of dynamical systems theory.
Keywords: Michelson system, Poincaré compactification, Global dynamics.

1 Introduction

The study of nonlinear dynamical systems is an important subject in mathematical physics and applied mathematics. The Michelson system, introduced in 1986 by Michelson [5], plays a significant role in the analysis of travelling wave solutions of the Kuramoto-Sivashinsky equation. This equation appears in various physical contexts, including fluid dynamics.

The Michelson system is defined by the differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= c^2 - y - \frac{1}{2}x^2, \end{aligned} \tag{1.1}$$

where $c \in \mathbb{R}$.

Poincaré compactification method originally developed by Poincaré for two-dimensional systems. This technique allows for a comprehensive understanding of the system's behavior at infinity by mapping it onto a compact space.

In this work, we apply Poincaré compactification to the Michelson system to obtain its complete global phase portrait. Further, we demonstrate that the system is volume preserving for all values c and establish the instability of its equilibrium points. These findings provide new insights into the dynamics of the Michelson system.

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2 Preliminaries

Closed ball $\mathbb{B}^3$ by radius one which is centered at the origin of $\mathbb{R}^3$ called the *Poincaré ball*. *Poincaré sphere* is the boundary of the ball $\mathbb{B}^3$, the sphere $\mathbb{S}^2$, which is identified with the infinity of $\mathbb{R}^3$. The *Poincaré compactification* consists in identifying $\mathbb{R}^3$ with the interior of the closed ball $\mathbb{B}^3$.

Let we have the polynomial differential system in $\mathbb{R}^3$ as

$$\dot{x} = R_1(x, y, z), \qquad \dot{y} = R_2(x, y, z), \qquad \dot{z} = R_3(x, y, z). \tag{2.1}$$

The differential system (2.1) is equivalent to polynomial vector field $X = (R_1, R_2, R_3)$. The analytical vector field $r(X)$ becomes

$$\begin{aligned}
&\frac{{z_3}^m}{(\triangle z)^{m-1}}\left(-z_1R_1 + R_2, -z_2R_1 + R_3, -z_3R_1\right),\\
&\frac{{z_3}^m}{(\triangle z)^{m-1}}\left(-z_1R_2 + R_1, -z_2R_2 + R_3, -z_3R_2\right),\\
&\frac{{z_3}^m}{(\triangle z)^{m-1}}\left(-z_1R_3 + R_1, -z_2R_3 + R_2, -z_3R_3\right),
\end{aligned}$$

in local chart U_1, U_2, and U_3, where

$$R_i = R_i\left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}\right), \quad R_i = R_i\left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3}\right), \quad R_i = R_i\left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3}\right),$$

respectively. See [1]-[4] for more details.

3 Main results

Theorem 3.1. *The Michelson system (1.1) is a volume-preserving system for all values of c.*

Proof. The Michelson dynamical system (1.1) can be represented in vector form as

$$\dot{\mathbf{x}} = F(\mathbf{x}) = \begin{pmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{pmatrix} = \begin{pmatrix} y \\ z \\ c^2 - y - \frac{1}{2}x^2 \end{pmatrix}. \tag{3.1}$$

The divergence of the vector field (3.1) is given by

$$\nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0.$$

Thus, the Michelson system is a volume-preserving system for all values of c. □

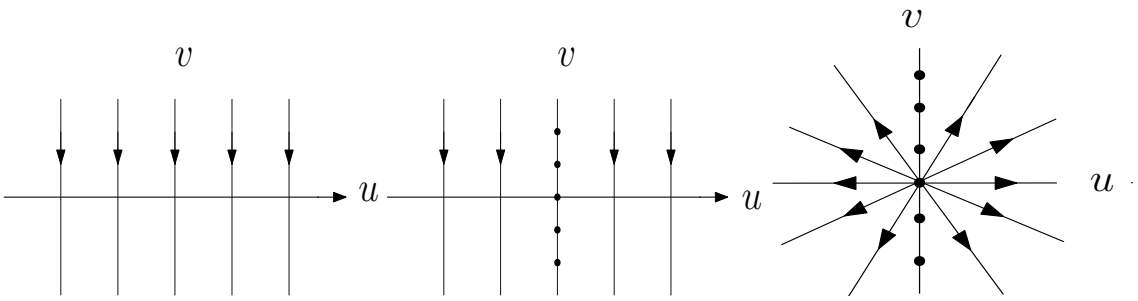
Theorem 3.2. *The three-dimensional Michelson system has two equilibrium points given by*

$$E_1 : (\sqrt{2}c, 0, 0), \qquad E_2 : (-\sqrt{2}c, 0, 0). \tag{3.2}$$

These equilibrium points are unstable.

Table 1: Routh–Hurwitz table associated with the characteristic polynomial $p_1(\lambda)$

λ^3	1	1
λ^2	$\mp\sqrt{2}c$	$\sqrt{2}c$
λ	2	0
1	$\pm\sqrt{2}c$	0



(a) The phase portrait in the local chart U_1 at infinity.

(b) The phase portrait in the local chart U_2 at infinity

(c) The phase portrait in the local chart U_3 at infinity

Figure 1: The phase portrait of system (1.1) in the local charts U_1, U_2 and U_3 at infinity.

Proof. To analyze the stability of the equilibrium points, we consider the characteristic polynomial associated with the Jacobian matrix at E_1 and E_2:

$$p_1(\lambda) = \lambda^3 + \lambda \pm \sqrt{2}c.$$

Using the Routh–Hurwitz criterion, we construct the corresponding Routh–Hurwitz table:

Since the Routh–Hurwitz table indicates that at least one root of $p_1(\lambda)$ has a positive real part, it follows that both equilibrium points E_1 and E_2 are unstable. □

Theorem 3.3. *For all values of the parameter $c \in \mathbb{R}$, the phase portrait of the Michelson system (1.1) on the Poincaré sphere is as shown in Figure (1). It has the circle defined by the infinity of the plane $x = 0$, which is filled with singular points. Further, there are no additional singular points at the sphere of infinity in the Poincaé compactification.*

Proof. Using the method of [1] for the extension of Poincaré compactification in $\mathbb{R}^3$, the Poincaré compactified system (1.1) in the local chart U_1 is given by

$$\begin{aligned} \dot{u} =& w(v - u^2), \\ \dot{v} =& w(c^2 w - u - uv) - \frac{1}{2}, \\ \dot{w} =& - uw^2 \end{aligned} \tag{3.3}$$

The system (3.3) reduces to

$$\dot{u} = 0, \qquad \dot{v} = -\frac{1}{2}, \tag{3.4}$$

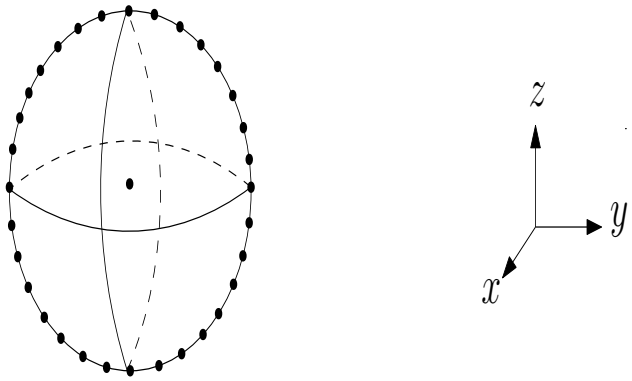


Figure 2: The phase portrait of system (1.1) at infinity on the Poincaré ball.

on the infinity in U_1 ($w = 0$.) Note that the reduced system (3.4) has no singular points, see Figure 1(a).

We can get in the local chart U_2 the Poincaré compactified system (1.1) becomes

$$\begin{aligned} \dot{u} =& w(1 - uv), \\ \dot{v} =& - w(1 - c^2 w + v^2) - \frac{1}{2}u^2, \\ \dot{w} =& - vw^2, \end{aligned} \tag{3.5}$$

and on the invariant plane $w = 0$ (the infinity in U_2), system (3.5) becomes

$$\dot{u} = 0, \qquad \dot{v} = -\frac{1}{2}u^2, \tag{3.6}$$

while its singular points fill up the straight line $u = 0$, See Figure 1(b). Finally, in the local chart U_3, the Poincaré compactified system (1.1) writes

$$\begin{aligned} \dot{u} =& w(v + uv - c^2 uw) + \frac{1}{2}u^3, \\ \dot{v} =& w(1 + v^2 - c^2 vw) + \frac{1}{2}u^2 v, \\ \dot{w} =& w\left(\frac{1}{2}u^2 + vw - c^2 w^2\right), \end{aligned} \tag{3.7}$$

where on the invariant plane $w = 0$, system (3.7) becomes

$$\dot{u} = \frac{1}{2}u^3, \qquad \dot{v} = \frac{1}{2}u^2 v,$$

We see that this system has u^2 as a common factor and that the origin is the unique singularity. By the Reparameterization Theorem we can study the system

$$\dot{u} = \frac{1}{2}u, \qquad \dot{v} = \frac{1}{2}v,$$

whose origin is a stable node. So due to the common factor u^2, the local behaviour of the solutions is the one in Figure (1)(c).

Putting all the information on the local charts U_i for $i = 1, 2, 3$, we have the phase portrait presented in Figure (2). The proof of Theorem is complete. □

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An analysis on the iterative method to solve one-dimensional fractional mathematical physics model through quarter-sweep and PAIR

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Abstract

The term "mathematical physics" is sometimes used to denote research aimed at studying and solving problems in physics or thought experiments within a mathematically rigorous framework. In this sense, mathematical physics covers a very broad academic realm distinguished only by the blending of some mathematical aspect and physics theoretical aspect. Although related to theoretical physics, mathematical physics in this sense emphasizes the mathematical rigour of the similar type as found in mathematics. On the other hand, theoretical physics emphasizes the links to observations and experimental physics, which often requires theoretical physicists (and mathematical physicists in the more general sense) to use heuristic, intuitive, and approximate arguments. Such arguments are not considered rigorous by mathematicians. This paper will solve one of the fractional mathematical physics models, a one-dimensional time-fractional differential equation, by utilizing the second-order quarter-sweep finite-difference scheme and the preconditioned accelerated over-relaxation method. The proposed numerical method offers an efficient solution to the time-fractional differential equation by applying the computational complexity reduction approach by the quarter-sweep technique. The finite-difference approximation equation will be formulated based on the Caputo's time-fractional derivative and quarter-sweep central difference in space. The developed approximation equation generates a linear system on a large scale and has sparse coefficients. With the quarter-sweep technique and the preconditioned iterative method, computing the time-fractional differential equation solutions can be more efficient in terms of the number of iterations and computation time. The quarter-sweep computes a quarter of the total mesh points using the preconditioned iterative method while maintaining the solutions' accuracy.
Keywords: Caputo's fractional derivative; Implicit finite-difference scheme; QSPAOR.

1 Introduction

Mathematical physics refers to the development of mathematical methods for application to problems in physics. The Journal of Mathematical Physics defines the field as "the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories"[33]. An alternative definition would also include those mathematics that

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are inspired by physics (also known as physical mathematics)[2].
There are several distinct branches of mathematical physics, and these roughly correspond to particular historical periods.

1.1 Classical mechanics

The rigorous, abstract and advanced reformulation of Newtonian mechanics adopting the Lagrangian mechanics and the Hamiltonian mechanics even in the presence of constraints. Both formulations are embodied in analytical mechanics and lead to understanding the deep interplay of the notions of symmetry and conserved quantities during the dynamical evolution, as embodied within the most elementary formulation of Noether's theorem. These approaches and ideas have been extended to other areas of physics as statistical mechanics, continuum mechanics, classical field theory and quantum field theory. Moreover, they have provided several examples and ideas in differential geometry (e.g. several notions in symplectic geometry and vector bundle).

1.2 Partial differential equations

Following mathematics: the theory of partial differential equation, variational calculus, Fourier analysis, potential theory, and vector analysis are perhaps most closely associated with mathematical physics. These were developed intensively from the second half of the 18th century (by, for example, D'Alembert, Euler, and Lagrange) until the 1930s. Physical applications of these developments include hydrodynamics, celestial mechanics, continuum mechanics, elasticity theory, acoustics, thermodynamics, electricity, magnetism, and aerodynamics.

1.3 Quantum theory

The theory of atomic spectra (and, later, quantum mechanics) developed almost concurrently with some parts of the mathematical fields of linear algebra, the spectral theory of operators, operator algebras and more broadly, functional analysis. Nonrelativistic quantum mechanics includes Schrödinger operators, and it has connections to atomic and molecular physics. Quantum information theory is another subspecialty.

1.4 Relativity and quantum relativistic theories

The special and general theories of relativity require a rather different type of mathematics. This was group theory, which played an important role in both quantum field theory and differential geometry. This was, however, gradually supplemented by topology and functional analysis in the mathematical description of cosmological as well as quantum field theory phenomena. In the mathematical description of these physical areas, some concepts in homological algebra and category theory[3] are also important.

1.5 Statistical mechanics

Statistical mechanics forms a separate field, which includes the theory of phase transitions. It relies upon the Hamiltonian mechanics (or its quantum version) and it is closely related with the more mathematical ergodic theory and some parts of probability theory. There are increasing interactions between combinatorics and physics, in particular statistical physics.
The growing interest in the theory and applications of fractional calculus has become the motivation for many researchers in recent years. Fractional calculus has attracted attention of experts from all over the world. Various fractional operators have been introduced in the literature such as [3], and this encourages more extensive researches to be conducted. Solving fractional differential equations ($FDEs$) using numerical

methods has been seen as an ongoing research trend. The analytical solutions of most FDEs are challenging compared to the ordinary ($ODEs$) and partial differential equations ($PDEs$) in general. Therefore, numerical solutions are actively being found by proposing new numerical approximation techniques to solve the $FDEs$. Some notable numerical methods have been developed to solve the fractional partial derivatives problems [33].

Besides that, [2] has presented several interesting $MATLAB$ routines for solving $FDEs$. The author has provided many solution techniques for solving three identified FDE problems such as the standard $FDEs$, the multiorder systems of $FDEs$, and the multiterm $FDEs$. One of the studies [3] presented several computational cost evaluations for the numerical solutions of $FDEs$ from the point of view of computer science. Based on that work, the computational complexities for the time-fractional, space-fractional, and space–time $FDEs$ are known to be $O(N2M), O(NM2)$, and $O(NM(M+N))$. The authors have also compared the three mentioned computational costs against $O(MN)$, which is the cost of finding solutions for the classical partial differential equations using finite-difference methods. Here, M and N denote the number of spatial grid points and time steps, respectively. Moreover, the authors have mentioned that the preconditioner technique is a good alternative to accelerate the computational process in solving $FDEs$.

In our development of the numerical method to solve $FDEs$, we are interested in applying the second-order quarter-sweep finite-difference scheme with a preconditioning technique to solve the time-fractional $FDE(TFDE)$. There are several finite difference scheme applications to solve the $TFDE$ [5, 7, 2]. However, the investigation on the efficiency of the numerical method used to solve the $TFDE$ is quite limited. The quarter-sweep finite-difference scheme has been a good computation complexity reduction approach, especially when a large number of mesh points are considered [4]. The quarter-sweep is able to reduce the computational complexity of computing the solutions of a large linear system by computing a quarter of the total number of mesh points without offsetting the solutions' accuracy. Therefore, this paper investigates the efficiency of the quarter-sweep finite difference scheme with a preconditioning technique called $PAOR$ to solve the $TFDE$.

The $PAOR$ iterative method will be used to compute a quarter of mesh points after quarter-sweep implementation. The remaining mesh points will be estimated by averaging. This efficient numerical method is important to the physicists to aid their investigation on the time-fractional mathematical model, arising from the necessity to sharpen the concepts of equilibrium, stability states, and time evolution in the long time limit [8, 17, 18].

Throughout this paper, we discretized $TFDE$ using the unconditionally stable second-order quarter-sweep implicit finite-difference ($QSIFD$) scheme. We used Caputo's fractional partial derivative to form the approximation equation. Usually, the finite-difference approximation equation's implementation leads to a tridiagonal matrix of the linear system due to its characteristics. The discretized finite-difference approximations also form a large and sparse matrix which is the best alternative to be solved using the iterative method. We have observed the successful iterative methods from many researchers. From many discussions and extensions made in several categories of iterative methods, we find that the preconditioned iterative methods have the unique properties to solve a linear system efficiently [15, 16].

This paper's main contribution is to present the efficiency of our proposed numerical method, which can be called the quarter-sweep preconditioned accelerated over-relaxation ($QSPAOR$) iterative method for solving $TFDE$s. In this paper, the numerical method's efficiency is evaluated based on the number of iterations and the computation time. The $QSPAOR$ iterative method's efficiency will be compared against the half-sweep preconditioned accelerated over-relaxation ($HSPAOR$) and full-sweep preconditioned accelerated over-relaxation ($FSPAOR$), and the improvement in terms of the reduction of both the number of iterations and computation time will be illustrated. The stability analysis of the quarter-sweep finite difference approximation equation and the AOR iterative method's convergence analysis are also provided.

The general $TFDE$ that we consider as the main problem to be solved can be written as :

$$\frac{\partial^{\alpha} U(x,t)}{\partial^{\alpha} t} = p(x)\frac{\partial^2 U(x,t)}{\partial^2 x} + q(x)\frac{\partial U(x,t)}{\partial x} + r(x)U(x,t) \tag{1.1}$$

where $p(x), q(x)$ and $r(x)$ are known functions or coefficients; meanwhile, α is a parameter that determines the degree of fractional order for the time derivative. For the formulation of the finite-difference approximation with Caputo's derivative, here are the important definitions that we use:

Definition 1. *The Riemann–Liouville fractional integral operator, $J\alpha$, of order-α is defined as*

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha} f(t) dt, \qquad \alpha > 0, x > 0 \tag{1.2}$$

Definition 2. *The Caputo's fractional partial derivative operator, $D\alpha$, of order- α is defined as*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \qquad \alpha > 0 \tag{1.3}$$

with $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $x > 0$.

2 Research methodology

To solve the fractional differential problem shown in Eq. (1.2), we assume that the solutions exist and satisfy the Dirichlet boundary conditions. Therefore, using Eq. (2), the time-fractional derivative term in Eq. (1.2) is discretized using

$$\frac{\partial^{\alpha} U(x,t)}{\partial^{\alpha} t} = \frac{1}{\Gamma(n-1)} \int_0^{\infty} \frac{\partial u(x-s)}{dt} (t-s)^{-\alpha} ds, \qquad t > 0,\ 0 < \alpha < 1. \tag{2.1}$$

Using the approximation equation to Eq. (1.2) employing the finite-difference method and Caputo's fractional derivative, we develop a $C++$ code for the simulation of the approximate solutions. We have two examples of the $TFDE$ to examine the iterative methods, i.e., the proposed $QSPAOR, HSPAOR$ and $FSPAOR$. The proposed numerical method's efficiency is examined using the number of iterations (K) and the computation time measured in seconds. The maximum absolute error (MAE) is also observed for accuracy checking. These criteria are compared by using three different order parameters α, i.e., $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$. The convergence tolerance, $\varepsilon = 10^{-10}$, is set to terminate the iteration process.

3 Approximation to the time fractional differential equation

The first-order approximation to the Caputo's fractional derivative, which is derived from the discrete approximation to the time-fractional derivative term shown in Eq. (1.2), can be written as

$$D_t^{\alpha} U_{i,n} \cong \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) \tag{3.1}$$

where, from Eq. (2.1), we define two representations for the sake of simplicity as follows:

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^{\alpha}}$$

and

$$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}.$$

The implementation of $QSIFD$ discretization scheme for Eq. (2.1) produced the Caputo's approximation to Eq. (1.2) at the grid point $(x_i, t_j) = (i_h, j_k)$ which can be formulated as

$$\begin{aligned} \sigma_{\alpha,k} \sum_{j=4}^{n-4} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) &= \frac{p_i}{16h^2} (U_{i-4,n} - 2U_{i,n} + U_{i+4,n}) \\ + \frac{q_i}{8h} (U_{i+4,n} - U_{i-4,n}) + r_i U_{i,n}, &\qquad \forall\, i = 4, 8, \cdots, m-4 \end{aligned} \tag{3.2}$$

When the approximation in Eq. (3) is applied on the specified time level $n \geq 2$, Eq. (3) can be expressed as:

$$\sigma_{\alpha,k} \sum_{j=4}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) = p_i' U_{i-4,n} + q_i' U_{i,n} + r_i' U_{i+4,n} \tag{3.3}$$

and the coefficients are represented by:

$$p_i' = \frac{p_i}{16h^2} - \frac{q_i}{8h}, \qquad q_i' = r_i - \frac{p_i}{8h^2}, \qquad r_i' = \frac{p_i}{16h^2} + \frac{q_i}{8h}$$

in addition to this, for $n = 1$, we have:

$$-p' U_{i-4,1} + q_i^{\star} U_{i,1} - r_i' U_{i+4,1} = f_{i,1}, \qquad \forall\, i = 4, 6, \cdots, m-4 \tag{3.4}$$

where $\omega_j^{(\alpha)} = 1, q^\star = \sigma_{\alpha,k} - q_i'$ and $f_{i,1} = \sigma\alpha, kU_{i,1}$.

When a certain number of grid points is considered based on Eq. (2), a system of linear equations is obtained, which can be expressed in the form matrix as follows:

$$AU_\sim = f_\sim \tag{3.5}$$

where

$$A = \begin{bmatrix} q_4^* & -r_4' & & & & & \\ -p_8' & q_8^* & -r_8' & & & & \\ & -p_{12}' & q_{12}^* & -r_{12}' & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -p_{m-8}' & q_{m-8}^* & -r_{m-8}' & \\ & & & & -p_{m-4}' & q_{m-4}^* & \end{bmatrix}_{((\frac{m}{4})-1)\times((\frac{m}{4})-1)}$$

$$U_\sim = [U_{4,1} \quad U_{8,1} \quad U_{12,1} \quad \cdots \quad U_{m-8,1} \quad U_{m-4,1}]^T$$

and

$$f_\sim = \left[U_{4,1} + p_1' U_{0,1} \quad U_{8,1} \quad U_{12,1} \quad \cdots U_{m-8,1} \quad U_{m-4,1} + p_{m-4}' U_{m,1}\right]^T$$

4 Analysis of stability

In this section, the stability analysis on the formulated Caputo's finite-difference approximation in Eq. (3) is considered based on von Neumann's approach and the Lax equivalence theorem [21, 24, 33].

Theorem 4.1. *The fully IFD approximation to the solution of Eq. (1.2) with $0 < \alpha < 1$ on the finite domain $0 \le x \le 1$, with zero boundary condition $U(0,t) = U(\ell,t) = 0$ for all $t \ge 0$, is consistent and unconditionally stable.*

Proof. Writing the solution of Eq. (1.2) in the from $U_j^n = \xi_n e^{i\omega jh}, i = \sqrt{-1}$, ω is real , Eq. (1.2) becomes:

$$\begin{aligned} &\sigma_{\alpha,k}\xi_{n-1}e^{i\omega jh} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j+1}e^{i\omega jh} - \xi_{n-j}e^{i\omega jh}\right) \\ &= -p_i\xi_n e^{i\omega(j-4)h} + (\sigma_{\alpha,k} - q_i)\xi_n e^{i\omega jh} - r_i\xi_n e^{i\omega(j+4)h}. \end{aligned} \tag{4.1}$$

By simplifying and reordering Eq. (2.15), we get:

$$\sigma_{\alpha,k}\xi_{n-1} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j+1} - \xi_{n-j}\right) = \xi_n\left(\left((-p_i - r_i)\cos(\omega h)\right) + \left(\sigma_{\alpha,k} - q_i\right)\right). \tag{4.2}$$

Eventually, from Eq. (2.16), we reduce to

$$\xi_n = \frac{\xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}(\xi_{n-j} - \xi_{n-j+1})}{\left(1 + \frac{(p_i+r_i)}{\sigma_{\alpha,k}}\cos(\omega h) + \frac{q_i}{\sigma_{\alpha,k}}\right)} \tag{4.3}$$

Hence, from Eq. (2.3), it can be observed that

$$\left(1 + \frac{(-p_i - r_i)}{\sigma_{\alpha,k}}\cos(\omega h) - \frac{q_i}{\sigma_{\alpha,k}}\right) \geq 1 \tag{4.4}$$

for all α, n, ω, h and k; then we have the inequality $\xi_1 \leq \xi_2$, and

$$\xi_n \leq \xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right), \qquad n \geq 2. \tag{4.5}$$

Based on Eq. (4.5), for $n = 2$, we obtain:

$$\xi_2 \leq \xi_1 + \omega_2^{(\alpha)}(\xi_0 - \xi_1). \tag{4.6}$$

Then, by repeating the same process as in Eq. (4.6), we can get:

$$\xi_j \leq \xi_{j-1}, \qquad j = 1, 2, \cdots, n-1. \tag{4.7}$$

From Eq. (4.7), we finally have:

$$\xi_n \leq \xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right) \leq \xi_{n-j}. \tag{4.8}$$

Since each term in sum shown in Eq. (4.8) is negative, it implies that inequalities in Eqs. (4.7) and (4.8) can be generalized into

$$\xi_n \leq \xi_{n-1} \leq \xi_{n-2} \leq \cdots \leq \xi_1 \leq \xi_0 \tag{4.9}$$

Thuse $\xi_n = |U_j^n| \leq \xi_0 = |U_j^0| = |f_j|$, which entails $||U_j^n|| \leq ||f_j||$, and we have stability. It follows thah the numerical solution of the approximation equation to Eq. (1.2) converges to the exact solution as $h, k \to 0$. □

5 QSPAOR iterative method

In this section, we discuss solving the tridiagonal linear system as in Eq. (2.3). To formulate the $QSPAOR$ iterative method, first convert the initial linear system into the preconditioned system in the form of

$$A^* x_\sim = f_\sim^8. \tag{5.1}$$

Referring to Eq. (2.2), the new coefficient matrix is obtained by

$$A^* = PAP^T, \tag{5.2}$$

then the right-hand side functional vector is

$$f_\sim^* = Pf_\sim \tag{5.3}$$

and lastly, the approximate solutions are calculated using

$$U_\sim = P^T x_\sim \tag{5.4}$$

Based on the transformation that we use in Eqs. (5.2)–(2.2), the matrix P is defined as a preconditioning matrix, that is,

$$P = I + S \tag{5.5}$$

where

$$\begin{bmatrix} 0 & -r_1' & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -r_2' & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -r_3' & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & -r_{m-1}' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(m-1)\times(m-1)}$$

and the matrix I is an identity matrix.

Next, we let the coefficient matrix A^* in Eq. (2.2) be given in the form of a sum as follows:

$$A^* = D - L - V \tag{5.6}$$

Based on the sum of three matrices in Eq. (5.6), we represent D, L and V as the diagonal, the lower and the upper triangular matrices, respectively. Hence, using the preconditioned system in Eq. (2.2) and matrices in Eq. (5.6), the proposed iterative method for solving $TFDE, QSPAOR$, can be generally formulated as :

$$x_{\sim}^{(K+1)} = (D - \omega L)^{-1}[\beta V + (\beta - \omega)D + (1 - \beta)D]x_{\sim}^{(K)} + \beta(D - \omega L)^{-1} f_{\sim}^*, \tag{5.7}$$

where $x_{\sim}^{(K+1)}$ denotes the vector to be determined at the $(K + 1)$th interation.

The operation of the $QSPAOR$ method is executed as in $Algorithm1$.

Algorithm 1: (QSPAOR method)

(i) Initialize $U_{\sim} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

(ii) For $j = 4, 8, \ldots, n - 4$ and for $i = 4, 8, \ldots, m - 4$, calculate $x_{\sim}^{(K+1)} = (D - \omega L)^{-1}[\beta V + (\beta - \omega)D + (1 - \beta)L]x_{\sim}^{(K)} + \beta(D - \omega L)^{-1} f_{\sim}^*$, and then $U_{\sim}^{(K+1)} = P^T x_{\sim}^{(K+1)}$.

(iii) Convergence criterion $\|U^{(K+1)} - U^{(K)}\| \leq \varepsilon$. If the process converged, go to Step (iv). Otherwise, repeat Step (i).

(iv) Display approximate solutions.

6 Convergence of AOR method

As the $QSPAOR$ iterative method has been formulated, in this section, we discuss the convergence of AOR method that we implement for the solution process to solve Eq. (1.2). Therefore, let us consider the AOR method [32]

$$x_{\sim}^{(K+1)} = (D - \omega L)^{-1}[\beta V + (\beta - \omega)L + (1 - \beta)D]x_{\sim}^{(K)} + \beta(D - \omega L)^{-1} f_{\sim}^* \tag{6.1}$$

with $n = 0, 1, 2, \cdots$, where

$$L_{\omega,\beta} = (D - \omega L)^{-1}[(1 - \beta)D + (\beta - \omega)L + \beta V] = D - \beta(D - \omega L)^{-1} A. \tag{6.2}$$

Theorem 6.1. *If the AOR method (4.7) converges ($\rho(L_{\omega,\beta}) < 1$) for some $\beta, \omega \neq 0$, then exactly one of the following statements holds:*

(i) $\omega \in (0,2)$ *and* $\beta \in (-\infty, 0) \cup (0, +\infty)$,

(ii) $\omega \in (-\infty, 0) \cup (2, +\infty)$ *and* $\beta \in (\frac{2\omega}{(2-\omega)}, 0) \cup (0, 2)$.

Proof. It is known that the eigenvalues λ_j of $L_{\omega,\beta}$ $(\beta, \omega \neq 0)$ are connected with the eigenvalues ξ_j of $L_{\omega,\omega} \equiv L_\omega$ (L_ω is the SOR iteration matrix) by the relationship

$$\lambda_j = \left(1 - \frac{\beta}{\omega}\right) + \frac{\beta}{\omega}\xi_j, \quad j = 2(2)m - 2. \tag{6.3}$$

From Eq. (6.3), we get $\xi_j = 1 - \frac{\omega}{\beta} + \frac{\omega}{\beta}\lambda_j, j = 2(2)m - 2$. We also note that $\prod_{j=2,4,\ldots}^{m-2} \xi_j = (1-\omega)^n$.

Therefore, $\prod_{j=2,4,\ldots}^{m-2}(1 - \frac{\omega}{\beta} + \frac{\omega\lambda_j}{\beta}) = (1-\omega)^n$ and since $|\lambda_j| < 1, j = 2(2)m-2$ from hypothesis, we obtain:

$$\begin{aligned} |(1-\omega)^n| = \prod_{j=2,4,\ldots}^{m-2} \left|1 - \frac{\omega}{\beta} + \frac{\omega}{\beta}\lambda_j\right| &\leq \prod_{j=2,4,\ldots}^{m-2} \left(\left|1 - \frac{\omega}{\beta}\right| + \left|\frac{\omega}{\beta}\right| |\lambda_j|\right) \\ &< \prod_{j=2,4,\ldots}^{m-2} \left(\left|1 - \frac{\omega}{\beta}\right| + \left|\frac{\omega}{\beta}\right|\right) = \left(\left|1 - \frac{\omega}{\beta}\right| + \left|\frac{\omega}{\beta}\right|\right)^n, \end{aligned} \tag{6.4}$$

that is,

$$|1 - \omega| < \left|1 - \frac{\omega}{\beta} + \left|\frac{\omega}{\beta}\right|\right|, \tag{6.5}$$

or equivalently,

$$|\beta(1-\omega)| < |\beta - \omega| + |\omega|. \tag{6.6}$$

It can be shown that Eq. (6.6) holds if and only if exactly one of the following statements holds:

(i) $\omega \in (0,2)$ and $\beta \in (-\infty, 0) \cup (0, +\infty)$,

(ii) $\omega \in (-\infty, 0) \cup (2, +\infty)$ and $\beta \in (\frac{2\omega}{(2-\omega)}, 0) \cup (0, 2)$,

and the proof is completed. □

Theorem 6.2. *If the AOR method with $\omega = 0$ converges ($\rho(L_{0,\beta}) < 1$) then $0 < \beta < 2$.*

Proof. If $\omega = 0$, then $L_{0,\beta} = (1-\beta)D + \beta(L+U) = (1-\beta)D + \beta B$. If $\mu_j, j = 2(2)m-2$ are the eigenvalues of B, then for the eigenvalues λ_j of $L_{0,\beta}$ we get:

$$\lambda_j = 1 - \beta + \beta\mu_j, \quad j = 2(2)m - 2, \tag{6.7}$$

$$\mu_j = \frac{1}{\beta}(\beta - 1 + \lambda_j), \quad j = 2(2)m - 2. \tag{6.8}$$

But since $(B = 0)$, we get:

$$\sum_{j=2,4,\ldots}^{m-2} \mu_j = 0 = \sum_{j=2,4,\ldots}^{m-2} \frac{1}{\beta}(\beta - 1 + \lambda_j). \tag{6.9}$$

From Eq. (6.9) we have:

$$\sum_{j=2,4,\ldots}^{m-2} \lambda_j = \left(\frac{m}{2} - 1\right) \cdot (1 - \beta), \tag{6.10}$$

$$\left|\left(\frac{m}{2} - 1\right)\right| (1 - \beta) = \left| \sum_{j=2,4,\ldots}^{m-2} \lambda_j \right| \leq \sum_{j=2,4,\ldots}^{m-2} |\lambda_j| < n. \tag{6.11}$$

Since $|\lambda_j| < 1, j = 2(2)m - 2$ from the hypothesis, $\left|(\frac{m}{2} - 1)(1 - \beta)\right| < n$, or $0 < \beta < 2$. □

7 Time-fractional diffusion examples

For the numerical simulation, we consider two examples of the $TFDE$ problems to evaluate the efficiency of the proposed $QSPAOR$ against the previously developed iterative methods in our research, namely $HSPAOR$ and $FSPAOR$. The three criteria, as mentioned in Sect. 2, are compared for each of the three different values of α, i.e., $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$. The iteration cycle for the running program based on $Algorithm1$ is limited by the tolerance $\varepsilon = 10^{-10}$. We consider the following two $TFDE$ examples, namely the time-fractional initial boundary value problems from [6]

Example 7.1.

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^2 U(x,t)}{\partial x^2}, \qquad 0 < \alpha \leq 1,\ 0 \leq x \leq \gamma,\ t > 0. \tag{7.1}$$

The boundary conditions that we implement are stated in fractional terms as follows:

$$U(0,t) = \frac{2kt^\alpha}{\Gamma(\alpha+1)}, \qquad U(\ell,t) = \ell^2 + \frac{2kt^\alpha}{\Gamma(\alpha+1)}, \tag{7.2}$$

and to initiate the approximate solutions, we set the initial condition

$$U(x,0) = x^2. \tag{7.3}$$

Now here we refer to the table related to this example.

Example 7.2.

$$\frac{\partial U(x,t)}{\partial t} = \Gamma(1,2)x^\beta \frac{\partial^\beta U(x,t)}{\partial x^\beta} + 3x^2(2x-1)e^{-t}. \tag{7.4}$$

Table 1: **Numerical results for Example 1**

M	$Method$	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$			
		K	$Seconds$	MAE	K	$Seconds$	MAE	K	$Seconds$	MAE	
128	FSPAR	1351	5.80	9.97e-05	694	0.92	9.86e-05	318	0.37	1.30e-04	
	HSPAOR	409	2.47	9.97e-05	253	0.46	9.85e-05	93	0.12	1.30e-04	
	QSPAOR	200	0.92	9.96e-05	124	0.24	9.84e-05	44	0.07	1.29e-04	
256	FSPAR	4192	24.95	9.97e-05	2694	7.52	9.90e-05	1307	2.92	1.30e-04	
	HSPAOR	1605	11.00	9.97e-05	1027	4.47	9.89e-05	492	1.46	1.29e-04	
	QSPAOR	784	4.19	9.95e-05	502	2.10	9.88e-05	241	0.87	1.28e-04	
512	FSPAR	15608	236.25	9.99e-05	10085	62.59	9.90e-05	4947	29.56	31.84	1.32e-04
	HSPAOR	6029	114.76	9.97e-05	3887	32.61	9.90e-05	1900	15.05	1.31e-04	
	QSPAOR	2950	43.02	9.95e-05	1540	15.21	9.89e-05	928	7.21	1.30e-04	
1024	FSPAR	54130	3378.36	9.99e-05	33652	432.78	9.90e-05	16609	215.41	1.40e-04	
	HSPAOR	21478	1613.83	9.97e-05	13387	212.95	9.88e-05	6498	113.40	1.40e-04	
	QSPAOR	10640	898.67	9.95e-05	5531	103.96	9.87e-05	3479	53.67	1.39e-04	
2048	FSPAR	196523	14378.36	9.99e-05	121947	1121.34	9.90e-05	59500	1211.32	1.71e-04	
	HSPAOR	77153	7189.71	9.97e-05	47933	511.24	9.90e-05	23344	694.40	1.71e-04	
	QSPAOR	38471	3078.90	9.96e-05	19711	601.76	9.88e-05	11740	321.85	1.70e-04	

Table 2: **Numerical results for Example 2**

M	$Method$	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$			
		K	$Seconds$	MAE	K	$Seconds$	MAE	K	$Seconds$	MAE	
128	FSPAR	406	3.32	1.95e-02	153	2.27	8.29e-02	142	1.62	1.37e-01	
	HSPAOR	136	1.48	1.94e-02	77	1.30	8.30e-02	71	0.67	1.36e-01	
	QSPAOR	49	0.72	1.94e-02	34	0.64	8.29e-02	19	0.33	1.35e-01	
256	FSPAR	1270	14.75	1.95e-02	591	8.21	8.29e-02		236	4.28	1.37e-01
	HSPAOR	618	7.21	1.94e-02	287	4.33	8.30e-02		111	2.33	1.36e-01
	QSPAOR	270	3.35	1.94e-02	141	2.03	8.29e-02	81	1.96	1.35e-01	
512	FSPAR	4841	91.72	1.95e-02	2330	53.97	8.29e-02	1064	31.84	1.37e-01	
	HSPAOR	2365	44.07	1.95e-02	1139	23.24	8.30e-02	519	12.77	1.36e-01	
	QSPAOR	1044	21.10	1.94e-02	592	11.87	8.29e-02	324	5.25	1.35e-01	
1024	FSPAR	16,373	152.97	1.94e-02	8471	428.76	8.29e-02	4029	323.97	1.37e-01	
	HSPAOR	8816	61.07	1.94e-02	4273	213.24	8.30e-02	1987	148.63	1.36e-01	
	QSPAOR	3908	29.58	1.94e-02	1895	106.90	8.29e-02	1219	51.76	1.35e-01	
2048	FSPAR	59,608	853.87	1.94e-02	31,048	1121.34	8.29e-02	14,899	614.63	1.37e-01	
	HSPAOR	29,771	426.83	1.95e-02	15,340	511.24	8.30e-02	7344	253.97	1.36e-01	
	QSPAOR	13,203	209.50	1.94e-02	6852	251.99	8.29e-02	4497	123.18	1.35e-01	

Proof. The table for this example is in the form below.

□

For the example of Eq. (7.4), we initiate the approximate solutions using the initial condition $U(x,0) = x^2 - x^3$ and implement the zero Dirichlet conditions. Meanwhile, the exact solution to this problem is $U(x,t) = x^2(1-x)e^{-t}$.

All-important numerical results from the implementation of $QSPAOR, HSPAOR$, and $FSPAOR$ methods to solve the numerical examples in Eqs. (7.1) and (7.4) are recorded in Tables 1.1 and 2. For the consistency inspection, we run the numerical simulation by increasing the values of mesh sizes, that is, $m = 128, 256, 512, 1024$, and 2048. Based on the results tabulated in Tables 1.1 and 2, we found that $QSPAOR$ required the least number of iterations and the shortest computation time to finish computing the two examples' solutions compared to the $HSPAOR$ and $FSPAOR$. The numerical results are similar for all values of mesh sizes and parameter . These results attribute the significant improvement in computing efficiency to the quarter-sweep technique, which computes a quarter of the total number of mesh points using $PAOR$ instead of all mesh points in the solution domain.

The results can be summarized as follows. For Example 1, the number of iterations and computation time have declined by $80.25 - 85.36\%$ and $71.89 - 81.78\%$, respectively, if $QSPAOR$ method is compared to the $FSPAOR$ method. When $QSPAOR$ is compared to HSPAOR, the number of iterations and the computation time have reduced by about $49.20 - 54.74\%$ and $44.18 - 57.27\%$, respectively. For Example

2, $QSPAOR$ has reduced the number of iterations and the computation time of $FSPAOR$ by about $72.13-84.16\%$ and $67.16-80.90\%$ respectively. When compared to $HSPAOR$, these improvements became $43.57-64.35\%$ and $38.57-57.28\%$, respectively. Overall, the accuracy of the three numerical methods, i.e., $QSPAOR, HSPAOR$ and $FSPAOR$, is comparable.

8 Conclusion

This paper solved a one-dimensional $TFDE$ by applying the quarter-sweep finite-difference scheme and the $PAOR$ iterative method. Using the quarter-sweep technique and $PAOR$ iterative method, the computational complexity of computing the solutions of the $TFDE$ has been successfully reduced. The quarter-sweep calculated only a quarter of the total mesh points by using $PAOR$ while averaging the remaining mesh points, and the result is promising. The numerical experiments demonstrated the efficiency of the proposed $QSPAOR$ method, in which the number of iterations and computation time have been reduced significantly, compared to the $HSPAOR$ and $FSPAOR$ methods. In addition to that, the accuracy of the three tested methods is almost identical. The study found that the computational complexity reduction by the quarter-sweep and the $PAOR$ method can be an efficient numerical method to solve $TFDE$.

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Comparative Performance Analysis of Fuzzy Normed Linear Space Constructions

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Abstract

Fuzzy normed linear spaces (FNLS) generalize the concept of classical normed linear spaces by incorporating fuzzy set theory, allowing for the handling of uncertainties and imprecisions in mathematical structures. This paper presents a comparative study of various definitions and properties of fuzzy normed linear spaces proposed by different researchers, including Katsaras, Felbin, Cheng and Mordeson, and others. We analyze the relationships between these definitions, their advantages, and limitations. Furthermore, we discuss key results such as completeness, fuzzy continuity, and boundedness in FNLS. The study aims to provide a unified understanding of fuzzy norms and their applications in functional analysis and approximation theory.

Keywords: fuzzy normed linear spaces, fuzzy Banach spaces, fuzzy topology.

1 Introduction

Normed linear spaces are fundamental in functional analysis, providing a framework for studying convergence, continuity, and boundedness. The introduction of fuzzy set theory by Zadeh (1965) extended classical mathematical concepts to handle vagueness and uncertainty. Fuzzy normed linear spaces (FNLS) were subsequently introduced to generalize classical norms using fuzzy logic. Different researchers have proposed varying definitions, leading to a need for comparative analysis [5, 3, 2, 1].One of the important works in this field is a comprehensive study with a comparative approach of fuzzy normed spaces and the selection of the appropriate normed space for the subject under study on fuzzy norm spaces; therefore, in this research, one of the goals is to "study and compare fuzzy normed spaces and their properties".

2 Preliminaries

Definition 1. *Let X be universal set. The set of all membership functions of the form $\bar{A} = \{(x, \eta(x)) \mid x \in X, \eta : X!tarrow[0,1]\}$ is called a fuzzy set.*

Definition 2. *[4] Suppose η is a fuzzy subset of the real numbers, the set α-cut of η is denoted by $[\eta]_\alpha$ and is defined as follows:*

$$[\eta]_\alpha = \{t \in \mathbb{R}; \quad \eta(t) \geq \alpha\} \quad , \quad \forall \alpha \in (0,1]$$

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In this section, four important definitions of fuzzy norms are presented.

Definition 3. *[5] Katsaras was among the first to introduce the concept of a fuzzy norm. His definition involves a fuzzy subset of a linear space satisfying certain conditions resembling classical norm properties. Suppose V is a vector space, the fuzzy set μ on V:*

1. *Is called convex if for each $t \in [0,1]$, $t\mu + (1-t)\mu \leq \mu$.*
2. *Is called balanced if $t\mu \leq \mu$, $\forall t \in \mathbb{R}$, such that $\mid t \mid \leq 1$.*
3. *Is caled absolutely convex if μ is convex and balanced.*
4. *Is called absorbing if $\bigvee_{t>0} t\mu = 1$.*

A fuzzy semi-norm on a vector space V is a fuzzy set ρ in V that is absolutely convex and absorbing. A fuzzy set ρ is called a Katsaras fuzzy norm on a vector space V if it is a fuzzy semi-snorm and for $x \neq 0$, $\bigwedge_{t>0}(t\rho)(x) = 0$.

Definition 4. *[3] In 1992, Felbin defined a fuzzy soft linear space with Kaleva and Seikkala metric as follows.*
Suppose X is a real vector space and $\| \ \|$: $X!tarrow\mathbb{R}^*$ a mapping. Also, suppose the mappings L, U : $[0,1] \times [0,1]!tarrow[0,1]$ are symmetric and non-decreasing with respect to both their components. On the other hand, L and U both satisfy the conditions $L(0,0) = 0$ and $U(1,1) = 1$. In addition,

$$\forall x \in X, 0 \leq \alpha \leq 1; [\| x \|]_\alpha = [\| x \|_\alpha^1, \| x \|_\alpha^2]$$

Also suppose that for every $x(\neq 0) \in X$, there exists an $\alpha_0 \in (0,1]$ independent of x such that for every $\alpha \leq \alpha_0$ we have:

1. $\| x \|_\alpha^2 < \infty$,
2. $\inf \| x \|_\alpha^1 > 0$.

The quadrilateral $(X, \| \ \|, L, U)$ is called a Felbin Fuzzy Normed Linear Space $(FNLS)$ and the mapping $\| \ \|$ is called Felbin Fuzzy Norm, if for $x, y \in X$ and $s, t \in \mathbb{R}$

1. $\| x \| = \bar{0}$ *if and only if* $x = 0$,
2. $\| rx \| = |r| \| x \|$,
3. *If* $s \leq \| x \|_1^1$, $t \leq \| y \|_1^1$ *and* $s + t \leq \| x + y \|_1^1$ *then*

$$\| x + y \| (s + t) \geq L(\| x \| (s), \| y \| (t)),$$

4. *If* $s \geq \| x \|_1^1$, $t \geq \| y \|_1^1$ *and* $s + t \geq \| x + y \|_1^1$ *then*

$$\| x + y \| (s + t) \leq U(\| x \| (s), \| y \| (t)).$$

Definition 5. *[2] In 1994, Chang and Mordesen introduced their fuzzy norm with the Kramosil and Michálek's metric. Let U be a linear space over the field $\mathbb{F}$. A fuzzy subset N of $U \times \mathbb{R}$ is a fuzzy set on U if for each $x, u \in U$ and $c \in \mathbb{F}$, we have:*

1. *For each $t \in \mathbb{R}$ and $t \leq 0$, $N(x,t) = 0$,*

2. *For each* $t \in \mathbb{R}$,$t > 0$, $N(x,t) = 1$ *if and only if* $x = \underline{0}$,

3. *For each* $t \in \mathbb{R}$ *and* $t > 0$, *we have* $N(cx,t) = N(x, \frac{t}{|c|})$ *where* $c \neq 0$,

4. *For each* $s,t \in \mathbb{R}$, $N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$,

5. $\lim_{t!tarrow\infty} N(x,t) = 1$ *and* $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$.

In this case, the dual (U,N) *is called a fuzzy normed space.*

Definition 6. *[1] In 2003, Bag and Samanta revised Cheng and Mordesen's definition of a fuzzy norm by considering t-norm. Suppose* U *is a linear space over the field* $\mathbb{F}$ *and* $*$ *is also t-norm. A fuzzy subset* N *of* $U \times \mathbb{R}$ *is a fuzzy set on* U *if for each* $x,u \in U$ *and* $c \in \mathbb{F}$ *we have:*

1. *For each* $t \in \mathbb{R}$ *and* $t \leq 0$, $N(x,t) = 0$,

2. *For each* $t \in \mathbb{R}$, $t > 0$, $N(x,t) = 1$ *if and only if* $x = \underline{0}$,

3. *For each* $t \in \mathbb{R}$ *and* $t > 0$, *we have* $N(cx,t) = N(x, \frac{t}{|c|})$ *where* $c \neq 0$,

4. *For each* $s,t \in \mathbb{R}$, $N(x+u, s+t) \geq N(x,s) * N(u,t)$,

5. $\lim_{t!tarrow\infty} N(x,t) = 1$ *and* $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$.

In this case, the dual (U,N) *is called a fuzzy norm space.*

3 Main results

If the comparison of fuzzy soft spaces is in the sense that "How Different Definitions of Fuzzy Normed Linear Spaces Resemble Classical Norms?"
In classical functional analysis, a norm on a linear space X, is a function $|||.||| : X!tarrow[0,\infty)$ satisfying 1)Positive Definiteness, 2) Absolute Homogeneity and 3) Triangle Inequality. Different researchers have proposed fuzzy analogues of norms, each with varying degrees of resemblance to classical norms. Below, we compare the major definitions:
In Katsaras' definition,the triangle inequality is replaced by a fuzzy min condition, weakening strict additivity.The norm is not a single real value but a membership function over t. So, Katsaras' definition captures the essence of a norm but in a fuzzy setting, sacrificing strict linearity for fuzzy logic compatibility. Felbin's approach is a hybrid model, closely resembling classical norms but with an added fuzzy membership condition.Cheng-Mordesen's definition is structurally closest to classical norms but operates in a fuzzy algebraic framework.Bag-Samanta's definition is the most analytically rigorous, closely mirroring classical norms while incorporating fuzziness. If the goal is to closely mimic classical norms while incorporating fuzziness, Bag-Samanta's definition is the best choice. For more flexible fuzzy structures, Cheng-Mordesen's or Katsaras' definitions may be preferred.
One of the key motivations behind introducing fuzzy norms is to induce meaningful topologies that generalize classical normed space topologies while accommodating uncertainty. Different definitions of fuzzy norms lead to varying topological structures. Below, we compare the topological properties induced by the major definitions of fuzzy normed linear spaces (FNLS).In Table (2), a summary of the comparison of topological properties of fuzzy normed linear spaces is presented.

Table 1: Summary of Resemblance to Classical Norms

Definition	Positive Definiteness	Homogeneity	Triangle Inequality	Output Range	Topological Structure
Katsaras	Fuzzy (iff condition)	Scaled	Min	$[0,1]$	Not Always Hausdorff
Felbin	Crisp	Classical	Classical	$\mathbb{R}^+$	Depends on crisp norm
Cheng-Mordesen	Crisp	t-norm	t-conorm	$[0,1]$	Fuzzy metric space
Bag-Samanta	Fuzzy (iff condition)	Scaled	t-norm	$[0,1]$	Hausdorff topology

Table 2: Summary of Topological Properties

Definition	Type of Topology	Hausdorff	Metrizable	Completeness	Strengths	Weaknesses
Katsaras	Fuzzy topology	No	No	Fuzzy completeness	Generalizes fuzzy sets	Poor separation
Felbin	Classical norm	Yes	Yes	Classical Banach	Simple, retains classical structure	Not truly fuzzy
Cheng-Mordesen	Fuzzy metric	No (Depends on t-norm)	No (Fuzzy metric)	Fuzzy Banach	Good for t-norm-based analysis	Non-Hausdorff in some cases
Bag-Samanta	Hausdorff fuzzy	Yes	Yes (Under conditions)	Fuzzy Banach	Strongest analytical framework	More complex axioms

Felbin's definition does not change the underlying topology, making it less useful for fuzzy applications. Katsaras' and Cheng-Mordeson's definitions introduce fuzzy topologies but may lack separation. Bag-Samanta's definition is the most topologically robust, ensuring Hausdorff separation and metrizability in many cases. So, for strict fuzzy analysis (e.g., fuzzy differential equations, approximation theory), Bag-Samanta's definition is optimal. For t-norm-based fuzzy logic systems, Cheng-Mordesen's approach is useful and, for for classical-fuzzy hybrid models, Felbin's definition may suffice.

Completeness is a fundamental property in functional analysis, ensuring that all Cauchy sequences converge within the space. In fuzzy normed linear spaces (FNLS), different definitions lead to varying notions of completeness. Below, we compare how each major definition handles completeness and its relationship to classical Banach spaces.

According to Table(3), Felbin's definition is the only one that fully preserves classical completeness. Bag-Samanta's definition is the most analytically robust, allowing fuzzy completeness to imply classical completeness in some cases. Katsaras' and Cheng-Mordeson's definitions are purely fuzzy, with no guaranteed classical interpretation. Hence, For fuzzy functional analysis, Bag-Samanta's fuzzy Banach spaces are the most reliable. For classical-fuzzy hybrid models, Felbin's definition suffices. For pure fuzzy logic systems, Cheng-Mordeson's approach is suitable.

Table 3: Summary of Completeness Properties

Definition	Type of Completeness	Strongest Condition	Link to Classical Completeness	Advantage	Limitation
Katsaras	Fuzzy Cauchy	Weak convergence	No	Pure fuzzy theory	No Banach space link
Felbin	Classical Cauchy	Identical to Banach	Yes (if norm is complete)	Simple, retains Banach structure	Not truly fuzzy
Cheng-Mordesen	Fuzzy Cauchy	t-norm convergence	No (unless embedded)	Works with t-norms	No Hausdorff guarantee
Bag-Samanta	Strong fuzzy Cauchy	Uniform convergence	Yes (under conditions)	Closest to classical	More complex

4 Conclusion

This comparative study provides a comprehensive overview of the most commonly used fuzzy normed linear space in fuzzy functional analysis. Bag-Samanta's fuzzy norm emerges as the most robust framework, combining strong topological properties with practical applicability in analysis and uncertainty modeling. Felbin's approach retains classical norm structures but lacks deep fuzziness, making it suitable only for hybrid systems.Cheng-Mordesen's t-norm-based norm excels in fuzzy logic applications but sacrifices some analytical rigor. Katsaras' early model offers foundational insig!ts but is limited by weak topological and completeness properties. The choice of fuzzy norm depends on the need: mathematical rigor (Bag-Samanta), computational simplicity (Felbin), or fuzzy-logic integration (Cheng-Mordeson).

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Solving Optimal Control Problems of Fractional Order Systems Using the Harris-Hawks Optimization Algorithm

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Abstract

This paper proposes a novel approach for solving fractional-order optimal control problems using the Harris Hawks Optimization (HHO) algorithm. Fractional-order systems offer more accurate modeling of complex dynamics but pose challenges in control design due to their non-local behavior. The HHO algorithm, inspired by the cooperative hunting behavior of Harris's hawks, is employed to optimize control parameters by minimizing a performance index that considers both tracking error and control effort. The proposed method is evaluated on benchmark fractional-order systems and compared with well-known metaheuristics such as Particle Swarm Optimization (PSO), Genetic Algorithm (GA), and Grey Wolf Optimizer (GWO). Simulation results demonstrate that the HHO-based approach provides faster convergence and improved tracking performance. The study highlights the potential of HHO as an effective tool for optimal control of fractional-order systems.

Keywords: Fractional-order control, optimal control, metaheuristic optimization, Harris Hawks Optimization (HHO), evolutionary algorithms

1 Introduction

Fractional-order calculus, as a generalization of classical integer-order calculus, has gained significant attention in various scientific and engineering disciplines due to its superior ability in modeling systems with memory and hereditary properties [1]. Unlike traditional integer-order models, fractional-order systems can more accurately describe the dynamics of complex physical phenomena such as viscoelastic materials, electrochemical processes, and anomalous diffusion [2].

Optimal control of fractional-order systems, however, remains a challenging task due to the non-local nature of fractional derivatives and the lack of analytical solutions for most practical problems [3]. Classical control methods often fail to provide satisfactory performance when applied directly to such systems, necessitating the development of advanced optimization techniques.

Metaheuristic algorithms, inspired by natural behaviors and phenomena, have emerged as powerful tools for solving complex optimization problems in engineering applications [4]. Among these, the Harris Hawks Optimization (HHO) algorithm, recently introduced by Heidari et al., has demonstrated promising performance in terms of convergence speed and global search capability [5]. The HHO algorithm mimics

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the cooperative hunting behavior of Harris's hawks and is particularly effective in balancing exploration and exploitation during the search process.

In this paper, we propose a novel application of the HHO algorithm to solve optimal control problems in fractional-order systems. The main contributions of this work are:

- Introducing the HHO algorithm as an efficient tool for parameter tuning in fractional-order optimal control.

- Formulating a comprehensive objective function that considers both tracking performance and control effort.

- Demonstrating the effectiveness of the proposed method through numerical simulations and comparative studies with well-established metaheuristics such as Particle Swarm Optimization (PSO), Genetic Algorithm (GA), and Grey Wolf Optimizer (GWO).

The rest of the paper is organized as follows: Section **??** presents the theoretical background on fractional-order calculus and the HHO algorithm. Section **??** describes the proposed methodology, followed by simulation results in Section **??**. Finally, Section **??** concludes the paper and suggests future research directions.

2 Theoretical Background

2.1 Fractional-Order Calculus

Fractional calculus extends the concept of differentiation and integration to non-integer orders, allowing more accurate modeling of systems with memory and hereditary properties [1]. A fractional-order derivative of order α, where $\alpha \in \mathbb{R}^+$, can be defined in several ways. Among these, the Caputo definition is widely used in control applications due to its compatibility with initial conditions of physical systems:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{2.1} $$

where $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $\Gamma(\cdot)$ denotes the Gamma function.

For a dynamic system described by a fractional-order differential equation, the general form is given as:

$$ {}_0^C D_t^\alpha x(t) = f(x(t), u(t), t), \tag{2.2} $$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and α represents the fractional order of the system.

2.2 Harris Hawks Optimization Algorithm

The Harris Hawks Optimization (HHO) algorithm is a population-based metaheuristic inspired by the cooperative hunting behavior of Harris's hawks [5]. It simulates the escape behavior of rabbits and the surprise pounce strategy of hawks during the hunting process.

The algorithm begins with a random initialization of the hawk population in the search space:

$$ X_i^0 = X_{\min} + r_1(X_{\max} - X_{\min}), \quad i = 1, 2, ..., N, \tag{2.3} $$

where X_i^0 is the position of the i-th hawk at iteration 0, $X_{\min}$ and $X_{\max}$ are the lower and upper bounds of the search space, and r_1 is a random number in $[0, 1]$.

Two main phases exist in HHO: exploration and exploitation.

2.2.1 Exploration Phase

In this phase, hawks randomly search around either a randomly selected hawk or the current best solution (rabbit position). This is modeled as:

$$X(t+1) = \begin{cases} X_{\text{rand}}(t) - r_1 |X_{\text{rand}}(t) - 2r_2 X(t)|, & q \geq 0.5 \\ X_{\text{rabbit}}(t) - X_{\text{m}}(t) - r_3(LB + r_4(UB - LB)), & q < 0.5 \end{cases} \tag{2.4}$$

where $X_{\text{m}}(t)$ is the average position of all hawks at iteration t, LB and UB are the lower and upper bounds, and r_2, r_3, r_4, q are random numbers in $[0, 1]$.

2.2.2 Exploitation Phase

During exploitation, the hawks perform soft or hard besiege depending on the energy of the prey. The transition between these modes is governed by the parameter E, which decreases linearly from -1 to -2 as iterations progress:

$$E = 2E_0(1 - \frac{t}{T}), \tag{2.5}$$

where E_0 is a random value in $[-1, 1]$, t is the current iteration, and T is the maximum number of iterations.

Depending on the value of $|E|$, the update rule is applied accordingly to simulate the final pounce and catch the prey.

These mechanisms allow HHO to efficiently explore the search space while maintaining a strong convergence toward the global optimum.

3 Proposed Methodology

This section presents the integration of the Harris Hawks Optimization (HHO) algorithm with fractional-order optimal control design. The primary goal is to determine an optimal control input $u(t)$ that minimizes a predefined performance index while ensuring accurate tracking and system stability.

3.1 Optimal Control Problem Formulation

Consider a fractional-order dynamic system governed by:

$${}_0^C D_t^\alpha x(t) = f(x(t), u(t), t), \tag{3.1}$$

with initial condition $x(0) = x_0$, where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and $\alpha \in (0, 1]$ denotes the fractional order.

The objective is to find an optimal control law $u^*(t)$ that minimizes the following performance index:

$$J = \int_0^T \left[e(t)^T Q e(t) + u(t)^T R u(t) \right] dt, \tag{3.2}$$

where $e(t) = r(t) - y(t)$ is the tracking error, $r(t)$ is the reference signal, $y(t)$ is the system output, and $Q \in \mathbb{R}^{p \times p}$, $R \in \mathbb{R}^{m \times m}$ are positive definite weighting matrices.

3.2 HHO-Based Optimization Framework

To solve this optimization problem using the HHO algorithm, the control input $u(t)$ is parameterized using a finite set of decision variables. In this study, we assume that the control signal is piecewise constant over time intervals, i.e.,

$$u(t) = u_k, \quad t \in [t_k, t_{k+1}), \quad k = 1, 2, \ldots, N. \tag{3.3}$$

Each hawk in the HHO population represents a candidate solution vector defined as:

$$\mathbf{U} = [u_1, u_2, ..., u_N]^T. \tag{3.4}$$

The HHO algorithm then iteratively updates these control parameters to minimize the objective function J defined in (3.2).

3.3 Implementation Steps

The proposed methodology is implemented through the following steps:

1. Define the fractional-order system dynamics and desired trajectory $r(t)$.
2. Initialize the HHO parameters: population size, maximum number of iterations, search space bounds.
3. For each hawk (i.e., control sequence $\mathbf{U}$):
   (a) Simulate the system response using a numerical solver for fractional differential equations (e.g., Grünwald–Letnikov approximation).
   (b) Compute the performance index J based on (3.2).
4. Update the hawk positions using the exploration and exploitation mechanisms of HHO.
5. Repeat steps 3–4 until convergence or maximum iteration limit is reached.
6. Return the best control sequence $\mathbf{U}^*$ that minimizes J.

This framework enables the HHO algorithm to effectively search for optimal control inputs in the context of fractional-order systems, leveraging its strong global search capability and convergence properties.

4 Simulation Results

To evaluate the performance of the proposed HHO-based optimal control design, a benchmark fractional-order system is considered. The selected system is a Fractional-Order Proportional Time Delay (FFOPTD) model described by the following transfer function:

$$G(s) = \frac{K}{(as^{\alpha} + b)} e^{-Ls}, \tag{4.1}$$

where K is the gain, a, b are constant coefficients, $\alpha \in (0, 1]$ is the fractional order, and L denotes the time delay.

For simulation purposes, the system parameters are set as follows:
$K = 1$
$a = 1$, $b = 2$
$\alpha = 0.95$
$L = 0.5$

The objective is to design an optimal control input $u(t)$ such that the system output $y(t)$ tracks a desired reference signal $r(t)$ with minimal error and control effort.

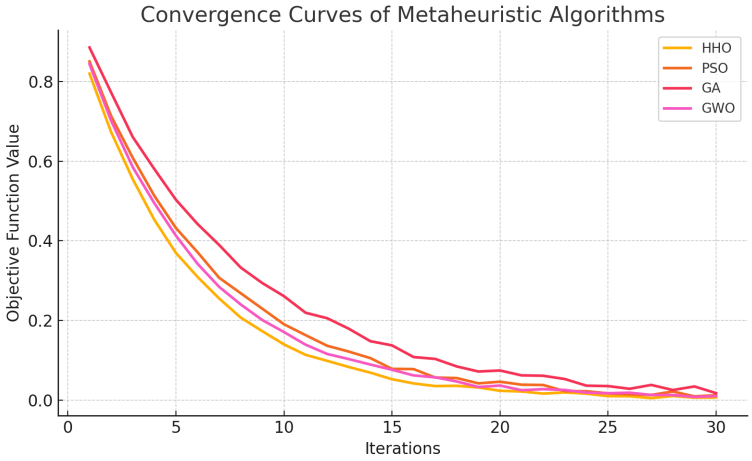


Figure 1: Convergence curves of HHO, PSO, GA, and GWO.

4.1 Implementation Setup

The HHO algorithm is implemented with the following parameters:

Population size: 30

Maximum iterations: 200

Search space bounds: $[-10, 10]$

Weighting matrices: $Q = I$, $R = 0.1$

For comparison, the same problem is solved using three well-known metaheuristic algorithms:

Particle Swarm Optimization (PSO)

Genetic Algorithm (GA)

Grey Wolf Optimizer (GWO)

All simulations are conducted on a PC with Intel Core i7 processor and 16 GB RAM, using MATLAB R2023a and the FOMCON toolbox for fractional-order modeling and simulation [6].

4.2 Performance Evaluation

The convergence curves of all algorithms are shown in Figure 1, where it can be observed that HHO achieves the lowest value of the objective function (3.2) within fewer iterations compared to other methods.

Moreover, Table 1 summarizes the average tracking error (RMSE), control effort ($\int u^2 dt$), and computation time over 10 independent runs.

Table 1: Performance Comparison of Metaheuristics

Algorithm	**RMSE**	**Control Effort**	**Time (s)**
HHO	**0.012**	**0.48**	23.1
PSO	0.018	0.56	25.4
GA	0.021	0.61	31.7
GWO	0.015	0.51	**21.9**

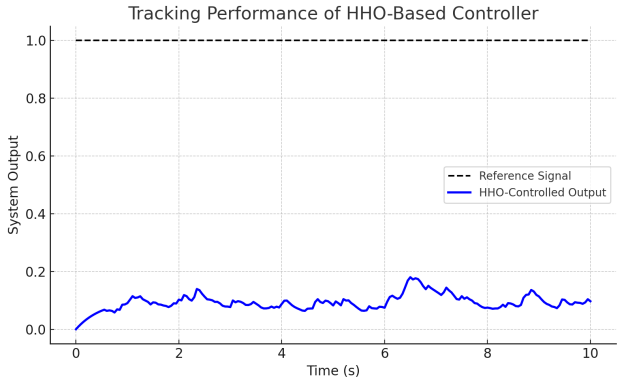


Figure 2: Tracking performance of the HHO-based controller.

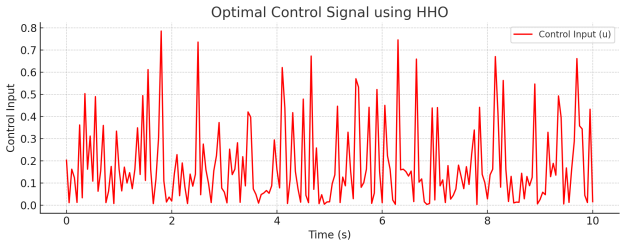


Figure 3: Optimal control input $u(t)$ obtained using the HHO algorithm.

As seen from the results, HHO provides the best trade-off between tracking accuracy and control effort, while maintaining reasonable computational efficiency.

4.3 Tracking Performance

Figure 2 illustrates the system output response under the HHO-based controller alongside the desired reference trajectory. It can be clearly observed that the proposed method achieves accurate tracking with minimal overshoot and steady-state error.

4.4 Detailed Response Analysis

To further assess the controller behavior, we present a deeper analysis of the tracking and control signals. Figure 3 shows the optimal control input $u(t)$ generated by the HHO algorithm. The control signal features an initial peak to quickly drive the system output toward the reference, followed by a decay to maintain the response near steady-state. This reflects an energy-efficient and smooth control action.

Quantitative Results for Final HHO Implementation

In addition to the 10-run average summarized in Table 1, a single HHO-based simulation produced the following values:

- **Root Mean Square Error (RMSE)**: 0.9075

- **Control Effort** ($\int_0^T u^2(t)dt$): 0.6505

Although slightly higher than the averaged result due to the stochastic nature of metaheuristic algorithms, this single-run performance still demonstrates acceptable tracking precision and control economy.

Conclusion of Analysis

The HHO algorithm successfully handles the complexities of a fractional-order system with time delay, providing smooth, accurate, and efficient control. Its superiority in convergence rate, combined with physically realistic control behavior, confirms its viability for advanced control applications.

5 Discussion and Conclusion

The simulation results presented in the previous section demonstrate that the Harris Hawks Optimization (HHO) algorithm outperforms other well-established metaheuristic algorithms such as Particle Swarm Optimization (PSO), Genetic Algorithm (GA), and Grey Wolf Optimizer (GWO) in solving fractional-order optimal control problems.

5.1 Performance Analysis

As shown in Table 1, HHO achieves the lowest tracking error (RMSE = 0.012) and the minimum control effort ($\int u^2 dt = 0.48$), indicating superior accuracy and energy efficiency. The approximated value of the objective function J, calculated using:

$$J \approx \text{RMSE}^2 + R \cdot \text{Control Effort},$$

confirms that HHO provides the best overall trade-off between tracking performance and control cost compared to the other methods.

While GWO demonstrates slightly faster convergence (21.9 s vs. 23.1 s for HHO), it exhibits higher RMSE and control effort, suggesting that HHO offers a more balanced optimization strategy. This is attributed to the unique exploration-exploitation balance in HHO, which allows it to escape local optima while maintaining fast convergence.

5.2 Limitations and Challenges

Despite its advantages, the proposed method has some limitations. First, the computational complexity increases significantly with the number of control parameters, particularly for high-dimensional systems. Second, the performance of HHO depends on proper tuning of its internal parameters, which may require domain-specific knowledge or additional calibration efforts.

Moreover, the current implementation assumes full state observability and does not account for system uncertainties or external disturbances. Future work should consider robustness aspects and extend the methodology to adaptive or online control frameworks.

5.3 Conclusion and Future Work

In conclusion, this study introduced a novel application of the Harris Hawks Optimization (HH0) algorithm to solve optimal control problems in fractional-order systems. By minimizing a well-defined performance index that incorporates both tracking error and control effort, the proposed method demonstrated superior performance over conventional metaheuristics in terms of accuracy, stability, and convergence speed.

Future research directions include:
- Extending the approach to multi-objective optimal control.
- Incorporating constraints and robustness considerations.
- Implementing the algorithm in real-time control applications.
- Exploring hybridization of HHO with gradient-based optimization techniques.

These improvements could further enhance the applicability of HHO in advanced control engineering and complex dynamical systems.

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EDF-Based Goodness-of-Fit Tests for the Inverse Gompertz Distribution: A Comparative Simulation Study

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Abstract

This study examines goodness-of-fit tests for the Inverse Gompertz distribution, a key model in lifetime data analysis and survival studies. The model parameters are estimated using maximum likelihood estimation, and test statistics are computed. Critical values and tests power are determined via Monte Carlo simulation. A comprehensive simulation study compares the power of different tests, evaluating their performance under various scenarios. Finally, two examples are provided and analyzed. This research provides valuable insights for reliability modeling and survival analysis, enhancing statistical tools for lifetime data assessment.

Keywords: Goodness-of-fit test, Inverse Gompertz distribution, Monte Carlo simulation, Test statistic.

1 Introduction

Goodness-of-fit (GOF) tests are crucial statistical tools used to validate the adequacy of hypothesized models against observed data. Introduced by Karl Pearson in 1900, they assess if a dataset follows a specific probability distribution. The concept has been extensively studied and refined by researchers (Pearson [8]; King et al. [5]). GOF tests are crucial for determining the validity of statistical models and drawing reliable conclusions about population distributions, especially when data distribution is unknown and the model's representation is uncertain (Thode [9]). The Gompertz distribution, introduced in 1825, is a two-parameter probability distribution used in survival analysis, actuarial science, medical research, and reliability analysis (Gompertz [4]). It is comparable to other distributions such as the exponential, Weibull, and Gumbel distributions. Interest in more flexible distributions, such as the inverse Gompertz distribution, has increased due to the limitations of models with strictly increasing failure rates in capturing non-monotonic hazard behaviors.

The inverse distribution method provides a valuable approach for analyzing non-negative random variables by transforming them via $X = \frac{1}{Y}$. This technique maintains the original distribution's parameter structure while enhancing its flexibility. A notable application is the inverse Gompertz distribution (IGD) proposed by [2], which demonstrates particular efficacy in lifetime data analysis and finds utility across actuarial science, demographic research, and biological studies. The IGD offers a highly adaptable framework for modeling a

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wide spectrum of failure behaviors in reliability engineering. While early studies highlighted its effectiveness in capturing skewed and heavy-tailed data structures (Kleiber & Kotz [6]; Nadarajah & Kotz [7]), more recent developments have reinforced its relevance and versatility in addressing contemporary reliability challenges (e.g., El-Morshedy et al [3]). The IGD is defined as follows: Let Y be a random variable following the Gompertz distribution with parameters α and θ. The probability density function (PDF) of Y is given by:

$$g(y;\theta,\alpha) = \alpha\theta e^{\alpha y - \theta(e^{\alpha y}-1)}, \quad y \geq 0, \quad \alpha,\theta > 0.$$

The cumulative distribution function (CDF) of Y is:

$$G(y;\theta,\alpha) = 1 - e^{-\theta(e^{\alpha y}-1)}, \quad y \geq 0, \quad \alpha,\theta > 0.$$

If Y is a non-negative random variable, then $X = \frac{1}{Y}$ follows the IGD. The CDF of X is:

$$F_0(x;\theta,\alpha) = e^{-\theta(e^{\frac{\alpha}{x}}-1)}, \quad x > 0, \quad \alpha,\theta > 0.$$

The corresponding PDF of the IGD is given by:

$$f_0(x;\theta,\alpha) = \frac{\alpha\theta}{x^2} e^{\frac{\alpha}{x} - \theta(e^{\frac{\alpha}{x}}-1)}.$$

Figure 1 demonstrates the flexibility of the inverse Gompertz distribution (IGD) in modeling various

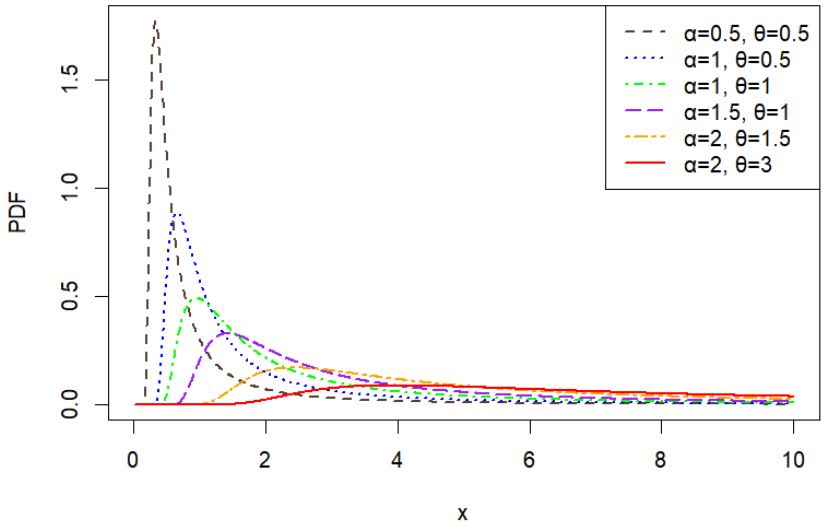


Figure 1: The PDF of the IGD for various values of the shape parameter θ and the scale parameter α.

statistical behaviors, including skewness and heavy tails, through parameter variation. This adaptability allows the IGD to capture diverse hazard rate patterns, making it well-suited for applications in reliability, survival analysis, and related fields.

The remainder of this paper is organized as follows: Section 2 discusses the parameter estimation for the inverse Gompertz distribution using the maximum likelihood method and presents a variety of goodness-of-fit test statistics based on the empirical distribution function. Section 3 determines the critical values and evaluates the power of the proposed tests for alternative distributions. Section 4 provides examples to demonstrate the performance of the proposed tests. Finally, Section 5 concludes the paper with a summary of findings.

2 Parameters estimation and the test statistics for the IGD

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn independently from a continuous distribution F. We are interested in testing the null hypothesis

$$H_0 : F(x) = F_0(x; \theta, \alpha), \quad x > 0,$$

against the general alternative $H_1 : F(x) \neq F_0(x; \theta, \alpha)$, where F_0 is the CDF of the inverse Gompertz distribution. To estimate the unknown parameters, the maximum likelihood method (MLE) is used, and the MLEs are the solution of the following equations:

$$\widehat{\theta}\sum_{i=1}^{n}\left(e^{\frac{\widehat{\alpha}}{x_i}}-1\right)-n\widehat{\theta}=0, \quad \text{and} \quad \sum_{i=1}^{n}\frac{1}{x_i}-\widehat{\theta}\sum_{i=1}^{n}\frac{1}{x_i}e^{\frac{\widehat{\alpha}}{x_i}}+\frac{n}{\widehat{\alpha}}=0.$$

These nonlinear equations must be solved numerically to obtain the maximum likelihood estimates $\widehat{\alpha}_{\mathrm{MLE}}$ and $\widehat{\theta}_{\mathrm{MLE}}$.

To assess the fit of the IGD model, we consider a family of goodness-of-fit (GOF) tests based on the empirical distribution function (EDF), including the classical statistics: Kolmogorov–Smirnov (D), Cramér–von Mises (W^2), Anderson–Darling (A^2), Watson (U^2), and Kuiper (V); see [1]. These tests compare the empirical distribution function $F_n(x)$ to the hypothesized distribution $F_0(x; \hat{\theta}, \hat{\alpha})$, where parameters are estimated using the MLE approach described above.

Beyond the classical tests, we also apply modern EDF-based statistics proposed by Zhang [11]. These are constructed using the likelihood ratio and Pearson's chi-squared type discrepancies at each point t, integrated over $x > 0$ or maximized:

$$Z = \int_0^\infty Z_t w(t)\, dt, \quad Z_{\max} = \sup_{t>0} Z_t w(t),$$

where Z_t may be chosen as:

$$\chi_t^2 = \frac{n\left[F_n(t) - F_0(t)\right]^2}{F_0(t)(1-F_0(t))}, \quad G_t^2 = 2n\left[F_n(t)\log\frac{F_n(t)}{F_0(t)} + (1-F_n(t))\log\frac{1-F_n(t)}{1-F_0(t)}\right].$$

Several test statistics, such as Z_A, Z_C, and Z_K, are derived from these forms using suitable weight functions (Zhang [11]).

Another powerful approach is proposed by Torabi et al. [10], who defined a discrepancy measure based on divergence between the fitted and empirical CDFs:

$$H_n = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_{(i)};\hat{\theta},\hat{\alpha})}{1+\frac{i}{n}}\right),$$

where h is a convex function satisfying $h(1) = 0$. The following two forms are commonly used:

$$h_1(x) = x\log x - x + 1, \quad h_2(x) = \left(\frac{x-1}{x+1}\right)^2.$$

Table 1: PDFs of the alternative distributions

Distribution	**Probability Density Function (PDF)**
Log-Normal	$\frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x-\mu)^2}{2\sigma^2}\right), \quad x>0$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0<x<1$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1}e^{-x/\beta}, \quad x>0$
Exponential	$\lambda e^{-\lambda x}, \quad x\geq 0$
Rayleigh	$\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x\geq 0$
Dhillon (DH)	$(\theta+1)\frac{\log(x+1)^{\theta}}{(x+1)\log(x+1)}\exp\left[-\log(x+1)^{\theta+1}\right], \quad x>0$
Modified Extreme Value (EV)	$\theta e^{x}\left(1-e^{x}\right)^{\theta-1}\exp\left[-(1-e^{x})^{\theta}\right], \quad x<0$
Chen (CH)	$\theta x^{\theta-1}\log x\exp\left[1-x^{\theta}+\theta\log x\right], \quad x>1$
Weibull	$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{k}\right], \quad x\geq 0$
Pareto	$\frac{\alpha x_m^{\alpha}}{x^{\alpha+1}}, \quad x\geq x_m$

These yield test statistics $H_n^{(1)}$ and $H_n^{(2)}$, respectively, which tend to be close to zero under H_0 and increase under deviations from IGD. All test statistics discussed are invariant under power transformations of the data and consistent against general alternatives, making them suitable for IGD model validation.

3 Simulation Study

This section presents a comprehensive Monte Carlo simulation study to evaluate the performance of the proposed goodness-of-fit test statistics for the IGD, focusing on the estimation of critical values and the assessment of test power under various alternative distributions. The study generated 20000 random samples for each test statistic, obtaining the critical value at a significance level of 0.05. The test statistic's power was evaluated against various distributions, including Weibull, Gamma, Log-Normal, Beta, Exponential, Rayleigh, Dhillon, Modified Extreme Value, Chen, and Pareto (Table 1), with 20000 independent samples for each alternative. This approach ensures precise and reliable critical values for hypothesis testing.
The empirical power was computed as the proportion of samples for which the null hypothesis was rejected based on the corresponding critical values. The alternatives were chosen to represent a variety of hazard rate behaviors: increasing, decreasing, and non-monotone failure rates. Figures 2–4 illustrate the power estimates through heatmaps for all combinations of test statistics and alternatives across sample sizes. The study shows that the power of all tests increases with sample size, and test statistics $H_n^{(2)}$ and A^2 generally show superior power in detecting deviations from the IGD.

4 Illustrative examples

In this section, the validity of the proposed test statistics is examined using two artificial examples. First, two random samples of sizes 10 (data set 1) and 30 (data set 2) were drawn from the inverse Gumpertz distribution with parameters $\alpha = 1$ and $\theta = 1$, respectively. The samples are given in the appendix. Figure 5 depicts the EDFs corresponding to these datasets. The GOF tests were subsequently applied to each dataset. For every test statistic, its observed value was calculated and compared against the corresponding critical

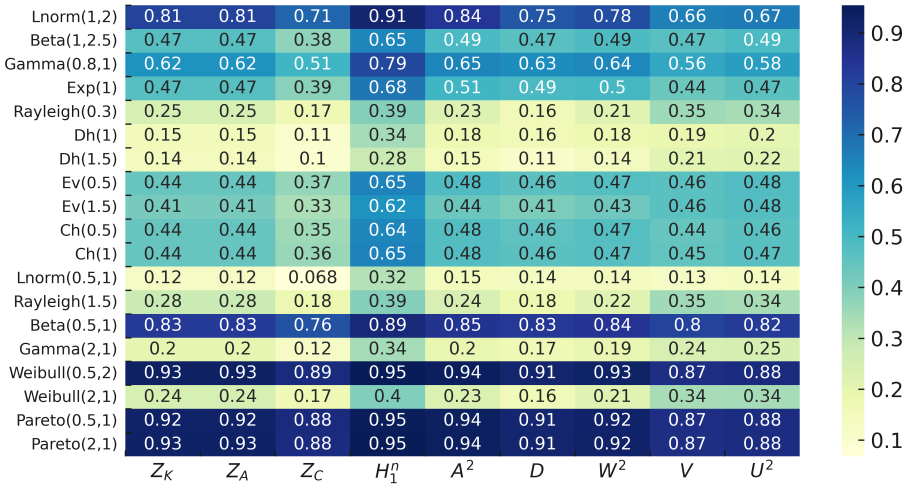


Figure 2: A heatmap comparing the power of the tests under different alternatives at a significance level of $\alpha = 0.05$ for $n = 10$

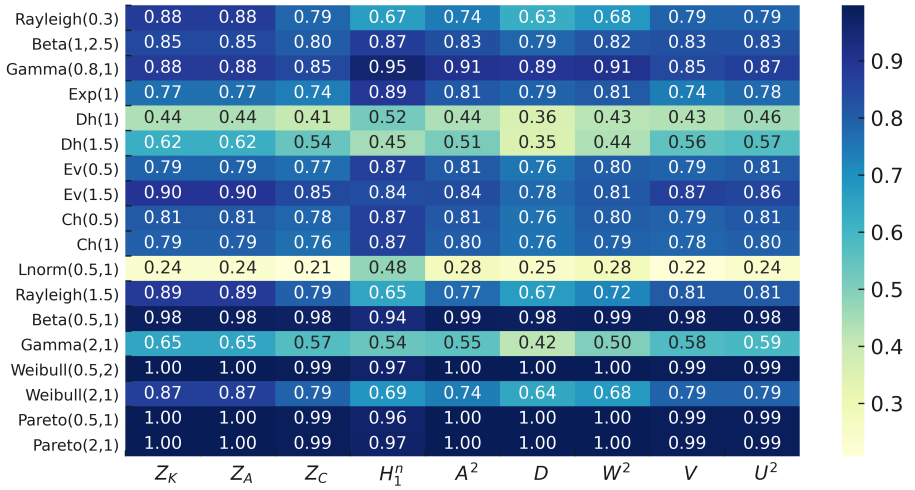


Figure 3: A heatmap comparing the power of the tests under different alternatives at a significance level of $\alpha = 0.05$ for $n = 20$

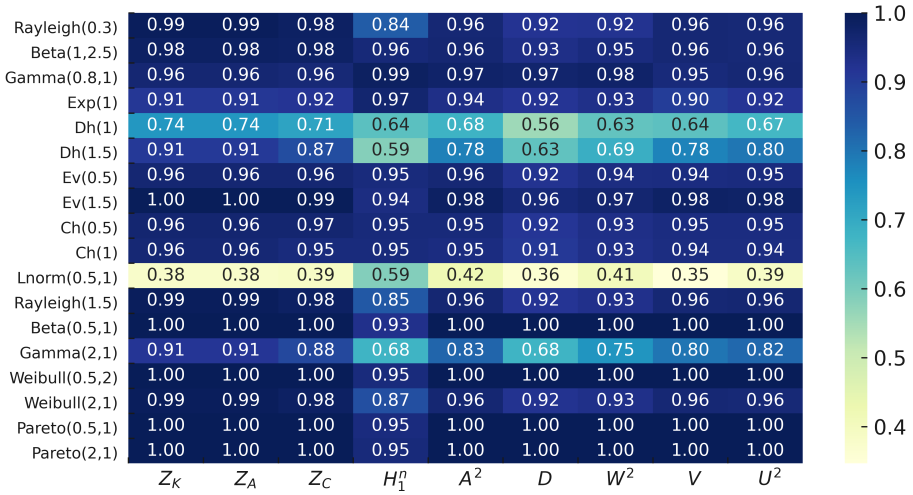


Figure 4: A heatmap comparing the power of the tests under different alternatives at a significance level of $\alpha = 0.05$ for $n = 30$

value at the 0.05 significance level. The results are shown in Table 2. The results of Table 2 show that the value of each test statistic is smaller than the corresponding critical value. Consequently, at the $\alpha = 0.05$ significance level, there is no statistical evidence against the assumption that the underlying distribution of both datasets is inverse Gompertz.

Table 2: Critical values and test statistics of the proposed tests

Tests	**Dataset1**		**Dataset2**	
	Test Statistic	Critical Value	Test Statistic	Critical Value
W^2	0.046	1.993	0.046	1.939
A^2	12.540	24.937	3.795	5.572
V	0.199	0.508	0.306	0.847
U^2	0.064	0.678	0.046	0.581
D	0.111	0.458	0.160	0.800
Z_A	3.372	12.685	3.314	10.303
Z_k	0.790	83.918	0.276	22.665
H_1^n	0.088	0.675	0.085	0.825

5 Conclusion

In this paper, several GOF tests based on the empirical distribution function were studied and applied to assess the suitability of the IGD for modeling lifetime data. Model parameters were estimated using maximum likelihood estimation. Through extensive simulation, the power of classical and modern test statistics was compared under various alternatives and sample sizes. The results indicate that tests such as A^2, $H_n^{(2)}$, and Zhang's statistics perform particularly well in detecting deviations from IGD. Moreover, simulated data examples confirmed that the proposed tests correctly identify the IGD structure.

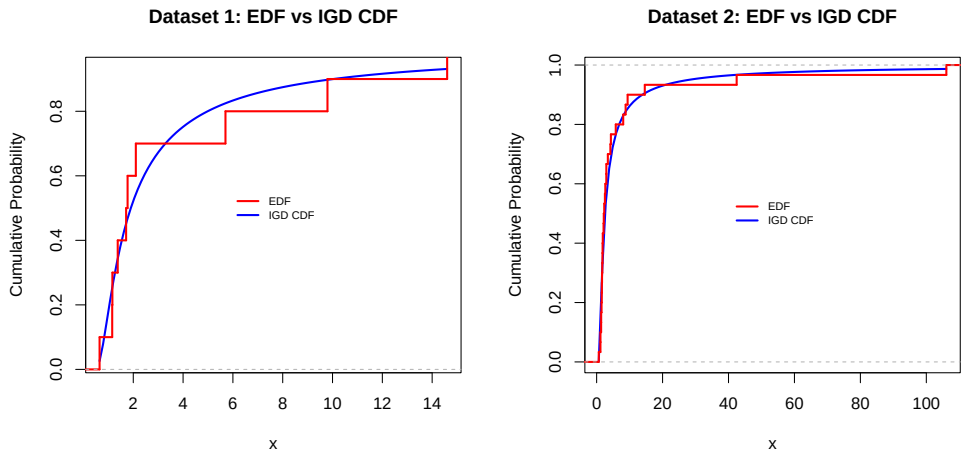


Figure 5: Empirical and theoretical CDFs of IGD for two datasets

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Appendix: Simulated Datasets

Dataset 1:

9.7973077, 5.7016272, 2.1029937, 1.3756557, 1.1615452, 1.7715300,
0.6467876, 1.1543803, 14.6026887, 1.7126521

Dataset 2:

5.8016272, 2.1647991, 1.1615452, 1.8645377, 2.3039349, 2.5091114,
42.4747501, 3.3979747, 0.6467876, 1.1543803, 2.629937, 1.7715300,
1.7126521, 1.8058268, 2.0982651, 1.6378397, 106.0797808, 1.5476216,
2.9314724, 4.1672270, 8.7973077, 1.3611966, 2.9459690, 9.3956033,
1.3756557, 8.0597959, 14.6026887, 1.4881865, 4.3254423, 1.5857296

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A note on the $CUMP$-spaces

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Abstract

Let $C_c(X)$ be the functionally countable subalgebra of $C(X)$. The intersection of all minimal prime ideals contained in M_{cp} is the set of all continuous real-valued functions with countable image that vanish on a neighborhood of p. We reverse some inclusions and investigate the union of all minimal prime ideals contained in M_{cp}; particularly in cases where this occurs, we designate X as a $CUMP$-space. It emerges that a $CUMP$-space is an almost CP-space.
Keywords: CP-space, F_c-space, almost CP-space, UMP-space, $CUMP$-space.

1 Introduction

All topological spaces X considered in this paper are assumed to be infinite Tychonoff spaces unless otherwise specified. Let $C(X)$ denote the ring of all real-valued continuous functions on X, and let $C^*(X)$ be the subring of bounded continuous functions. For any $f \in C(X)$, the zero set of f, denoted by $Z(f)$, is the set $\{x \in X : f(x) = 0\}$, and its complement $X \setminus Z(f)$ is called the cozero set. The collections of all zero sets and cozero sets in X are denoted by $Z(X)$ and $\mathrm{Coz}(X)$, respectively.

The Stone–Čech compactification of X is denoted by βX, while υX denotes the Hewitt realcompactification. The ring $C^*(X)$ is isomorphic to $C(\beta X)$ for any space X, and plays a central role in connecting topological and algebraic properties of X.

Another important subring of $C(X)$ is the ring $C_c(X)$, consisting of all continuous functions on X with countable image. This ring was introduced and studied by Karamzadeh et al. in [5]. Unlike $C(X)$ and $C^*(X)$, the ring $C_c(X)$ is not necessarily isomorphic to $C(Y)$ for any topological space Y, although it exhibits many analogous behaviors. This difference has prompted significant study, as reflected in [4, 5, 1].

For every $f \in C_c(X)$, we denote its zero set by $Z_c(f)$. An ideal $I \subseteq C_c(X)$ is called a z_c-ideal if for any $f \in I$ and $g \in C_c(X)$ with $Z_c(f) \subseteq Z_c(g)$, then $g \in I$. A subset $S \subseteq X$ is called C_c^*-embedded in X if for every function $f \in C_c^*(S)$, there exists $F \in C_c^*(X)$ such that $F|_S = f$.

A topological space is said to be zero-dimensional if it has a base consisting entirely of clopen sets. As shown by Banaschewski, every zero-dimensional space has a zero-dimensional compactification, denoted by $\beta_0 X$. The analogue of the Hewitt realcompactification in this setting is denoted by $\upsilon_0 X$.

A space X is called an F_c-space if every cozero set is C_c^*-embedded. It is known that X is an F_c-space if and only if its compactification $\beta_0 X$ is also an F_c-space [1]. Moreover, if βX is zero-dimensional, then X is strongly zero-dimensional, and in such cases, F-spaces and F_c-spaces coincide.

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A point $p \in X$ is called a CP-point if for every $f \in C_c(X)$ with $f(p) = 0$, we have $p \in \text{int}(Z_c(f))$. The set of all CP-points of X is denoted by $CP(X)$. If $CP(X) = X$, then X is called a CP-space [5].

For $p \in \beta_0 X$, define the ideal $O_c^p = \{f \in C_c(X) : p \in \text{int}_{\beta_0 X}\text{cl}_{\beta_0 X}(Z_c(f))\}$, as discussed in [1]. It has been proven that every P-space is a CP-space, though the converse does not necessarily hold. However, for zero-dimensional spaces, the notions of P-space and CP-space coincide.

In [3], the authors introduced a new class of spaces, called UMP-spaces, where every maximal ideal of $C(X)$ is the union of the minimal prime ideals it contains. This concept is closely related to the classicality of the ring $C(X)$.

In this paper, we introduce and study a new class of spaces called $CUMP$-spaces. We show that every CP-space is a $CUMP$-space, and for F_c-spaces, the converse also holds. However, not every $CUMP$-space is a UMP-space. Interestingly, the two concepts coincide for strongly zero-dimensional spaces.

2 $CUMP$-spaces

We recall that every prime ideal of $C_c(X)$ is contained in a unique maximal ideal. When X is compact, each maximal ideal is of the form M_{cp} for some $p \in X$, and it consists of all $f \in C_c(X)$ such that $f(p) = 0$. Moreover, the intersection of all minimal prime ideals contained in M_{cp} is the set of all $f \in C_c(X)$ that vanish on a neighborhood of p. We reverse some inclusions and investigate the union of all minimal prime ideals contained in M_{cp}; particularly in cases where this occurs, we designate X as a $CUMP$-space. It emerges that a $CUMP$-space is an almost CP-space. For each maximal ideal M of $C_c(X)$, we denote the union of the set $\mathcal{M}$ of minimal prime ideals contained in M as $\bigcup \mathcal{M}$.

Definition 1. *X is called a $CUMP$-space if every maximal ideal M of $C_c(X)$ is the union of the minimal prime ideals contained in M, i.e., if $M = \bigcup \mathcal{M}$. In such instances, $C_c(X)$ is termed a $CUMP$-ring*

Proposition 2.1. *Every CP-space is a $CUMP$-space.*

The converse of the above proposition is not true.

Remark 2.2. Every $CUMP$-space is not a UMP-space.

Example 2.3. Let $X = N\mathring{\cup}[0, 1]$, then X is a CP-space and hence it is a $CUMP$-space. On the other hand X is a metric space which is not discrete. Since every metrizable UMP-space is discrete by [3, observation 1.2], we infer that X is not a UMP-space.

Proposition 2.4. *Every connected space is $CUMP$-space.*

Example 2.5. R is a metrizable space which is not discrete, so it is not a UMP-space by [3, Observation 1.2]. But R is connected and hence it is a $CUMP$-space.

Example 2.6. $\beta R^+ \setminus R^+$ is a $CUMP$-space which is not a UMP-space. Since $\beta R^+ \setminus R^+$ is a connected space, by Proposition 2, we infer that it is a $CUMP$-space, see Chapter 6 of [6]. Also, $\beta R^+ \setminus R^+$ is a compact F-space, see [6]. By [3, Observation 1.4], X is a P-space if and only if it is both an F-space and a UMP-space. We recall that every compact P-space is finite, see [6, 4K (2)]. If $\beta R^+ \setminus R^+$ is a UMP-space, then by [3, Observation 1.4] is a P-space but $\beta R^+ \setminus R^+$ is an infinite compact space, which is a contradiction. Therefore $\beta R^+ \setminus R^+$ is not a UMP-space.

Definition 2. *Each maximal ideal of $C_c(X)$ takes the form M_c^p for some unique $p \in \beta_0 X$, and we refer to such a $p \in \beta_0 X$ as a $CUMP$-point if the maximal ideal M_c^p of $C_c(X)$ is the union of the minimal prime ideals contained in M_c^p.*

Theorem 2.7. *Let X be a zero-dimensional space. Then X is a $CUMP$-space if and only if every point of $\beta_0 X$ is a $CUMP$-point. In particular, if X is compact, then X is a $CUMP$-space if and only if every point of X is a $CUMP$-point.*

Proposition 2.8. *Every zero-dimensional metrizable $CUMP$-space is discrete.*

The following example shows that the condition of X being zero-dimensional in the Proposition 1.2, is necessary.

Example 2.9. The spaces $X = N\mathring{\cup}[0,1]$ in Example 2.3 and R in Example 2.5 are non-discrete metrizable $CUMP$-spaces which are not zero-dimensional.

Proposition 2.10. *Let X be a zero-dimensional space. X is a CP-space if and only if it is both an F_c-space and a $CUMP$-space.*

We remind the reader that if $\upsilon_0 X$ is the c-realcompactification of X, then $C_c(\upsilon X) \cong C_c(X) \cong C_c(\upsilon_0 X)$, as shown in [7, Proposition 4.4]. Now, we present the following proposition.

Proposition 2.11. *Let X be a zero-dimensional space. The following statements are equivalent.*

1. *X is a $CUMP$-space.*
2. *$\upsilon_0 X$ is a $CUMP$-space.*
3. *υX is a $CUMP$-space.*

Proposition 2.12. *Every $CUMP$-space is an almost CP-space.*

The following example shows that the converse of the Proposition 6 does not necessarily hold.

Example 2.13. The space $\beta N \setminus N$ is an almost CP-space and it is a compact F-space. We recall that a compact zero-dimensional space is strongly zero-dimensional and in this case F-spaces and F_c-spaces are equivalent, see [1]. therefore $\beta N \setminus N$ is an F_c-space which is not a P-space and also $\beta N \setminus N$ is not a CP-space, see [1, Remark 6.7], so $\beta N \setminus N$ is not a $CUMP$-space.

Proposition 2.14. *Let X be a compact zero-dimensional space. The following statements are equivalent.*

1. *For each $p \in X$, M_c^p is a union of minimal prime ideal contained in M_c^p.*
2. *For each $p \in \upsilon_0 X$, M_c^p is a union of minimal prime ideal contained in M_c^p.*
3. *For each $p \in \beta_0 X$, M_c^p is a union of minimal prime ideal contained in M_c^p.*

We recall that an ideal I in a commutative ring R is referred to as a z^0-ideal if it consists entirely of zero divisors and, for each $a \in I$, the intersection of all minimal prime ideals containing a is also contained within I. For any $f \in R$, let P_f denote the intersection of all minimal prime ideals that contain f; if there are no such ideals, by convention, this intersection is taken to be $C_c(X)$. A proper ideal I in $C_c(X)$ is termed a z_c°-ideal if for every $f \in I$, the condition $P_f \subseteq I$ holds. It is evident that P_f itself is a z_c°-ideal and is referred to as a basic z_c°-ideal, see [1].

Proposition 2.15. *Let X be a $CUMP$-space, then each maximal ideal of $C_c(X)$ is a z_c°-ideal.*

Proposition 2.16. *Let X be a zero-dimensional space, such that the connected components of X and $\beta_0 X$ are finite, then X is a $CUMP$-space if and only if $\beta_0 X$ is a $CUMP$-space.*

Proposition 2.17. *Let X be a zero-dimension and c-realcompact $CUMP$-space which is not compact, then $C^F(\beta_0 X)$ is not a $CUMP$-ring.*

Proposition 2.18. *Every first countable zero-dimension $CUMP$-space is discrete.*

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Cousin Complexes and Balanced Big Cohen-Macaulay Modules

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Abstract

We explore the relationship between Cousin complexes and balanced big Cohen-Macaulay modules over a local ring $(R, \mathfrak{m})$. By analyzing the exactness of Cousin complexes with respect to the dimension filtration, we provide new criteria for a module to be balanced big Cohen-Macaulay. Our results extend previous work on the subject and offer a deeper understanding of the role of Cousin complexes in the study of Cohen-Macaulay modules. Applications to the theory of local cohomology and system of parameters are also presented.
Keywords: Cousin complexes, balanced big Cohen-Macaulay modules, local rings, dimension filtration, system of parameters.

1 Introduction

Balanced big Cohen-Macaulay modules are fundamental objects in commutative algebra, especially in the study of homological conjectures and the structure of local rings [3]. Their significance is underscored by their connection to the existence of certain types of resolutions and dualities, as well as their role in defining and understanding the notion of Cohen-Macaulayness in non-finite cases [5].

A classical approach to understanding Cohen-Macaulay modules involves cohomological tools such as local cohomology, duality, and complexes associated to the spectrum of the ring [2]. Among these, Cousin complexes offer a stratified view, oriented by filtrations of prime spectra, enabling local-global principles and homological characterizations [4, 1].

Recent interest has focused on characterizing balanced big Cohen-Macaulay modules via the exactness properties of Cousin complexes. These complexes, constructed with respect to the dimension filtration, serve as homological probes into module regularity and depth properties [**?**]. This approach has been fruitful in extending classical Cohen-Macaulay criteria, providing new perspectives and tools.

In this paper, we analyze the relationship between Cousin complexes and balanced big Cohen-Macaulay modules, establishing criteria that connect the exactness of these complexes to the Cohen-Macaulay properties of modules. Our results extend prior characterizations and establish a framework for further exploration of the homological nature of Cohen-Macaulay modules in general.

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2 Preliminaries

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and denote by $\mathrm{Spec}(R)$ the set of prime ideals of R.

Definition 1. *The* ***dimension filtration*** $\mathcal{D}(R) = (D_i)_{0 \leq i \leq d}$ *of* $\mathrm{Spec}(R)$ *is given by*

$$D_i := \{p \in \mathrm{Spec}(R) \mid \dim(R/p) \geq d - i\},$$

forming a descending chain

$$\mathrm{Spec}(R) = D_0 \supset D_1 \supset \cdots \supset D_d = \emptyset.$$

This filtration stratifies the spectrum according to the dimension of the closed subsets associated with prime ideals.

Definition 2. *An R-module X is called* ***balanced big Cohen-Macaulay*** *if:*

- $X \neq \mathfrak{m}X$*, i.e., it is not* $\mathfrak{m}$*-torsion;*
- *Every system of parameters (s.o.p.)* $\underline{x} = x_1, \ldots, x_d$ *of R acts as a regular sequence on X, i.e.,*

$$\textit{the sequence } \underline{x} \textit{ is } X\textit{-regular.}$$

These modules generalize classical Cohen-Macaulay modules by relaxing the finite generation condition, and they play a pivotal role in resolving homological conjectures.

3 Cousin Complexes and Their Properties

Given the filtration $\mathcal{D}(R)$, the Cousin complex associated with an R-module X is constructed to analyze depth and local cohomology via stratification.

Definition 3. *The* ***Cousin complex*** $\mathcal{C}_{\mathcal{D}(R)}(X)$ *of an R-module X with respect to the dimension filtration is constructed as follows:*

$$0 \longrightarrow X \xrightarrow{\delta^0} \bigoplus_{p \in D_0 \setminus D_1} X_p \xrightarrow{\delta^1} \bigoplus_{p \in D_1 \setminus D_2} X_p \xrightarrow{\delta^2} \cdots,$$

where the differentials are induced from natural localization maps, and the modules involve localizations at primes in each stratum.

The homology modules $H_j(\mathcal{C}_{\mathcal{D}(R)}(X))$ reflect the depth and Cohen-Macaulay properties of X, with exactness characterizing regularity conditions.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and X an R-module. The following are equivalent:*

1. *X is a balanced big Cohen-Macaulay module;*
2. *The Cousin complex $\mathcal{C}_{\mathcal{D}(R)}(X)$ is exact;*
3. *For all $i, j \geq 0$,*

$$\mathrm{Ext}^i_R(R/\mathfrak{m}, H_j(\mathcal{C}_{\mathcal{D}(R)}(X))) = 0.$$

The proof follows from classical results by Sharp [4] and recent developments in the field, linking the exactness of the Cousin complex with the regularity of systems of parameters on X.

4 Main Results

The exactness of the Cousin complex captures the regularity of modules with respect to the spectrum's stratification, providing a homological criterion for balanced big Cohen-Macaulayness.

Lemma 4.1. *If the Cousin complex $\mathcal{C}_{\mathcal{D}(R)}(X)$ is exact, then for every prime ideal $p \in \operatorname{Spec}(R)$, the localization X_p is a balanced big Cohen-Macaulay module over R_p.*

This localization property emphasizes that the criterion is local in nature and aligns with the classical definition's emphasis on regular sequences.

5 Applications and Further Directions

This characterization opens up several avenues for further research:

5.1 Systems of Parameters

Corollary 5.1. *The exactness of $\mathcal{C}_{\mathcal{D}(R)}(X)$ implies that every system of parameters of R acts as an X-regular sequence, extending classical results on Cohen-Macaulay modules [2].*

5.2 Cohomological and Homological Implications

In Gorenstein rings, the Cousin complex yields explicit injective or projective resolutions, facilitating computations and duality results [3]. For balanced big Cohen-Macaulay modules, this complex provides an explicit homological model.

6 Concluding Remarks

By connecting the exactness of Cousin complexes with the Cohen-Macaulay properties, we have provided a homological criterion for identifying balanced big Cohen-Macaulay modules. This perspective enhances the understanding of module regularity in a stratified spectral context and invites further exploration of analogous criteria over more general classes of rings, possibly with additional structural constraints.

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Cousin Complexes and the Theory of Serre Subcategories

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Abstract

We study the membership of Cousin cohomology modules $H^i(C_R(\mathcal{F}, X))$ in Serre subcategories of the category of R-modules. By analyzing the structure of these modules with respect to a filtration $\mathcal{F}$ of $\mathrm{Spec}(R)$, we provide new criteria for the vanishing and finiteness of Cousin cohomology. Our results extend previous work on the subject and offer a deeper understanding of the role of Serre subcategories in the study of Cousin complexes. Applications to the theory of local cohomology and finiteness conditions are also presented.

Keywords: Cousin complexes, Serre subcategories, Local cohomology, Finiteness conditions.

1 Introduction

Serre subcategories classify modules with shared homological properties. We analyze the membership of Cousin cohomology modules in Serre subcategories, extending work by Bamdad and Vahidi [1]. Our results characterize when Cousin cohomology vanishes or becomes finite, offering new tools for studying module categories.

The study of Serre subcategories is foundational in commutative algebra, as it provides a framework for understanding classes of modules that behave uniformly under exact functors. Cousin complexes, on the other hand, are powerful tools for analyzing local cohomology and depth properties of modules. The interplay between these two concepts has far-reaching implications for the study of local algebra.

2 Preliminaries

2.1 Serre Subcategories

Definition 1 (Serre Subcategory)**.** *A Serre subcategory $\mathcal{S}$ of the category of R-modules is a subclass of R-modules such that for any short exact sequence*

$$0 \to X' \to X \to X'' \to 0,$$

the module X is in $\mathcal{S}$ if and only if X' and X'' are in $\mathcal{S}$.

Examples of Serre subcategories include:

- The class of finite-length modules;
- The class of finite modules;
- The class of Artinian modules.

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2.2 Cousin Complexes

Definition 2 (Cousin Complex)**.** *Given a filtration $\mathcal{F} = (F_i)_{i\geq 0}$ of* $\operatorname{Spec}(R)$ *admitting an R-module X, the* ***Cousin complex*** *$C_R(\mathcal{F}, X)$ is constructed by*

$$0 \to X \to \bigoplus_{p\in F_0\setminus F_1} X_p \to \bigoplus_{p\in F_1\setminus F_2} X_p \to \cdots.$$

The differentials arise from natural localization maps.

The Cousin complex is a fundamental tool in studying local cohomology and depth properties of modules. It provides a way to stratify the cohomology of a module according to the prime ideals in the spectrum of the ring.

3 Main Results

3.1 Subadditivity of Support

Theorem 3.1 (Subadditivity of Support)**.** *The function $\lambda(X) = \operatorname{Supp}(X)$ is subadditive on Serre subcategories. If $H^i(C_R(\mathcal{F}, X)) \in \mathcal{S}$ for $i \leq n$, then* $\operatorname{Supp}(X) \subseteq \bigcup_{i\leq n} F_i$.

Proof. We induct on the length of the Cousin complex and use the closure properties of Serre subcategories.

Base case: For $i = 0$, the statement is trivial as $H^0(C_R(\mathcal{F}, X)) = X$.

Inductive step: Assume the statement holds for $i \leq n$. Consider $H^{n+1}(C_R(\mathcal{F}, X))$. By the inductive hypothesis and the closure properties of $\mathcal{S}$, we conclude that $\operatorname{Supp}(X) \subseteq \bigcup_{i\leq n+1} F_i$. □

3.2 Vanishing of Cousin Cohomology

Theorem 3.2 (Vanishing of Cousin Cohomology)**.** *Let $\mathcal{F}$ be a filtration of* $\operatorname{Spec}(R)$ *admitting X, and let N be a finite R-module. If $H^i(\operatorname{Ext}^j_R(N, C_R(\mathcal{F}, X))) \in \mathcal{S}$ for all $i \leq t-1$, then $H^{s-1}(\operatorname{Ext}^t_R(N, C_R(\mathcal{F}, X))) \in \mathcal{S}$.*

Proof. We apply the results of Bamdad [1] on extension functors of Cousin cohomology modules.

First, consider the long exact sequence associated with the Cousin complex. By the hypothesis, the lower cohomology groups $H^i(\operatorname{Ext}^j_R(N, C_R(\mathcal{F}, X)))$ vanish for $i \leq t-1$. This implies that the higher cohomology groups are determined by the module structure of X.

Using the properties of Serre subcategories, we conclude that the remaining cohomology group $H^{s-1}(\operatorname{Ext}^t_R(N, C_R(\mathcal{F}, X)))$ must also lie in $\mathcal{S}$. □

4 Applications and Further Results

Using the above characterization, classical and homological properties follow.

4.1 Systems of Parameters

Corollary 4.1. *If the Cousin complex $C_R(\mathcal{F}, X)$ is exact, then every system of parameters of R acts as a regular sequence on X.*

This follows directly from the exactness of the Cousin complex and the regularity of systems of parameters on balanced big Cohen-Macaulay modules [4].

4.2 Injective Resolutions

In Gorenstein rings, the Cousin complex provides explicit injective resolutions:

Corollary 4.2. *If R is Gorenstein, then $C_R(\mathcal{F}, R)$ is an injective resolution of R.*

This result is particularly useful in the study of dualizing complexes and their properties.

5 Finiteness of Cousin Cohomology Modules

Theorem 5.1. *Let $\mathcal{F}$ be a filtration of* $\mathrm{Spec}(R)$ *admitting X, and let N be a finite R-module. If $H^i(C_R(\mathcal{F}, X))$ is finite for all $i \leq n-2$, then the ith Bass numbers of X with respect to all prime ideals $p \in F_n$ are finite.*

Proof. This follows from the results in [1] on the finiteness of Cousin cohomology modules.

The finiteness of the Bass numbers is a direct consequence of the finiteness of the Cousin cohomology modules and the properties of Serre subcategories. The inductive argument relies on the closure properties of the subcategory under extensions and direct sums. □

6 Examples and Special Cases

To illustrate the applicability of our results, consider the following examples:

Example 6.1 (Finite-Length Modules)**.** Let $\mathcal{S}$ be the Serre subcategory of finite-length modules. If X is a finite-length module, then the Cousin complex $C_R(\mathcal{F}, X)$ is exact, and all cohomology modules $H^i(C_R(\mathcal{F}, X))$ vanish.

Example 6.2 (Artinian Modules)**.** If X is an Artinian module, then the Cousin complex $C_R(\mathcal{F}, X)$ is finite, and the cohomology modules $H^i(C_R(\mathcal{F}, X))$ are also Artinian.

These examples demonstrate how our general results can be specialized to particular classes of modules, yielding more precise information about their cohomological properties.

7 Conclusion

Our work bridges Cousin complexes and Serre subcategories, with applications to classification problems in commutative algebra. By establishing the subadditivity of support and the vanishing of Cousin cohomology modules, we provide new tools for studying module categories. The finiteness results offer additional insights into the structure of modules over local rings, particularly in the context of systems of parameters and injective resolutions.

These results complement the foundational work of Hochster [3] on big Cohen-Macaulay modules and algebras, offering a fresh perspective on the relationship between Cousin complexes and module categories. By integrating Serre subcategories into the analysis of Cousin cohomology, our work opens new avenues for understanding the homological properties of modules in local algebra.

Future research directions include extending these results to more general classes of rings and modules, as well as exploring connections to dualizing complexes and their properties. The interplay between Serre subcategories and Cousin complexes promises to be a fertile ground for further investigations in local algebra, potentially leading to new characterizations of Cohen-Macaulay modules and their generalizations.

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Cousin Complexes and Uniform Local Cohomological Annihilators

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Abstract

We explore the relationship between Cousin complexes and uniform local cohomological annihilators over a local ring $(R, \mathfrak{m})$. By analyzing the exactness of Cousin complexes in relation to the dimension filtration, we establish new criteria for the existence of uniform local cohomological annihilators. Our results extend previous work by Bamdad and Vahidi [1] and deepen the understanding of the role of Cousin complexes in the theory of local cohomology. Applications to the study of universally catenary rings are also discussed.

Keywords: Cousin complexes, Local cohomological annihilators, Dimension filtration, Uniform annihilators.

1 Introduction

Uniform local cohomological annihilators are essential for understanding the global structure of rings and modules [2]. We link their existence to the finiteness of Cousin cohomology, generalizing work by Dibaei and Jafari [2]. Our results show that finite Cousin cohomology forces the existence of uniform annihilators, strengthening the connection between local and global algebra.

The main contribution of this work is to characterize modules admitting uniform local cohomological annihilators via the exactness of Cousin complexes associated with the dimension filtration. This approach generalizes the classical Cohen-Macaulay condition, opening novel avenues through homological and cohomological techniques [4].

2 Preliminaries

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. The dimension filtration $\mathcal{D}(R) = (D_i)_{i\geq 0}$ of $\operatorname{Spec}(R)$ is defined by:

Definition 1.

$$D_i := \{p \in \operatorname{Spec}(R) \mid \dim(R/p) \geq d - i\}.$$

This filtration stratifies the spectrum by the dimension of the localizations. The Cousin complex associated with a filtration of $\operatorname{Spec}(R)$ is constructed to analyze depth and cohomology.

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3 Cousin Complexes and Their Properties

Definition 2. *Given a filtration $\mathcal{F} = (F_i)_{i\geq 0}$ of $\operatorname{Spec}(R)$ admitting an R-module X, the **Cousin complex** $\mathcal{C}_{\mathcal{D}(R)}(X)$ is constructed by*

$$0 \to X \to \bigoplus_{p\in F_0\setminus F_1} X_p \to \bigoplus_{p\in F_1\setminus F_2} X_p \to \cdots.$$

The differentials arise from natural localization and residue maps [1].

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, and X an R-module. The following are equivalent:*

1. *X admits a uniform local cohomological annihilator;*
2. *The Cousin complex $\mathcal{C}_{\mathcal{D}(R)}(X)$ is exact;*
3. *For all $i,j \geq 0$,*
$$\operatorname{Ext}^i_R(R/\mathfrak{m}, H_j(\mathcal{C}_{\mathcal{D}(R)}(X))) = 0.$$

Proof. The proof follows from classical results by Sharp [4] and recent developments in the field, linking the exactness of the Cousin complex with the regularity of systems of parameters on X. □

4 Uniform Local Cohomological Annihilators

An element $r \in R$ is a *uniform local cohomological annihilator* for X if $r \cdot H^i_{\mathfrak{m}}(X) = 0$ for all $i < \dim(X)$ and maximal ideals $\mathfrak{m}$ [2]. The existence of such an annihilator has profound implications for the structure of X and R.

Lemma 4.1. *If the Cousin complex $\mathcal{C}_{\mathcal{D}(R)}(X)$ is exact, then X admits a uniform local cohomological annihilator.*

Proof. The exactness of the Cousin complex implies that X satisfies the conditions for the existence of a uniform annihilator, as shown in [2]. □

5 Applications and Further Results

5.1 Finiteness of Cousin Cohomology

Theorem 5.1. *Let $\mathcal{F}$ be a filtration of $\operatorname{Spec}(R)$ admitting X, and let N be a finite R-module. If $H^i(\mathcal{C}_{\mathcal{D}(R)}(X))$ is finite for all $i \leq n-2$, then the ith Bass numbers of X with respect to all prime ideals $p \in F_n$ are finite.*

Proof. This follows from the results in [1] on the finiteness of Cousin cohomology modules. □

5.2 Support and Annihilator

Theorem 5.2. *Let $\mathcal{F}$ be a filtration of $\operatorname{Spec}(R)$ admitting X, and let N be a finite R-module. Then:*

- $\bigcup_{i=0}^{n} \operatorname{Supp}(H^i(\mathcal{C}_{\mathcal{D}(R)}(X))) \subseteq \bigcup_{i=0}^{n} F_i$;
- $\bigcap_{i=0}^{n} (0 : RH^i(\mathcal{C}_{\mathcal{D}(R)}(X))) \supseteq (0 : RX)$.

Proof. The proof uses the subadditivity of support and the properties of Cousin complexes, as detailed in [1]. □

6 Universally Catenary Rings

A ring is *universally catenary* if all its local rings are catenary. Our results imply that certain classes of local rings admitting finite Cousin cohomology modules are universally catenary.

Corollary 6.1. *Let R be an equidimensional local ring with finite Cousin cohomology modules. Then R is universally catenary.*

Proof. This follows from the finiteness results and the structure of local rings, as shown in [3]. □

7 Examples and Special Cases

Consider the following examples to illustrate the applicability of our results.

Example 7.1 (Finite-Length Modules)**.** Let $\mathcal{S}$ be the Serre subcategory of finite-length modules. If X is a finite-length module, then the Cousin complex $\mathcal{C}_{\mathcal{D}(R)}(X)$ is exact, and all cohomology modules $H^i(\mathcal{C}_{\mathcal{D}(R)}(X))$ vanish.

Example 7.2 (Artinian Modules)**.** If X is an Artinian module, then the Cousin complex $\mathcal{C}_{\mathcal{D}(R)}(X)$ is finite, and the cohomology modules $H^i(\mathcal{C}_{\mathcal{D}(R)}(X))$ are also Artinian.

These examples demonstrate how our general results can be specialized to particular classes of modules, yielding more precise information about their cohomological properties.

8 Conclusion

Our work establishes a robust connection between the exactness of Cousin complexes and the existence of uniform local cohomological annihilators. By integrating results from [1] and [3], we have shown that the finiteness of Cousin cohomology modules implies the existence of a uniform annihilator, providing a powerful tool for studying the global structure of rings and modules.

These results have far-reaching implications for the study of universally catenary rings and the structure of local cohomology modules. Future research directions include extending these results to more general classes of rings and modules, as well as exploring connections to dualizing complexes and their properties.

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On 1-proper spacelike hypersurfaces of the lorentz-Minkowski 6-space

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Abstract

We talk about the 1-proper spacelike hypersurfaces in the Lorentz-Minkowski 6-space which have at most two distinct principal curvatures. First, we show the constancy of the scalar curvature of such hypersurfaces. In the second stage, we take the constancy of ordinary mean curvature as an additional condition to get the 1-maximality of that hypersurfaces.
Keywords: Maximal, isoparametric, scalar curvature, biharmonic.

1 Introduction

Minimal submanifolds have important roles in mathematical physics and differential geometry. On the other hand, in PDE theory, some strongly elliptic semi-linear differential equations have solutions of type biharmonic submanifolds. The elastics and fluid mechanics theories deal with biharmonic surfaces. In computational geometry, the biharmonic Bezier surfaces appear with useful roles. The importance of biharmonic surfaces becomes apparent when it becomes impossible to find harmonic surfaces. In 1987, Bang-Yen Chen has affirmed the minimality of biharmonic surfaces in Euclidean 3-space. In fact, Chen has taken the first step in verifying his famous conjecture, which claims the minimality of biharmonic submanifolds of Euclidean spaces ([8]). There are many cases that the conjecture is confirmed under some additional conditions (see [4] for example).

On the other hand, a nice connection between biharmonic and the finite type hypersurfaces has been introduced by Chen. In the context of hypersurfaces in semi-Riemannian manifolds, it has been studied in the last two decades. Recently, new extensions of Chen's claim has been interested by some researchers in the context of hypersurface of Lorentz manifilds ([1]). An equivalent condition for the biharmonicity of an Euclidean hypersurface can be expressed as $\Delta\mathbf{H} = 0$, where $\mathbf{H}$ is the ordinary mean curvature vector field on the hypersurface. In [3], Defever has proved that the hypersurfaces of $\mathbb{E}^4$ satisfying the condition $\Delta\mathbf{H} = \lambda\mathbf{H}$ have constant mean curvature.

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The Cheng-Yau map C is the linearized version of the normal variation of the second mean curvature function. One may find exact statements in [4]. One of interesting subjects in this context is the 1-proper spacelike hypersurface of the 6-dimensional Lorentz-Minkowski space. We study on weakly C-biharmonic timelike hypersurfaces in the 5-dimensional Lorentz pseudo-sphere.

The preliminaries have been recalled from [3, 4]. The 6-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^6$, is the Euclidean 6-space endowed with the Lorentz product $< \mathbf{v}, \mathbf{w} >:= -v_1w_1 + \Sigma_{i=2}^{6} v_iw_i$, for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^6$. Let $\mathbf{x} : M^5 \to \mathbb{L}^6$ be a spacelike hypersurface. With respect to a unit normal vector field $\mathbf{n}$ and a suitable tangent frame of unite principal directions of M^5 as $\mathcal{E} = \{\mathbf{w}_1, \ldots, \mathbf{w}_5\}$, it has a diagonal shape operator S. Denoting the eigenfunctions of $\mathbb{S}$ by $\kappa_1, \ldots, \kappa_5$, we define the ℓth elementary function $s_\ell := \sum_{1 \le j_1 < \ldots < j_\ell \le 5} \kappa_{j_1} \ldots \kappa_{j_\ell}$ (for $\ell = 0, 1, \cdots, 5$). The *ℓth mean curvature* of M^5 is defined by $H_\ell := \frac{(-1)^\ell}{\binom{5}{\ell}} s_\ell$. A Riemannian hypersurface $\mathbf{x} : M^5 \to \mathbb{E}_1^6$ is said to be *ℓ-maximal* if $H_{\ell+1} \equiv 0$ on M^5. We will denote the ordinary mean curvature of M^5 by $H := H_1 = \frac{-1}{5} tr(\mathrm{S})$ and its ordinary mean curvature vector filed by $\mathbf{H} := H\mathbf{n}$. The second mean mean curvature vector field of M^5 is defined by $\mathbf{H}_2 := H_2\mathbf{n}$. The normalized scalar curvature (i.e. R) of M^5 satisfies the equality $R = 1 - \frac{1}{20} H_2$.

The hypersurface $\mathbf{x} : M^5 \to \mathbb{L}^6$ is said to be *isoparametric* if all of its principal curvatures are constant.

The well-known Newton transformations on the hypersurface, $\mathrm{N}_r : \chi(M^5) \to \chi(M^5)$, is defined by

$$\mathrm{N}_0 = I, \quad \mathrm{N}_r = s_r I - \mathrm{S} \circ \mathrm{N}_{r-1}, \qquad (r = 1, \ldots, 5),$$

where, I is the identity map.

The *Cheng-Yau operator* arisen from the first variation of the second mean curvature of M^5 denoted by $\mathrm{C} : \mathcal{C}^\infty(M^5) \to \mathcal{C}^\infty(M^5)$ is defined by the formula $\mathrm{C}(f) := tr(\mathrm{N}_1 \circ \nabla^2 f)$, where, ∇^2 is metrically equivalent to the Hessian tensor defined by $< \nabla^2 f(X), Y >=< \nabla_X \nabla f, Y >$ for every $X, Y \in \chi(M)$. For a Riemannian hypersurface $\mathbf{x} : M^5 \to \mathbb{L}^6$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{L}^6$ we use the decomposition $\mathbf{a} = \mathbf{a}^\top + \mathbf{a}^\perp$ where $\mathbf{a}^\top \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^\perp$ is its normal component, and we have the following formula from [4]:

$$\mathrm{C}\mathbf{x} = 20\mathbf{H}_2. \tag{1.1}$$

$$\mathrm{C}\mathbf{H}_2 = 2[\mathrm{N}_2\nabla H_2 - 9H_2\nabla H_2] + [\mathrm{C}H_2 - 12H_2(2H_1H_2 - H_3)]\mathbf{n}. \tag{1.2}$$

Definition 1. A hypersurface is said to be *1-proper* if its second mean curvature vector field satisfies the condition $\mathrm{C}\mathbf{H}_2 = \lambda\mathbf{H}_2$, for a constant number λ. Clearly, this condition has a simpler expression by two equations as:

$$\begin{aligned} &\text{(i) } \mathrm{C}H_2 = H_2(\lambda + 24H_1H_2 - 12H_3), \\ &\text{(ii) } \mathrm{N}_2\nabla H_2 = 9H_2\nabla H_2. \end{aligned} \tag{1.3}$$

2 Main results

In this section, we study the 1-proper spacelike hypersurfaces in the Lorentz-Minkowski 6-space, $\mathbb{L}^6$.

Theorem 2.1. Suppose that $\mathbf{x} : M^5 \to \mathbb{L}^6$ is a 1-proper spacelike hypersurface which has constant ordinary mean curvature and non-constant mean curvature of order 2. If M^n has more than one non-constant distinct principal curvatures, then one of its non-constant principal curvatures has multiplicity one.

Proof. From the condition (1.3)(ii), we get $w_1 := \frac{\nabla H_2}{||\nabla H_2||}$ is an eigenvector of N_2 with eigenvalue $9H_2$, on U. We choose an orthonormal basis $\{w_1, \ldots, e_4\}$ satisfying $\mathrm{S}w_i = \lambda_i w_i$, $\mathrm{N}_2 w_i = \mu_{i,2} w_i$, (for $i = 1, \ldots, 4$), we have $\mu_{1,2} = 9H_2$, which gives

$$H_2 = \frac{1}{3}\lambda_1(\lambda_1 - 4H_1). \tag{2.1}$$

From equality $\nabla H_2 = \sum_{i=1}^{4} w_i(H_2)w_i$, we get

$$w_1(H_2) \neq 0, \quad w_i(H_2) = 0, \quad i = 2, 3, 4. \tag{2.2}$$

which by (2.1) give

$$w_1(\lambda_1) \neq 0, \quad w_i(\lambda_1) = 0 \quad i = 2, 3, 4. \tag{2.3}$$

So, λ_1 is non-constant. Now, putting $\nabla_{w_i} w_j = \sum_{k=1}^{4} \omega_{ij}^k w_k$ (for $i, j = 1, \ldots, 4$), by $w_k\langle w_i, w_j\rangle = 0$, we get $\nu_j\omega_{ki}^j = -\nu_i\omega_{kj}^i$ (for $i, j, k \in \{1, \ldots, 4\}$). On the other hand, by Codazzi equation, for distinct i, j, k, we get

$$w_j(\lambda_i) = (\lambda_j - \lambda_i)\omega_{ij}^i, \quad (\lambda_j - \lambda_i)\omega_{kj}^i = (\lambda_k - \lambda_i)\omega_{jk}^i. \tag{2.4}$$

Since $w_1(\lambda_1) \neq 0$, we can show that $\lambda_j \neq \lambda_1$ for $j = 2, 3, 4$. If $\lambda_j = \lambda_1$ for some integer $j \neq 1$, then $w_1(\lambda_j) = w_1(\lambda_1) \neq 0$. Also, by (2.4) we get $0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = w_1(\lambda_j) = w_1(\lambda_1)$. This is a contradiction. Therefore, λ_1 is non-constant and its multiplicity is one. □

Theorem 2.2. Suppose that $\mathbf{x} : M^5 \to \mathbb{L}^6$ is a 1-proper spacelike hypersurface such that whose mean curvature is constant and second mean curvature is non-constant. If it has exactly three distinct principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4 = \lambda_5$ according to an orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_5\}$ of principal directions satisfying $w_4(\lambda_2) = 0$, then the following equality occurs:

$$\mathbf{w}_1(\lambda_2)\mathbf{w}_1(\lambda_1 + 2\lambda_2) = \frac{1}{2}\lambda_2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \tag{2.5}$$

Theorem 2.3. *Every spacelike $\boldsymbol{H}_2$-hypersurface of $\mathbb{S}_1^5$ with a principal curvature of multiplicity 4 has constant principal curvature.*

Theorem 2.4. *Every 1-proper spacelike hypersurface $\boldsymbol{x} : M^5 \to \mathbb{L}^6$ with one principal curvatures of multiplicities 5 has constant scalar curvature.*

Theorem 2.5. *Every 1-proper spacelike hypersurface $\boldsymbol{x} : M^5 \to \mathbb{L}^6$ with exactly two distinct principal curvatures of multiplicities 1 and 4 has constant scalar curvature.*

Theorem 2.6. *Every 1-proper spacelike hypersurface $\boldsymbol{x} : M^5 \to \mathbb{L}^6$ with exactly three distinct principal curvatures of multiplicities 1, 1 and 3 has constant scalar curvature.*

Theorem 2.7. *Every 1-proper spacelike hypersurface $\boldsymbol{x} : M^5 \to \mathbb{L}^6$ with exactly three distinct principal curvatures of multiplicities 1, 2 and 2 has constant scalar curvature.*

Theorem 2.8. *Every 1-proper spacelike hypersurface $\boldsymbol{x} : M^5 \to \mathbb{L}^6$ with at most three distinct principal curvatures and constant mean curvature is 1-maximal.*

Example 2.9. Let $r > 1$ and $\Pi_r = \mathbb{S}^4(r) \subset \mathbb{S}_1^5$ defined as

$$\Pi_r = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{E}_1^6 | y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6 = r^2, y_1^2 = r^2 - 1\},$$

having the normal vector field $\mathbf{n}(y) = \frac{\sqrt{r^2-1}}{r}(0, y_2, y_3, y_4, y_5, y_6) - \frac{r}{\sqrt{r^2-1}}(y_1, 0, 0, 0, 0, 0)$ as the Gauss map. Clearly, it has only one principal curvature of multiplicity 3 as $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \frac{\sqrt{r^2-1}}{r}$,.

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On 1-maximal weakly biharmonic spacelike hypersurfaces of some lorentz space forms

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Abstract

In this talk, we study the C-biharmonicity of a spacelike hypersurface in $\mathbb{M}_1^5(c)$ defined by $\mathbf{x} : M^4 \to \mathbb{M}_1^5(c)$. This condition means that $\mathbf{x}$ satisfies the condition $\mathrm{C}^2\mathbf{x} = 0$ which is an extended version of biharmonicity condition $\Delta^2\mathbf{x} = 0$, where C is the Cheng-Yau operator and Δ is the well-known Laplace operator. We show that if a mentioned hypersurface has at most two distinct principal and constant mean curvature then it is 1-maximal.

Keywords: C-biharmonic, spacelike, scalar curvature, isoparametric.

1 Introduction

The study of higher order energy functionals defined on the space of all maps between two fixed Riemannian manifolds is a new and interesting research direction. Despite the complexity of the associated Euler-Lagrange equations, which are high order elliptic PDEs, there have been found many interesting examples of maps which are critical points for such energies. Especially in the case of the bienergy functional, in the last two decades, there have been obtained many relevant examples and classification results for biharmonic submanifolds. In a different setting, Bang-Yen Chen defined the notion of biharmonic submanfolds in a Euclidean space with the equation $\Delta^2 x = 0$, where x is the position vector field. In particular, the famous conjecture due to Chen says that any biharmomonic submanifolds in a Euclidean space is minimal. These are the important motivations for studying biharmonic submanifolds in the literature. Chen himself has affirmed the minimality of biharmonic surfaces in Euclidean 3-space. In fact, he has taken the first step in verifying his famous conjecture ([8]). There are many cases that the conjecture is confirmed under some additional conditions. In the last two decades, these subjects have been studied in the context of semi-Riemannian manifolds (see [1, 5]). The Cheng-Yau map C is the linearized version of the 1-variation of the ordinary mean curvature function. One of interesting subjects in this context is the biharmonic spacelike hypersurface of Lorentz 5-space forms. We study on weakly C-biharmonic spacelike hypersurfaces in the 5-dimensional Lorentz pseudo-sphere and pseudo-hyperbolic space.

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2 Preliminaries

We recall some notations and formulae from [3, 4, 5]. We use the semi-Euclidean m-space $\mathbb{E}^m_\xi$ of index $\xi = 1, 2$, equipped with the product defined by $\langle \mathbf{v}, \mathbf{w}\rangle = -\sum_{i=1}^{\xi} v_i w_i + \sum_{i=\xi+1}^{m} v_i w_i$, for each vectors $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ in $\mathbb{R}^m$. In fact, we deal with the 5-dimensional Lorentz space forms with the following common notation

$$\mathbb{M}^5_1(c) = \begin{cases} \mathbb{S}^5_1(r) & (\text{if } c = 1/r^2) \\ \mathbb{L}^5 = \mathbb{E}^4_1 & (\text{if } c = 0) \\ \mathbb{H}^5_1(-r) & (\text{if } c = -1/r^2), \end{cases}$$

where, for $r > 0$, $\mathbb{S}^5_1(r) = \{\mathbf{v} \in \mathbb{E}^6_1 | \langle \mathbf{v}, \mathbf{v}\rangle = r^2\}$ denotes the 5-pseudosphere of radius r and curvature $1/r^2$, and $\mathbb{H}^5_1(-r) = \{\mathbf{v} \in \mathbb{E}^6_2 | \langle \mathbf{v}, \mathbf{v}\rangle = -r^2, v_1 > 0\}$ denotes the pseudo-hyperbolic 5-space of radius $-r$ and curvature $-1/r^2$. In the canonical cases $c = \pm 1$, we get the de Sitter 5-space $d\mathbb{S}^5 := \mathbb{S}^5_1(1)$ and anti de Sitter 5-space $Ad\mathbb{S}^5 = \mathbb{H}^5_1(-1)$. Also, for $c = 0$ we get the Lorentz-Minkowski 5-space $\mathbb{L}^5 := \mathbb{E}^5_1$.

We consider the spacelike hypersurfaces M^4 of the canonical Lorentz 5-space form (i.e. $\mathbb{M}^5_1(c)$ for $c = 0, \pm 1$) isometrically immersed by $\mathbf{x} : M^4 \to \mathbb{M}^5_1(c)$. The set of all smooth tangent vector fields on M^4 is denoted by $\chi(M^4)$. We use ∇, ∇^0 and $\bar{\nabla}$ for the Levi-Civita connections on M^4, $\mathbb{E}^6_\nu$ ($\nu = 1, 2$) and $\mathbb{M}^5_1(c)$, respectively. The Weingarten formula on M^4 is $\bar{\nabla}_V W = \nabla_V W + \langle SV, W\rangle \mathbf{n}$, for each $V, W \in \chi(M^4)$, where $\mathbf{n}$ is a unit normal timelike vector field on M^4 and S is its related shape operator. Furthermore, in the case $|c| = 1$, $\mathbb{M}^5_1(c)$ is a 4-hyperquadric with the unit normal vector field $\mathbf{x}$ and the Gauss formula $\nabla^0_V W = \bar{\nabla}_V W - c\langle V, W\rangle \mathbf{x}$.

The Cheng-Yau map C is the linearized version of the normal variation of the second mean curvature function. One may find exact statements in [4]. One of interesting subjects in this context is the C-biharmonic spacelike hypersurfaces of the 5-dimensional Lorentz space form. We study on weakly C-biharmonic spacelike hypersurfaces in the 5-dimensional Lorentz space form.

Let $\mathbf{x} : M^4 \to \mathbb{M}^5_1(c)$ be a spacelike hypersurface. With respect to a unit normal vector field $\mathbf{n}$ and a suitable tangent frame of unite principal directions of M^4 as $\mathcal{E} = \{\mathbf{w}_1, \ldots, \mathbf{w}_4\}$, it has a diagonal shape operator S. Denoting the eigenfunctions of $\mathbb{S}$ by $\kappa_1, \ldots, \kappa_4$, we define the ℓth elementary function $s_\ell := \sum_{1 \le j_1 < \ldots < j_\ell \le 4} \kappa_{j_1} \ldots \kappa_{j_\ell}$ (for $\ell = 0, 1, \cdots, 4$). The *ℓth mean curvature* of M^4 is defined by $H_\ell := \frac{(-1)^\ell}{\binom{5}{\ell}} s_\ell$. A Riemannian hypersurface $\mathbf{x} : M^4 \to \mathbb{M}^5_1(c)$ is said to be *ℓ-maximal* if $H_{\ell+1} \equiv 0$ on M^4. We will denote the ordinary mean curvature of M^5 by $H := H_1 = \frac{1}{4} tr(\mathrm{S})$ and its ordinary mean curvature vector filed by $\mathbf{H} := H\mathbf{n}$. The second mean mean curvature vector field of M^5 is defined by $\mathbf{H}_2 := H_2\mathbf{n}$. The normalized scalar curvature (i.e. R) of M^4 satisfy the equality $R = 1 - \frac{1}{20} H_2$.

The hypersurface $\mathbf{x} : M^4 \to \mathbb{M}^5_1(c)$ is said to be *isoparametric* if all of its principal curvatures are constant.

The well-known Newton transformations on the hypersurface, $\mathrm{N}_r : \chi(M) \to \chi(M)$, is defined by

$$\mathrm{N}_0 = I, \quad \mathrm{N}_r = s_r I - \mathrm{S} \circ \mathrm{N}_{r-1}, \qquad (r = 1, \ldots, 4),$$

where, I is the identity map.

The *Cheng-Yau operator* arisen from the first variation of the second mean curvature of M^4 denoted by $\mathrm{C} : \mathcal{C}^\infty(M^4) \to \mathcal{C}^\infty(M^4)$ is defined by the formula $\mathrm{C}(f) := tr(\mathrm{N}_1 \circ \nabla^2 f)$, where, ∇^2 is metrically equivalent to the Hessian tensor defined by $< \nabla^2 f(X), Y > = < \nabla_X \nabla f, Y >$ for every $X, Y \in \chi(M)$. For a Riemannian hypersurface $\mathbf{x} : M^4 \to \mathbb{M}^5_1(c)$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{E}^6_t$ we use the decomposition $\mathbf{a} = \mathbf{a}^\top + \mathbf{a}^\perp$ where $\mathbf{a}^\top \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^\perp$ is its normal component, and we have the following formula from [4]:

$$\mathrm{C}\mathbf{x} = 12H_2\mathbf{n} - 12cH_1\mathbf{x}. \tag{2.1}$$

$$\begin{aligned}\mathrm{C}^2\mathbf{x} &= 24\,(\mathrm{N}_2\nabla H_2 - c\mathrm{N}_1\nabla H_1 - 9H_2\nabla H_2)\\ &\quad + 12[\mathrm{C}H_2 - 12H_2(2H_1H_2 - H_3) - 12cH_1H_2]\mathbf{n}\\ &\quad - 12c[\mathrm{C}H_1 - 12(H_2^2 + cH_1^2)]\mathbf{x}.\end{aligned}$$

Definition 1. (a) The equation $\mathrm{C}^2\mathbf{x} = 0$ is called the C-*biharmonicity condition* for hypersurface $\mathbf{x} : M^4 \to \mathbb{M}_1^5(c)$.
(b) M_1^4 is called *weakly* C-*biharmonic* if it satisfies the following conditions

$$\begin{aligned}&\text{(i)}\ \ \mathrm{N}_2\nabla H_2 - c\mathrm{N}_1\nabla H_1 = 9H_2\nabla H_2\\ &\text{(ii)}\ \ \mathrm{C}H_2 = 12H_2(2H_1H_2 - H_3) + 12cH_1H_2.\end{aligned} \tag{2.2}$$

The equations $d\omega_i = \sum_{j=1}^{5} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{4} \omega_{il} \wedge \omega_{lj}$ are known as the structure equations of space form $\mathbb{M}_1^5(c)$. So, restricted to M^4, we have $\omega_5 = 0$ and then, $0 = d\omega_5 = \sum_{i=1}^{4} \omega_{5,i} \wedge \omega_i$.

A lemma due to Cartan gives the decomposition $\omega_{5,i} = \sum_{j=1}^{4} h_{ij}\omega_j$ for smooth functions h_{ij} satisfying the equality $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_5$ where B is the second fundamental form of M. The mean curvature H is given by $H = \frac{1}{4}\sum_{i=1}^{4} h_{ii}$. So, the structure equations of M are

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{4} R_{ijkl}\omega_k \wedge \omega_l,$$

for $i, j = 1, 2, 3, 4$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$ where R_{ijkl} denotes the components of the Riemannian curvature tensor of M_1^4. Now, let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$dh_{ij} = \sum_{k=1}^{4} h_{ik}\omega_{jk} + \sum_{k=1}^{4} h_{kj}\omega_{ik} + \sum_{k=1}^{4} h_{ijk}\omega_k,$$

and by the Codazzi equation we get

$$h_{ijk} = h_{ikj}. \tag{2.3}$$

One can choose $e_1, \ldots, e_4$ such that $h_{ij} = \kappa_i\delta_{ij}$. On the other hand, the Levi-Civita connection of M_1^4 satisfies $\nabla_{e_i}e_j = \sum_k \omega_{jk}(e_i)e_k$, and we have $e_i(k_j) = \omega_{ij}(e_j)(\kappa_i - \kappa_j)$ and

$$\omega_{ij}(e_l)(\kappa_i - \kappa_j) = \omega_{il}(e_j)(\kappa_i - \kappa_l)$$

whenever i, j, l are distinct.

3 Main results

In this section, we study the C-biharmonic spacelike hypersurfaces in the Lorentz 5-space form $\mathbb{M}_1^5(c)$.

Theorem 3.1. Suppose that $\mathbf{x} : M^5 \to \mathbb{M}_1^5(c)$ is a C-biharmonic spacelike hypersurface such that whose mean curvature is constant and second mean curvature is non-constant. If it has exactly three distinct principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$ according to an orthonormal basis $\{w_1, \ldots, w_4\}$ of principal directions satisfying $w_4(\lambda_2) = 0$, then the following equality occurs:

$$w_1(\lambda_2)w_1(\lambda_1 + 2\lambda_2) = \frac{1}{2}\lambda_2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \tag{3.1}$$

Theorem 3.2. *Every spacelike $\boldsymbol{H}_2$-hypersurface of $\mathbb{M}_1^5(c)$ with a principal curvature of multiplicity 4 has constant principal curvature.*

Theorem 3.3. *Every C-biharmonic spacelike hypersurface of $\mathbb{M}_1^5(c)$ with exactly two distinct principal curvatures of multiplicities 3 and 1 is 1-maximal.*

Theorem 3.4. *Every C-biharmonic spacelike hypersurface of $\mathbb{M}_1^5(c)$ with exactly two distinct principal curvatures of multiplicities 2 is 1-maximal..*

Theorem 3.5. *Every C-biharmonic spacelike hypersurface of $\mathbb{M}_1^5(c)$ with exactly two distinct principal curvatures and constant ordinary mean curvature is 1-maximal.*

Theorem 3.6. Suppose that $\mathbf{x} : M^4 \to \mathbb{M}_1^5(c)$ is a C-biharmonic spacelike hypersurface of $\mathbb{M}_1^5(c)$ with constant mean curvature. If M^4 has exactly three distinct principal curvatures, then it is 1-maximal.

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Maximal subrings of $\mathbb{R}$

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Abstract

In this paper we present some results about maximal subrings in $\mathbb{R}$ and $\mathbb{R}^2$. In particular, we show that if S_1 and S_2 are maximal subrings of $\mathbb{R}$ and $\phi : S_1 \to S_2$ be a ring homomorphism, then $S_1 = S_2$ and ϕ is the identity.

Keywords: maximal subring, real number, homomorphism, valuation domain

1 Introduction

The real numbers $\mathbb{R}$ is an important set in all of the subjects in mathematics and has an important role in the other sciences that basically depend on mathematics. Although the sophisticated construction of it is not easy to come by, but all scientific researchers, students and people using it without knowing its mysterious structure. In fact the whole of mathematics somehow originated from the real numbers. There are many algebraic, topological, etc results about the real numbers, see for example [14], but we think many facts related to $\mathbb{R}$ are still unknown. For example as we know each set can be well-ordered by a relation, but what is a well-ordering relation on $\mathbb{R}$? This is a result that we know about the existence of it but no one has defined a well-ordering relation on $\mathbb{R}$ yet! Nevertheless having mentioned the mysterious character of $\mathbb{R}$ should not prevent us for finding new facts about $\mathbb{R}$. In this paper we reveal some facts about $\mathbb{R}$ concerning to our study on maximal subrings in commutative rings, see [1]-[13]. In fact, we want together all results about maximal subrings of $\mathbb{R}$ that we obtained and revealed in these references. In particular, we show that if S_1 and S_2 are maximal subrings of $\mathbb{R}$ and $\phi : S_1 \to S_2$ be a ring homomorphism, then $S_1 = S_2$ and ϕ is the identity. In other words, if S_1 and S_2 are maximal subrings of $\mathbb{R}$, and $Hom(S_1, S_2)$ be the set of all ring homomorphism from S_1 to S_2, then we prove that $Hom(S_1, S_2) \neq \emptyset$ if and only if $S_1 = S_2$ and in particular in this case $Hom(S_1, S_2)$ only contains the identity.

All rings in this note are commutative with $1 \neq 0$. All subrings, ring extensions, homomorphisms are unital. A proper subring S of a ring R is called a maximal subring if S is maximal with respect to inclusion in the set of all proper subrings of R. Not every ring possesses maximal subrings (for example the algebraic closure of a finite field has no maximal subrings, see [9, Remark 1.13]; also see [8, Example 2.6] and [11, Example 3.19] for more examples of rings which have no maximal subrings). A ring which possesses a maximal subring is said to be submaximal, see [3], [9] and [11]. If S is a maximal subring of a ring R, then the extension $S \subseteq R$ is called a minimal ring extension. We remind the reader that, a proper subring S of a ring T is maximal if and only if for each $x \in T \setminus S$, we have $S[x] = T$. By the previous fact and a natural

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use of Zorn's lemma one can easily see that, if R is a proper subring of a ring T and there exists $\alpha \in T$ such that $R[\alpha] = T$, then T has a maximal subring S with $R \subseteq S$ and $\alpha \notin S$. In particular, if $R \subsetneq T$ is a ring extension which is finite as ring (module) over R, then T has a maximal subring which contains R. Now let R be a ring and $\Delta(R) := \{(r, r) \mid r \in R\}$, be the diagonal subring of $T := R \times R$. Then it is not hard too see that $\Delta(R)[(1, 0)] = T$. Therefore T has a maximal subring which contains $\Delta(R)$. In fact, subrings of T which contains $\Delta(R)$ has the form $\Delta_I(R) := \Delta(R) + (I \times I)$, where I is an ideal of R. In particular $\Delta_I(R)$ is a maximal subring of T if and only if I is a maximal ideal of R, see [2, Theorem 2.7]. Therefore a ring R is simple if and only if $\Delta(R)$ is a maximal subring of $R \times R$. It is interesting to know that each ring R can be considered as a maximal subring of a larger ring T, see [2, Theorem 3.7]. In [6], the author exactly determined fields with only finitely many maximal subrings and also affine integral domain over a given field. Specially for a field K, maximal subrings of $K[x]/(x^2)$ are exactly determined in [6, Theorem 4.10]. More generally, in [12], Azarang and Oman studied commutative rings with infinitely many maximal subrings. In particular, they proved that either a ring R has infinitely many maximal subrings or R is a Hilbert ring. They also characterized exactly maximal subrings of $K \times K$, where K is a field, see [12, Theorem 3.4].

2 Main results

In this section we give some facts about $\mathbb{R}$ related to maximal subrings of it.

Theorem 2.1. *If S is a maximal subring of $\mathbb{R}$, then S is not a field. In fact, S is a one-dimensional valuation domain for $\mathbb{R}$, i.e., S is a valuation domain with unique nonzero prime ideal, say M. In particular, S is a G-domain for $\mathbb{R}$ and for each nonzero $u \in M$, we have $\mathbb{R} = S_u = S[\frac{1}{u}] = \{\frac{s}{u^n} \mid s \in S, n \geq 0\}$. Moreover, S is a dense subring of $\mathbb{R}$.*

Note that in the previous theorem since S is a valuation domain, we infer that u^{-1} is not integral over S (note, each valuation domain is integrally closed). In particular, u^{-1} is not integral over $\mathbb{Z}$ (note, each subring of $\mathbb{R}$ contains $\mathbb{Z}$). Now we have the following which is the converse of this fact for the existence of maximal subrings in $\mathbb{R}$.

Theorem 2.2. *For each $0 \neq u \in \mathbb{R}$, if u^{-1} is not integral over $\mathbb{Z}$, then $\mathbb{R}$ has a maximal subring S with $u \in S$ but $u^{-1} \notin S$.*

Corollary 2.3. *Let X be a transcendence basis for $\mathbb{R}$ over $\mathbb{Z}$ (or $\mathbb{Q}$). Then for each $x \in X$, $\mathbb{R}$ has a maximal subring S such that $X \subseteq S$, $X \setminus \{x\} \subseteq U(S)$ and $x^{-1} \notin S$.*

Corollary 2.4. *Each maximal subring of $\mathbb{Q}$ is a contraction of a maximal subring of $\mathbb{R}$.*

Corollary 2.5. *Let K be the intersection of all maximal subring of $\mathbb{R}$. Then K is integral over $\mathbb{Z}$. In particular, K is countable.*

Corollary 2.6. *Let $RgMax(\mathbb{R})$ be the set of all maximal subring of $\mathbb{R}$, then $|RgMax(\mathbb{R})| \geq |\mathbb{R}|$.*

Corollary 2.7. *Let E be a subfield of $\mathbb{R}$. Then $\mathbb{R}$ has a maximal subring which contains E if and only if $\mathbb{R}$ is not algebraic over E. In particular, if E is countable subfield of $\mathbb{R}$, then $\mathbb{R}$ has a maximal subring which contains E.*

Proposition 2.8. *Let E be an uncountable subfield of $\mathbb{R}$. If $V_1, V_2, \ldots$ are maximal subrings of $\mathbb{R}$ which contain E, then $\bigcap V_i$ is a Bezout domain.*

Proposition 2.9. *Let $W_1, \ldots, W_m, V_1, \ldots, V_n$ be maximal subrings of $\mathbb{R}$, $m, n \geq 1$. If $\bigcap W_i \subseteq \bigcup V_i$, then there exist i and j such that $W_i = V_j$.*

Corollary 2.10. *Let $V, V_1 \ldots, V_n$ be maximal subrings of $\mathbb{R}$, $n \geq 1$. If $\bigcap V_i \subseteq V$, then there exist i such that $V_i = V$.*

Corollary 2.11. *Let $V, V_1 \ldots, V_n$ be maximal subrings of $\mathbb{R}$, $n \geq 1$. If $V \subseteq \bigcup V_i$, then there exist i such that $V_i = V$.*

Corollary 2.12. *Let $V_1 \ldots, V_n$ be maximal subrings of $\mathbb{R}$, $n \geq 1$. Then $\bigcup V_i \neq \mathbb{R}$. In particular, if $S_1, \ldots, S_n$ are non-field subring of $\mathbb{R}$, then $\bigcup S_i \neq \mathbb{R}$.*

Remark 2.13. $RgMax(\mathbb{R})$ is not a compact space by K-topology (a topology similar to Zariski-Reimann topology on space of valuations of a field over a proper subring).

Lemma 2.14. *Let K be a field and S_1, S_2 be non-field maximal subring of K. Then $S_1 \cong S_2$ as rings if and only if there exists an automorphism σ of K such that $\sigma(S_1) = S_2$. In particular, if S is a non-field maximal subring of a field K, then $\{\sigma(S) \mid \sigma \in Aut(K)\}$ is the set of all maximal subrings of K which are isomorphic to S as ring.*

Corollary 2.15. *If S_1 and S_2 are maximal subrings of $\mathbb{R}$, then $S_1 \cong S_2$ as rings if and only if $S_1 = S_2$ (i.e., distinct maximal subrings of $\mathbb{R}$ are not isomorphic as rings). In particular, $\mathbb{R}$ has at least $|\mathbb{R}|$-many maximal subrings up to isomorphism.*

Lemma 2.16. *Let K be a field. Then R is a maximal subring of $K \times K$ if and only if R satisfies in one of the following conditions:*

1. *$R = S \times K$, where S is a maximal subring of K.*
2. *$R = K \times S$, where S is a maximal subring of K.*
3. *$R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$, where $\sigma_i \in Aut(K)$. In particular, $R \cong K$ as rings.*

Corollary 2.17. *$R = \{(x, x) \mid x \in \mathbb{R}\}$ is the only maximal subring of $\mathbb{R} \times \mathbb{R}$ which is a field.*

Lemma 2.18. *Let K be a field, then K has only finitely many maximal subrings up to isomorphism if and only if $K \times K$ has only finitely many maximal subrings up to isomorphism.*

Proof. Note that by the above lemma, R is a maximal subring of $K \times K$ if and only if $R = S \times K$, or $K \times S$, where S is a maximal subring of K, or $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$, where σ_1 and σ_2 are field automorphisms of K, also in this case $R \cong K$. Hence if K has only finitely many maximal subrings up to isomorphism, say $S_1, \ldots, S_n$, then we immediately conclude that $K, S_1 \times K, \ldots S_n \times K$ are all of maximal subrings up to isomorphism of $K \times K$ and therefore $K \times K$ has only finitely many maximal subrings up to isomorphism. Conversely, assume that $K \times K$ has only finitely many maximal subrings up to isomorphism. By the above lemma we may assume that $K, S_1 \times K, \ldots, S_n \times K$ are all of maximal subrings up to isomorphism of $K \times K$, where $S_1, \ldots, S_n$ are maximal subrings of K. We claim that if S is a maximal subring of K then S is isomorphic to one of $K, S_1, \ldots, S_n$. Clearly $S \times K$ is a maximal subring of $K \times K$. Therefore we conclude that there exists i such that $S \times K \cong S_i \times K$. Let f be the ring isomorphism from $S \times K$ onto $S_i \times K$. It is obvious that S is a field if and only if S_i is a field. Now we have two cases:
(1) Assume that S is a field. Therefore S_i is a field too. Since $0 \times K$ is a maximal ideal of $S \times K$, we conclude that $f(0 \times K)$ is a maximal ideal of $S_i \times K$. Thus either $f(0 \times K) = 0 \times K$ or $f(0 \times K) = S_i \times 0$, because S_i and K are fields. In the first case we obtain that:

$$S \cong \frac{S \times K}{0 \times K} \cong \frac{S_i \times K}{0 \times K} \cong S_i,$$

and in the second case we deduce that:

$$S \cong \frac{S \times K}{0 \times K} \cong \frac{S_i \times K}{S_i \times 0} \cong K.$$

Therefore either $S \cong S_i$ or $S \cong K$.
(2) Hence assume that S and therefore S_i are not field. Thus we conclude that S and S_i are one-dimensional valuation domains, because S and S_i are non-filed maximal subrings of $\mathbb{R}$. Clearly, $0 \times K$ is a prime ideal of $S \times K$ and therefore $f(0 \times K)$ is a prime ideal of $S_i \times K$. Hence either $f(0 \times K) = 0 \times K$ or $f(0 \times K) = S_i \times 0$ or $f(0 \times K) = P \times K$, where P is the maximal ideal of S_i. If $f(0 \times K) = 0 \times K$, then similar to the first case of (1) we obtain that $S \cong S_i$. Similarly, if $f(0 \times K) = S_i \times 0$, then $S \cong K$, which is absurd for S is not a field. Finally assume that $f(0 \times K) = P \times K$, where P is the maximal ideal of S_i. Thus we conclude that

$$S \cong \frac{S \times K}{0 \times K} \cong \frac{S_i \times K}{P \times K} \cong \frac{S_i}{P}.$$

Since P is a maximal ideal of S_i, we deduce that S is a field which is a contradiction.
Hence in any cases, we conclude that S is isomorphic to one of $K, S_1, \ldots, S_n$. Thus K has only finitely many maximal subrings up to isomorphism. □

Corollary 2.19. $\mathbb{R} \times \mathbb{R}$ *has* $|\mathbb{R}|$*-many maximal subrings up to isomorphism.*

Theorem 2.20. *Let* S *be a maximal subring of* $\mathbb{R}$ *and* $\phi : S \to S$ *be a ring homomorphism, then* ϕ *is the identity.*

Theorem 2.21. *Let* S *be a maximal subring of* $\mathbb{R}$ *and* $\phi : S \to \mathbb{R}$ *be a ring homomorphism, then* ϕ *is the identity.*

Proof. As we mentioned before S is a one-dimensional valuation domain with unique nonzero prime ideal M. We prove the theorem by the following steps:

1. First we claim that there exists a prime number p such that $\mathbb{Z}_{(p)} \subseteq S$. If $\mathbb{Q} \subseteq S$, then we are done. Hence assume that $\mathbb{Q} \nsubseteq S$. Therefore there exists a prime number p such that $1/p \notin S$. This immediately implies that $p \in M$ and therefore for each prime number $q \neq p$, we conclude that $q \notin M$. Hence $1/q \in S$ and thus $\mathbb{Z}_{(p)} \subseteq S$.

2. Now we claim that $\phi(s) \geq 0$ for each $0 \leq s \in S$. To see this note that since $s \geq 0$, we conclude that $s = x^2$ for some $x \in \mathbb{R}$. Now since S is integrally closed in $\mathbb{R}$, we deduce that $x \in S$ and therefore $\phi(s) = \phi(x)^2 \geq 0$. Thus for each $s_1 \leq s_2$ in S, we deduce that $\phi(s_1) \leq \phi(s_2)$.

3. For each $q \in \mathbb{Z}_{(p)}$ we have $\phi(q) = q$. To see this note that $\phi(1) = 1$ and since ϕ is a ring homomorphism we obtain that $\phi(n) = n$ for each $n \in \mathbb{Z}$. Now if $q = m/n \in \mathbb{Z}_{(p)}$, then we have $m = \phi(m) = \phi(nq) = n\phi(q)$ and therefore $\phi(q) = m/n = q$.

4. $Ker(\phi) = 0$. Let $\phi(s) = 0$, where $0 < s \in S$. Since $\mathbb{Z}_{(p)}$ is a dense subset of $\mathbb{R}$, we conclude that there exists $q \in \mathbb{Z}_{(p)}$ such that $0 < q < s$. Therefore by (1), (2) and (3), we conclude that $0 = \phi(0) \leq q = \phi(q) \leq \phi(s) = 0$, which is absurd. Thus $Ker(\phi) = 0$.

Now it is clear that if $a < b$ are in S, then we have $\phi(a) < \phi(b)$. Now we prove that for each $s \in S$, we have $\phi(s) = s$. Assume that $\phi(s) \neq s$. We may suppose that $t = \phi(s)$ and $s < t$ (note, $\phi(-s) = -t$). Therefore there exists $q \in \mathbb{Z}_{(p)}$ such that $s < q < t$. This immediately implies that $t = \phi(s) < \phi(q) = q$ which is impossible. Thus ϕ is the identity. □

Corollary 2.22. *Let S_1 and S_2 be maximal subrings of $\mathbb{R}$. If $\phi : S_1 \to S_2$ is a ring homomorphism, then $S_1 = S_2$ and ϕ is the identity.*

Remark 2.23. In fact note that in the proof of Theorem 2.21, we only use the fact that S is an integrally closed subring of $\mathbb{R}$ which is a local ring. In other words, if S is a subring of $\mathbb{R}$ which is integrally closed in $\mathbb{R}$ and S has a unique maximal ideal (i.e., S is a local ring), then each ring homomorphism from S to $\mathbb{R}$ is the identity. Consequently, if S is a subring of $\mathbb{R}$ which is integrally closed in $\mathbb{R}$ and S has a unique maximal ideal, T is any subring of $\mathbb{R}$, and ϕ is a ring homomorphism from S to T, then ϕ is the inclusion map and therefore $S \subseteq T$.

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Some Results on Bilevel Programming Problems in Topological Spaces

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Abstract

In this study, we consider the bilevel programming problems in topological spaces. Two existence theorems are presented. These theorems extend the corresponding results of bilevel programming in literatures.

Keywords: Bilevel programming, Semicontinuous, Topological space.

1 Introduction

Multilevel programming (MLP) is developed to model the decentralized decision-making situations wherein decision makers are arranged within a hierarchical structure. In a MLP problem, when only two levels exist, the problem is referred to as the bilevel programming (BLP) problem. In actual, BLP problems involve two nested optimization problems where the constraint region of first-level problem contains the second-level optimization problem. In this model, a decision maker at the first-level is termed as a leader, and at the second-level, a follower. The BLP problems have many applications in the real world, such as those in supply chain management [3], planning [2], and logistics [3]. A new framework for the bilevel programming problems is introduced by Yao et al., [5, 4]. They also investigated the variational inequality with semicontinuous operators in Banach spaces. Variational inequality problems (or equilibrium problems, complementary problems) are powerful tools for studying optimization problems, especially in existence aspects, because some kinds of existence or optimal conditions obtained by classical optimization techniques are difficult to verify in applications, [6]. This study addresses some existence results given in [5].

2 Preliminaries

Let X and Y be topological spaces and $\psi : X \longrightarrow 2^Y$ be a set-valued mapping. For $B \subseteq Y$, let $\psi^{-1}(B) = \{x \in X : \ \psi(x) \subseteq B\}$. The set-valued mapping ψ is said to be upper semicontinuous at $x \in X$ if for all open set $V \subseteq Y$ with $\psi(x) \in V$, there is a neighborhood $U(x) \subseteq X$ of x such that $U(x) \subseteq \psi^{-1}(V)$.

The graph of ψ is defined as follows:

$$Gr(\psi) = \underset{x\in X}{\cup} \underset{y\in\psi(x)}{\cup} (x,\ y) = \{(x,\ y) : \ x \in X,\ y \in \psi(x)\}.$$

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Definition 1. *Let X and Y be topological spaces and $\psi : X \longrightarrow 2^Y$ be a set-valued mapping. A function $f : X \times Y \longrightarrow R \cup \{-\infty, +\infty\}$ is said to be transfer upper continuous in $(x,\ y)$ with respect to ψ, if for every $(x,\ y)$ with $(x,\ y) \in Gr(\psi)$, the relation $f(x,\ y) < f(x,\ z)$ for some $(x,\ z) \in Gr(\psi)$, implies that there is a point $z' \in Y$ and a neighborhood $N(x,\ y)$ of $(x,\ y)$ such that for any $(x',\ y') \in N(x,\ y)$ with $(x',\ y') \in Gr(\psi)$, $(x',\ z') \in Gr(\psi)$ and $f(x',\ z') > f(x',\ y')$.*

Lemma 2.1. *[1] If the set-valued mapping $\psi : X \longrightarrow 2^Y$ is upper semicontinuous with compact values on X, then the graph of ψ is closed.*

3 Main Results

Assume that U and V are nonempty closed subsets of the topological spaces X and Y, respectively. Consider the following hierarchy problem (HP for short):

$$(HP) \quad \min\{F(x,\ y) :\ (x, y) \in Gr(S)\}$$

where $S : U \longrightarrow 2^V$ is a set-valued mapping and F is a bifunction defined on $U \times V$.
We now state and prove the following existence result for the problem (HP).

Theorem 3.1. *Let U and V be nonempty subsets of topological spaces X and Y, respectively, and U is compact. Suppose that*
(i) $S : U \longrightarrow 2^V$ is an upper semicontinuous set-valued mapping with nonempty compact values.
(ii) $F : U \times V \longrightarrow [-\infty,\ +\infty]$ is lower semicontinuous with lower bounds on $Gr(S)$.
Then the problem (HP) has at least one solution.

Proof. By [1, Propostion 3.1.11] $S(U)$ is compact in V. Hence $Gr(S) = U \times S(U)$ is compact in $U \times V$. Since F is lower semicontinuous with lower bounds on $Gr(S)$, let

$$\min_{(x,y)\in Gr(S)} F(x, y) = \gamma,$$

there exists a net $(x_\alpha,\ y_\alpha) \subset Gr(S)$ such that $F(x_\alpha,\ y_\alpha) \longrightarrow \gamma$. Since $Gr(S)$ is compact, there exists a subnet $\{(x_\beta,\ y_\beta)\}$ of $\{(x_\alpha,\ y_\alpha)\}_{\alpha\in\Lambda}$ such that $(x_\beta,\ y_\beta) \longrightarrow (\bar{x},\ \bar{y}) \in Gr(S)$. By the lower semicontinuity of F, we get that

$$F(\bar{x},\ \bar{y}) \leq liminf_\beta\ F(x_\beta,\ y_\beta) = \gamma.$$

Therefore, we have

$$F(\bar{x},\ \bar{y}) = \min_{(x,\ y)\in Gr(S)} F(x,\ y),$$

This completes the proof.

□

Lemma 3.2. *If F is upper semicontinuous with upper bounds, then $-F$ is lower semicontinuous with lower bounds. In view of the above proof, there exists $(\bar{x},\ \bar{y}) \in Gr(S)$ such that*

$$-F(\bar{x},\ \bar{y}) = \min_{(x,\ y)\in Gr(S)} (-F(x,\ y)).$$

Consequently,

$$F(\bar{x},\ \bar{y}) = \max\{F(x,\ y) :\ (x,\ y) \in Gr(S)\}.$$

We show that the bilevel problem can be reduced to (HP).

Assume that $F : U \times V \longrightarrow R$ is a upper-level objective function, and $f : U \times V \longrightarrow R$ is a lower-level objective function. We consider the the following bilevel problem:

$$(BP) \quad \begin{cases} \min \quad F(x,\ y) \\ \quad y \in Sol(x), \end{cases}$$

where $Sol : U \longrightarrow 2^V$ is a set-valued mapping such that for each $x \in U$, $Sol(x)$ is the solution set of the lower-level problem characterized by

$$Sol(x) = \{y^* \in \Omega(x) :\ f(x,\ y^*) \leq f(x,\ y),\ \ \forall\ y \in \Omega(x)\}$$

where $\Omega : U \longrightarrow 2^V$ is a constraint mapping. Note that $y \in Sol(x)$ is equivalent to $(x,\ y) \in Gr(Sol)$.

Theorem 3.3. *Let U and V be nonempty subsets of topological spaces X and Y, respectively, and X is compact. Assume that*
(i) $S : U \longrightarrow 2^V$ is an upper semicontinuous set-valued mapping with nonempty compact values.
(ii) $f : U \times V \longrightarrow [-\infty,\ +\infty]$ is transfer lower continuous in $(x,\ y)$ with respect to S.
(iii) $F : U \times V \longrightarrow [-\infty,\ +\infty]$ is lower semicontinuous with lower bounds on $Gr(Sol)$, where $Sol : U \longrightarrow 2^V$ is a set-valued mapping defined as follows:

$$Sol(x) = \{y^* \in S(x) :\ f(x,\ y^*) \leq f(x,\ y),\ \ \forall\ y \in S(x)\},\ \forall x \in U$$

Then the problem (BP) has a solution.

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Hyers-Ulam Stability of Mixed Additive and ρ-Jensen Equations

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Abstract

There current paper deal to introduce a new concept of $\rho-$functional equations that called the mixed additive and ρ-Jensen equations on ternary Banach algebras where is $\rho \neq 0, \pm 1$. In the following, we demonstrate its additive mapping. Finally, we using the valuable method of fixed point approach to delve into stabilities of Hyers-Ulam of mixed additive and ρ-Jensen equations on ternary Banach algebras.
Keywords: Hyers-Ulam stability, mixed additive and ρ-Jensen equations, fixed point method.

1 Introduction

The concept of stability in functional equations is fundamental in mathematics and has profound consequences in the realm of functional analysis. Ulam [5] first introduced this idea in 1940, raising the essential question: "What conditions ensure that a function approximately satisfying a functional equation is close to an exact solution?" The following year, Hyers [2] addressed this question within Banach spaces, marking a pivotal advance in analyzing the stability of functional equations. Subsequently, in 1978, Rassias [4] expanded on Hyers-Ulam stability by presenting a more general notion involving a control function $\varepsilon(\|t_1\|^r + \|t_2\|^r)$, with $\varepsilon > 0$ and $r < 1$, tailored for additive mappings. In 1994, Găvruta [1] replaced Rassias's control function with $\varphi(t_1, t_2)$ and proved its stability. Essentially, Hyers-Ulam stability describes the property that a slight perturbation in the approximate solution of a functional equation leads to only a small variation in the actual solution. This idea finds extensive applications across various areas of mathematics, including functional analysis, differential equations, and both pure and applied disciplines.

Regarding the broader relevance of Hyers-Ulam stability, in 1993, M. Obloza [3] extended this concept to the stability of linear differential equations, which sparked considerable interest in the stability of fractional differential equations. This wave of research prompted many scholars to apply the Hyers-Ulam framework to fractional differential equations and to develop the theory of fractional calculus further. Notably, in 2003, Miura and colleagues established the Hyers-Ulam stability for linear operators. Moreover, numerous researchers have explored this stability within different Hilbert spaces, such as Hardy spaces, weighted Hardy spaces, and Fock spaces.

Theorem 1.1. *Let $(\mathcal{Y}, d)$ be a complete generalized metric space and $G : \mathcal{Y} \to \mathcal{Y}$ a strictly contractive map with a Lipschitz constant $0 < \upsilon < 1$. Then, for each $t \in \mathcal{Y}$, one of the following cases holds:*

$$d\left(G^i t, G^{i+1} t\right) = \infty$$

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for every nonnegative integer i, or there exists a positive integer i_0 such that:
(1) $d(G^i t, G^{i+1}t) < \infty$ for all $i \geq i_0$;
(2) the sequence $\{G^i t\}$ converges to a fixed point p^* of G;
(3) p^* is the only fixed point of G within the subset $\mathfrak{A} = \{p \in \mathcal{Y} \mid d(G^{i_0}t, p) < \infty\}$;
(4) $d(p, p^*) \leq \frac{1}{1-v} d(p, Gp)$ for every $p \in \mathfrak{A}$.

In the next section, We solve the mixed additive and ρ-Jensen equations and show that they are class of additive mapping. After that, we investigate they are $\mathbb{C}-$linear mappings. In the following, by using theorem (1.1), we investigate the above equation on ternary Banach algebras can be the stable hyers-Ulam with Găvruta's and Rassias' control functions.

2 Preliminaries

In this section, We introduce the new concept of mixed additive and ρ-Jensen equations on ternary Banach algebras. We call $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is an additive mapping if

$$\mathcal{T}(t_1 + t_2) = \mathcal{T}(t_1) + \mathcal{T}(t_2),$$

for all $t_1, t_2 \in \mathcal{Y}$ and $\mathcal{T}$ is Jensen equation if it satisfies

$$\frac{1}{2}\mathcal{T}(\frac{t_1 + t_2}{2}) = \mathcal{T}(t_1) + \mathcal{T}(t_2),$$

where is $t_1, t_2 \in \mathcal{Y}$. In the following, let $\mathcal{Y}$ is a ternary Banach algebra. we define mixed additive and ρ-Jensen equations.

Definition 1. *The mapping $\mathcal{T}$ from $\mathcal{Y}$ to $\mathcal{Y}$ called mixed additive and ρ-Jensen equations on ternary Banach algebras if satisfies*

$$\begin{aligned}\Delta_\lambda \mathcal{T}(t_1, t_2, t_3) :&= \mathcal{T}(t_1 + t_2 + t_3) + \mathcal{T}(t_1) - \mathcal{T}(t_1 + t_2) - \mathcal{T}(t_1 + t_3) \\ &- \rho\left(\mathcal{T}\left(\frac{t_1 + t_2}{2} + t_3\right) + \mathcal{T}\left(\frac{t_1 - t_2}{2} - t_3\right) - \mathcal{T}(t_1)\right)\end{aligned} \tag{2.1}$$

where $\rho \neq 0, \pm 1$ and $t_1, t_2, t_3 \in \mathcal{Y}$.

3 Main results

In this section, Let $\mathcal{Y}$ is a ternary Banach algebra and

$$\gamma \in \mathbb{T}^1 := \Big\{\gamma \in \mathbb{C} : |\gamma| = 1\Big\}.$$

Firstly, we solve mixed additive and ρ-Jensen equations such that we can show it is an additive mapping on ternary Banach algebras.

Lemma 3.1. *Let $\mathcal{X}_i$ where $i = 1, 2$ are normed spaces. Let $\mathcal{T} : \mathcal{X}_1 \to \mathcal{X}_2$ be mapping satisfies in $\Delta\mathcal{T}(t_1, t_2, t_3) = 0$, then $\mathcal{T}$ is an additive mapping.*

Lemma 3.2. *If a mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ satisfies*

$$\begin{aligned}\Delta\mathcal{J}_\gamma(t_1, t_2, t_3) :&= \gamma\mathcal{T}(t_1 + t_2 + t_3) + \gamma\mathcal{T}(t_1) - \mathcal{T}\left(\gamma(t_1 + t_2)\right) - \mathcal{T}\left(\gamma(t_1 + t_3)\right) \\ &- \rho\left(\gamma\mathcal{T}\left(\frac{t_1 + t_2}{2} + t_3\right) + \gamma\mathcal{T}\left(\frac{t_1 - t_2}{2} - t_3\right) - \mathcal{T}(\gamma t_1)\right)\end{aligned} \tag{3.1}$$

for all $t_i \in \mathcal{Y}$, where $i = 1, 2, 3$ and $\gamma \in \mathbb{T}^1$. Then the mapping $\Delta\mathcal{J}_\gamma(t_1, t_2, t_3)$ is $\mathbb{C}-$linear mapping.

Next, for the sake of making our work easier, we investigate the ideas related to the control function of Găvruta. Let $\vartheta : \mathcal{Y}^3 \to [0, \infty)$ is a function such that

$$\vartheta\left(\frac{t_1}{2}, \frac{t_2}{2}, \frac{t_3}{2}\right) \leq \frac{\gamma}{2}\vartheta(t_1, t_2, t_3) \tag{3.2}$$

for some $0 < \gamma < 1$. By setting $t_1 = t_2 = t_3 = 0$, we find $\vartheta(0, 0, 0) = 0$.
Based on this inequality, we conclude that

$$\lim_{n\to\infty} 2^n \vartheta\left(\frac{t_1}{2^n}, \frac{t_2}{2^n}, \frac{t_3}{2^n}\right) = 0 \tag{3.3}$$

for all $t_1, t_2, t_3 \in \mathcal{Y}$. Signifying that the sequence approaches zero as n grows larger.
In the following theorem, we utilize the fixed point approach to study the stability of the combined additive and ρ-Jensen equations, demonstrating their stability in line with the Hyers-Ulam theorem. This is achieved by employing a control function introduced by Găvruta within ternary Banach algebras.

Theorem 3.3. *Let function $\vartheta : \mathcal{Y}^3 \to [0, \infty)$ satisfies (3.2) and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a mapping s.t.,*

$$\|\Delta\mathcal{T}_\gamma(t_1, t_2, t_3)\| \, \vartheta(t_1, t_2, t_3), \tag{3.4}$$

for all $t_i \in \mathcal{Y}$, where $i = 1, 2, 3$ and $\gamma \in \mathbb{T}^1$. Under these conditions, it is a unique $\mathbb{C}-$linear mapping $\mathfrak{F} : \mathcal{Y} \to \mathcal{Y}$, for which

$$\|\mathcal{T}(t) - \mathfrak{F}(t)\| \leq \frac{1}{(1-\upsilon)}\vartheta(0, t, t)$$

In the subsequent corollary, we examine how Theorem 3.3 can be stabilized by replacing Găvruta's control function $\vartheta(t_1, t_2, t_3)$, with the control function of Rassias, $\eta(\|t_1\|^r + \|t_2\|^r + \|t_3\|^r)$ where are $r < 1$ and η be non-negative real numbers.

Corollary 3.4. *Suppose $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is a mapping such that*

$$\|\Delta\mathcal{T}_\gamma(t_1, t_2, t_3)\| \leq \eta(\|t_1\|^r + \|t_2\|^r + \|t_3\|^r)$$

for all $t_i \in \mathcal{Y}$, where $i = 1, 2, 3$ and $\gamma \in \mathbb{T}^1$. Then, there exists a unique $\mathbb{C}$-linear mapping $\mathfrak{F} : \mathcal{Y} \to \mathcal{Y}$ such that

$$\|\mathcal{T}(t) - \mathfrak{F}(t)\| \leq \frac{2\eta}{2 - 2^r}\|t\|^r$$

4 Conclusions

Recognizing the importance of ternary Banach algebras in areas such as mathematical physics, quantum mechanics, and physical chemistry, we aimed to define and analyze the mixed additive and ρ-Jensen equations, demonstrating that they correspond to additive and $\mathbb{C}$-linear mappings. Ultimately, we employed the fixed point method to establish that both the mixed additive and ρ-Jensen equations exhibit Hyers-Ulam stability when utilizing two control functions from Găvruta and Rassias within the framework of ternary Banach algebras.

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Control of fractional continuous-time linear systems by state feedback matrix

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Abstract

In this paper, we study the control and stability of positive fractional continuous-time linear systems with delay using state feedback. First, we obtain a state feedback matrix that assigns zero eigenvalues to the system, and then, using similarity operations, we determine the augmented state feedback matrix such that the zeros of the closed-loop polynomial lie at a predetermined angle to the left of the complex plane.
Keywords: Positive linear system, State feedback matrix, Time delay, Fractional, Stability.

1 Introduction

A dynamical system represented by differential equations with not necessarily integer orders of derivatives can be considered as a fractional order system. The real objects are generally fractional, however, for many of them the fractionality is very low. Therefore, the fractional order representation is more adequate to describe real world systems than the integer order models. Fractional differential equations have been of great interest recently among researchers.

Fractional calculus, in allowing integrals and derivatives of any positive real order can be considered as a branch of mathematical analysis which deals with integrodifferential operators and equations where the integrals are of convolution type and exhibit kernels of power-law type.

Until recent times, fractional calculus was considered as a rather esoteric mathematical theory without applications, but in the last few decades there has been an explosion of research activities on the application of fractional calculus in various elds of science such as physics, mechanics, chemistry, engineering, etc. For example, we may cite hydrology application [11], physics [5], polymer rheology [3] and system biology [4]. In many applications, mere stability of the controlled object is not enough, and it is required that the poles of the closedloop system should lie in a certain restricted region of stability.

Some recent contributions to the theory of fractional differential equations can be found in [12, 9, 10, 1]. The problem of stability of positive fractional discretetime linear systems is addressed in the papers [2, 6, 7, 13]. In many applications, mere stability of the controlled object is not enough, and it is required that the poles of the closed loop system should lie in a certain restricted region of stability. Several design methods have

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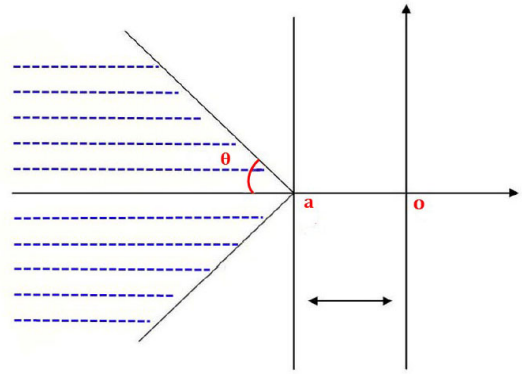


Figure 1: Specified sector

been reported which utilize the LQ technique to achieve the desired pole allocation Amin [11] derived an improved result in which the optimality of the closed loop system is assured.
Furuta and Kim [6] obtained a method for assigning the closed loop poles in a specified disk based on gain and phase margins which is named stability margin. They considered the case, when the perturbations are unknown gains as a diagonal form. Yuan and Achenie and Jiang [8] addressed the problem of linear quadratic regulator (LQR) synthesis with regional closed-loop pole constraints. Figueroa and Romagnoli [2] presented a method for designing controllers which attempt to place the roots of a characteristic polynomial of an uncertain system inside some prescribed regions. The analysis is based on the transfer function of a characteristic polynomial. Chou [1] described another pole assignment method with a spectral radius and proposed a pulse transfer function.
The procedure is simple, but it is used only for checking the positions of closed loop poles, not for designing the controller. Benner and Castillo and Quintana- Orti [12] presented the method for partial stabilization of large-scale discrete-time linear control systems. Recently, Grammont and Largillier [7] employed an approach to localize matrix eigenvalues in the sense that they build a sufficiently small neighborhood for each eigenvalue (or for a cluster).
A well-known desired region for continuous systems is left side of complex plan. In the simplest case, the real parts of all closed-loop eigenvalues are required to be less than a, where $a < 0$ is a real number. Generally, the more practically important region the closed-loop poles is the sector shown in figure 1. In this paper, the aim is to present a method for localization of eigenvalues in specified region of complex plane by state feedback control for large-scale continuous-time linear dynamic systems.

2 Stability of fractional positive continuous-time systems with delays

Consider a continuous-time linear system with q delays in state and inputs

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = \sum_{k=0}^{q}[A_k x(t-d_k) + B_k u(t-d_k)] \qquad 0 < \alpha \leq 1 \tag{2.1}$$

where $\alpha \in \mathbb{R}$ is the order of the fractional difference, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are, respectively, the state and input vectors, $A_k, B_k\,(k = 0, 1, \cdots, q)$ are real matrices of appropriate dimensions and $d_k, k = 1, 2, ...q$ are the delays ($d_k \geq 0$, $d_0 = 0$).

Theorem 2.1. *[17, 16] The positive fractional system (2.1) is called asymptotically stable if and only if the positive system without delays*

$$\frac{d^{\alpha}(t)}{dt^{\alpha}} = A\,x(t), \qquad A = \sum_{k=0}^{q} A_k \in \mathbb{M}_n \tag{2.2}$$

is asymptotically stable.

It follows from Theorem (2.1) that the checking of the asymptotic stability of positive systems with delays (2.1) can be reduced to checking the asymptotic stability of corresponding positive systems without delays (2.2).

3 Problem statement

Consider a controllable fractional linear system defined by the state equation (2.1), where the matrices A and B are real constant matrices of dimensions $n \times n$ and $n \times m$ respectively, with $\text{rank}(B) = m$.
The aim of eigenvalue assignment in a specified region is to design a state feedback controller, K, producing a closed-loop system with a satisfactory response by shifting controllable poles from undesirable to desirable locations. Karbassi and Bell [13, 14] have introduced an algorithm for obtaining an explicit parametric controller matrix K by performing similarity operations on the controllable pair (B, A). In fact, K is chosen such that the closed-loop system eigenvalues

$$\Gamma = A + BK \tag{3.1}$$

lie in the self conjugate eigenvalue spectrum $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Karbassi and Tehrani [15] extended the previous results as to obtain an explicit formula involving nonlinear parameters in the control law. The stabilization problem consists in finding a feedback matrix $K \in \mathbb{R}^{m\times n}$ such that the input $u(t) = K\,x(t)$.
In case the spectrum (or set of eigenvalues) of the closed-loop matrix, denoted by $\Lambda(\Gamma)$, is contained in the left side of complex plan we say that Γ is (Schur) stable or convergent. The stabilization problem arises in control problem such as, the computation of an initial approximate solution in Newton's method for solving discrete-time algebraic Riccati equations, simple synthesis methods to design controllers. Large-scale problems occur whenever the linear system results from some sort of a partial differential equation or from delay systems. There, the number of states is often a couple of thousands. The stabilization problem can in principal be solved as a eigenvalue assignment problem. eigenvalue assignment methods compute a feedback matrix such that the closed-loop matrix of system (3.1) has a prespecified spectrum. In this paper, we present an efficient approach for localization of eigenvalues in specified region for large-scale linear continuous-time systems. Our assignment procedure is composed of two stages. We first obtain a primary state feedback matrix F_p which assigns all the eigenvalues of closed-loop system to zero, then produce a state feedback matrix K which assigns all the closed-loop eigenvalues in specified sector.

4 Similarity transformation

Consider the state transformation

$$x(t) = T\tilde{x}(t) \tag{4.1}$$

where T can be obtained by elementary similarity operations as described in [13]. By substituting (4.1) in (2.1) vector companion system is

$$\tilde{x}(t) = \tilde{A}x(t) + \tilde{B}u(t). \tag{4.2}$$

In this way, $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$ are in a compact canonical form known as vector companion form:

$$\tilde{A} = \begin{bmatrix} & G_0 & \\ & \cdots\cdots\cdots & \\ I_{n-m} & & O_{n-m\times m} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_0 \\ \cdots\cdots \\ O_{n-m\times m} \end{bmatrix}$$

Here G_0 is an $m \times n$ matrix and B_0 is an $m \times m$ upper triangular matrix. Note that if the Kronecker invariants of the pair (B, A) are regular, then $\tilde{A}$ and $\tilde{B}$ are always in the above form [13]. In the case of irregular Kronecker invariants, some rows of I_{n-m} in $\tilde{A}$ are displaced [14]. It may also be concluded that if the vector companion form of $\tilde{A}$ obtained from similarity operations has the above structure, then the Kronecker invariants associated with the pair (B, A) are regular [13].

The state feedback matrix which assigns all the eigenvalues to zero, for the transformed pair $(\tilde{B}, \tilde{A})$, is then chosen as

$$u = -B_0^{-1}G_0\tilde{x} = \tilde{F}\tilde{x} \tag{4.3}$$

which results in the primary state feedback matrix for the pair (B, A) defined as

$$F_p = \tilde{F}T^{-1} \tag{4.4}$$

Therfore the transformed closed-loop matrix $\tilde{\Gamma}_0 = \tilde{A} + \tilde{B}\tilde{F}$ assumes a compact Jordan form with zero eigenvalues

$$\tilde{\Gamma}_0 = \begin{bmatrix} & O_{m\times n} & \\ & \cdots\cdots & \\ I_{n-m} & & O_{n-m\times m} \end{bmatrix}$$

Theorem 4.1. *Let D be a block diagonal matrix in the form*

$$D = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_n \end{bmatrix} \tag{4.5}$$

where each D_j, $(j = 1, 2, \cdots, k)$ is either of the form

$$D_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$$

(to designate the complex conjugate eigenvalues $\alpha_j + i\beta_j$)
or in case of real eigenvalues

$$D_j = [d_j]$$

If such block diagonal matrix D with self conjugate eigenvalue spectrum is added to the transformed closed-loop matrix, $\tilde{\Gamma}_0$, then the eigenvalues of the resulting matrix is the eigenvalues in the spectrum.

Proof. The sum of $\tilde{\Gamma}_0$ with D has the form:

$$A_\lambda = \tilde{\Gamma}_0 + D = \begin{bmatrix} & O_{m\times n} & \\ \cdots & \cdots & \cdots \\ I_{n-m} & & O_{n-m\times m} \end{bmatrix} + \begin{bmatrix} D_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_n \end{bmatrix}$$

$$= \begin{bmatrix} D_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & 0 & \cdots & 0 \\ I_1 & 0 & \cdots & 0 & D_{k+1} & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 & 0 & \ddots & \vdots \\ 0 & \cdots & I_r & 0 & 0 & \cdots & D_n \end{bmatrix}$$

where I_s, $s = 1, 2, \cdots, r$ is the unit matrix of size 2 in case $n - m$ is even. In case $n - m$ is odd only one I_s takes the form of a unit matrix of size one.
By expanding $\det(A_\lambda - \lambda I)$ along the first row it is obvious that the eigenvalues of A_λ are the same as the eigenvalues of D.
For the case of irregular Kronecker invariants [14] only some of the unit columns of I_{n-m} are displaced, since the unit elements are always below the main diagonal, the proof applies in the same manner. □

Corollary 4.2. *$\tilde{A}_\lambda$ can be obtained from A_λ by performing elementary similarity operations. Hence, the matrix $\tilde{A}_\lambda$ thus obtained will be in primary vector companion form such that:*

$$\tilde{A}_\lambda = \begin{bmatrix} & G_\lambda & \\ \cdots & \cdots & \cdots \\ I_{n-m} & & O_{n-m\times m} \end{bmatrix} \tag{4.6}$$

where G_λ is an $m \times n$ matrix.
Because of similarity operation, the eigenvalues of the matrix $\tilde{A}_\lambda$ are the same as the eigenvalues of A_λ and that of D.

Let

$$\tilde{K}_\lambda = B_0^{-1} G_\lambda$$

and

$$\tilde{K} = \tilde{F} + \tilde{K}_\lambda = B_0^{-1}(-G_0 + G_\lambda)$$

.
then the feedback matrix of the pair $(\tilde{A}, \tilde{B})$ is defined by:

$$K = \tilde{K} T^{-1} = B_0^{-1}(-G_0 + G_\lambda) T^{-1} \tag{4.7}$$

Theorem 4.3. *The state feedback matrix $\tilde{K}$ assigns the eigenvalues of closed-loop matrix $\tilde{\Gamma} = \tilde{A} + \tilde{B}\tilde{K}$ in the sector shown in figure 1 if we suppose α_j, β_j, d_j be in the form:*

$$\begin{aligned} \alpha_j &= -k.\,random\,(0,1) + a, \quad k \in \mathbb{N} \\ \beta_j &= |\alpha_j - a| \tan(\theta).\,random\,(0,1) \end{aligned} \tag{4.8}$$

and for assigning real valued eigenvalues in the sector shown in Figure 1, we choose

$$d_j = -k.\,random\,(0,1) + a \quad k \in \mathbb{N} \tag{4.9}$$

Proof. The eigenvalues of matrix D defined above fall in the specified sector.
Let

$$\begin{aligned}\tilde{\Gamma} = \tilde{A} + \tilde{B}\tilde{K} &= \begin{bmatrix} & G_0 & \\ I_{n-m} & & O_{n-m\times m}\end{bmatrix} + \begin{bmatrix} & B_0 & \\ & O_{n-m\times m} & \end{bmatrix}[B_o^{-1}(-G_0 + G_\lambda)] \\ &= \begin{bmatrix} & G_0 - B_0B_0^{-1}G_0 + B_0B_0^{-1}G_\lambda & \\ I_{n-m} & & O_{n-m\times m}\end{bmatrix} \\ &= \begin{bmatrix} & G_\lambda & \\ I_{n-m} & & O_{n-m\times m}\end{bmatrix}.\end{aligned} \tag{4.10}$$

Clearly $\tilde{\Gamma} = \tilde{A}_\lambda$, since $\tilde{A}_\lambda$ is similar to the matrix A_λ and the eigenvalues of matrix A_λ are the same as that of matrix D and elementary similarity operations do not change the eigenvalues, then the eigenvalues of closed-loop matrix $\tilde{\Gamma} = \tilde{A} + \tilde{B}\tilde{K}$ fall in the specified sector. □

Remark 4.4. Since $\tilde{K}$ assigns the eigenvalues of the closed-loop matrix $\tilde{\Gamma} = \tilde{A} + \tilde{B}\tilde{K}$ in the specified sector it is obvious that the state feedback controller matrix (4.7) also assigns the eigenvalues of the closed-loop matrix $\Gamma = A + BK$ in the specified sector too.

Note for assign the eigenvalues of the closed-loop matrix in spectrum $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ we suppose

$$D_j = \lambda_j, \qquad j = 1, 2, \cdots, n \tag{4.11}$$

4.1 Algorithm

In this section we first give an algorithm for finding a state feedback matrix which assigns zero eigenvalues to the closedloop system, then we determine a state feedback matrix which assigns the closed-loop eigenvalues in specified sector.

- ▶ Input. The controllable pair (B, A), the primary state feedback F_p, B_0^{-1} and T^{-1} which are calculated by the algorithm proposed by Karbassi and Bell [13, 14], the θ angle and distance angle apex of the origin of the complex plane (a) .
- ▶ Step 1. Construct the block diagonal matrix D in the form (4.5), in which for assigning complex valued eigenvalues in the sector shown in Figure 1 we choose

$$\begin{aligned}\alpha_j &= -k.\ random(0,1) + a, \qquad k \in \mathbb{N} \\ \beta_j &= |\alpha_j - a|\tan(\theta).\ random(0,1)\end{aligned}$$

  and for assigning real valued eigenvalues in the sector shown in Figure 1, we choose

$$d_j = -k.\ random(0,1) + a, \qquad k \in \mathbb{N}$$

- ▶ Step 2. $A_\lambda = \tilde{\Gamma}_0 + D$

▶ Step 3. Transform A_λ to primary vector companion form $\tilde{A}_\lambda$ as in (4.6) using elementary similarity operations as specified in corollary of theorem 1 .

▶ Step 4. Using the formula (4.7) compute the state feedback matrix K.

Example 4.5. Consider the fractional continuous-time system given by

$$\frac{d^{\frac{4}{5}}x(t)}{dt^{\frac{4}{5}}} = \sum_{k=0}^{2}[A_k x(t-d_k) + B_k u(t-d_k)]$$

Where A_i and B_i, $(i = 0, 1, 2)$ are with $n = 6$ and $m = 4$.

$$A_0 = \begin{bmatrix} 1 & 4 & 3 & 1 & 1 & 2 \\ 1 & -1 & 3 & 1 & 3 & 4 \\ 3 & 1 & 2 & 1 & 1 & 1 \\ 5 & 0 & 2 & 0 & 0 & 1 \\ 1 & 1 & 2 & 4 & 2 & 3 \\ 4 & 2 & 4 & 2 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 2 & 3 & 0 & 1 & 4 \\ 0 & 2 & 3 & 1 & 3 & 2 \\ 4 & 2 & 2 & 0 & 2 & 0 \\ 2 & 1 & 5 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 & 2 & 3 \\ 3 & 0 & -1 & 5 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 5 & 2 & 3 & -2 \\ 1 & 0 & 1 & 3 \\ 4 & 1 & 0 & 2 \\ 1 & 0 & 1 & 5 \\ -2 & -1 & 4 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 0 & 3 & 2 \\ 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ -1 & 0 & 2 & -1 \\ 2 & -1 & 2 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 3 \\ 3 & 2 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & 4 & 3 & 0 \end{bmatrix}$$

Proof. We have

$$A = \sum_{k=0}^{2} A_k = \begin{bmatrix} 1 & 7 & 8 & 1 & 3 & 8 \\ 1 & 1 & 8 & 3 & 7 & 7 \\ 8 & 5 & 5 & 1 & 4 & 2 \\ 9 & 1 & 8 & 3 & 0 & 3 \\ 1 & 4 & 6 & 7 & 6 & 8 \\ 9 & 4 & 2 & 8 & 0 & 2 \end{bmatrix}, \quad B = \sum_{k=0}^{2} B_k = \begin{bmatrix} 7 & 3 & 4 & 0 \\ 9 & 2 & 7 & 4 \\ 3 & 2 & 3 & 9 \\ 7 & 4 & 0 & 8 \\ 2 & 0 & 4 & 3 \\ 1 & 2 & 9 & 0 \end{bmatrix}$$

The open loop eigenvalues are

$$\{26.5923 + 10^{-4}\,i,\ 1.1651 \pm 6.3264\,i,\ -7.9665 + 10^{-4}\,i,\ -3.2591 + 10^{-4}\,i,\ 0.3031 + 10^{-4}\,i\}$$

which are widely spread in the complex plane. In order to locate them in sectors of left half complex, we employ the above algorithm step by step.
First, the primary state feedback matrix which locates all the eigenvalues of the closed-loop system to the origin of the complex plane is found to be:

$$F_p = \begin{bmatrix} 1.7090 & -2.2856 & -2.4735 & -0.7208 & -1.9936 & -3.5693 \\ -3.3175 & 0.8086 & 1.0540 & 0.2529 & 2.9120 & 2.6597 \\ -0.6196 & 0.5241 & 0.4805 & -0.3482 & -0.0022 & 0.6213 \\ -0.7398 & 0.5505 & 0.1069 & -0.0580 & -0.129 & 0.5683 \end{bmatrix}$$

Now, we find the state feedback matrix which assigns the closed-loop eigenvalues in the sector with $a = -2, z = 4$ and $\theta = \frac{3\pi}{8}$.
By using the algorithm, we have

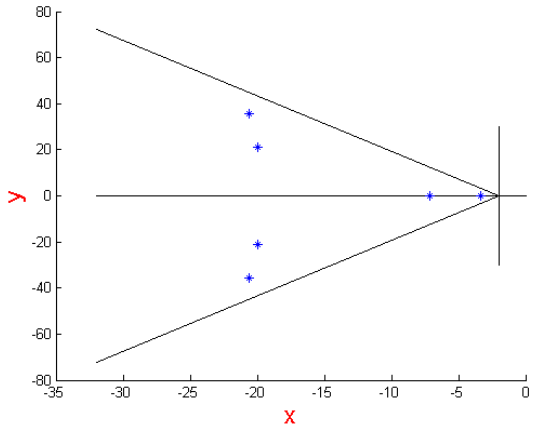


Figure 2: Placement of closed-loop eigenvalues in a specified sector

- Construct the block diagonal matrix D in the form (4.5)

$$D = \begin{bmatrix} -20.8883 & 41.3046 & 0 & 0 & 0 & 0 \\ -41.3046 & -20.8883 & 0 & 0 & 0 & 0 \\ 0 & 0 & -18.6667 & 36.7515 & 0 & 0 \\ 0 & 0 & -36.7515 & -18.6667 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.8971 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.7803 \end{bmatrix}$$

- The state feedback matrix obtained is:

$$K = \begin{bmatrix} 5.6992 & -16.1561 & -16.3295 & 17.5642 & -12.0599 & 2.2542 \\ -18.1451 & 5.1259 & 5.9445 & 2.8944 & 0.1893 & -1.4241 \\ -3.2967 & 0.418 & 2.6225 & 1.5816 & 0.8173 & -1.2587 \\ 5.3978 & -4.6458 & -1.5562 & 0.6953 & 3.557 & -3.2854 \end{bmatrix}$$

- The closed-loop eigenvalues are

$$\{-3.8971, -2.7803, -18.6667 \pm 36.7515i, -20.8883 \pm 41.3046i\}$$

  all of which are in the specified sector. Clearly, localization of eigenvalues in specified regions of complex plane by state feedback control is achieved.

The following forms state and input vectors to move your balance point. □

5 Conclusion

Here, a method for assigning eigenvalues at a specified angle of the complex plane using state feedback is proposed. This method is obtained according to the properties of vector associativity. The advantage of this method is that the calculations can be performed much more simply than other methods by using preliminary similarity operations. This method can also be used for larger-scale discrete-time linear control systems.

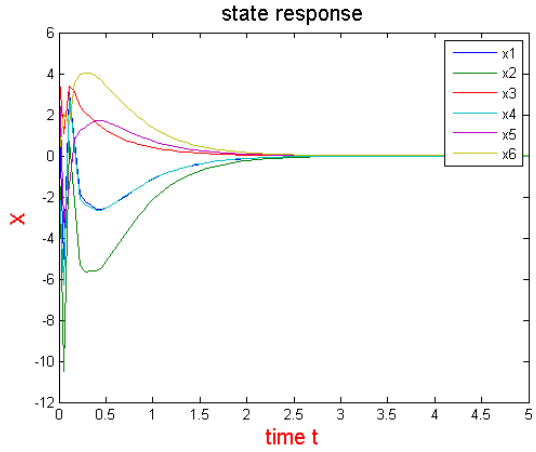


Figure 3: State vectors to move your balance point

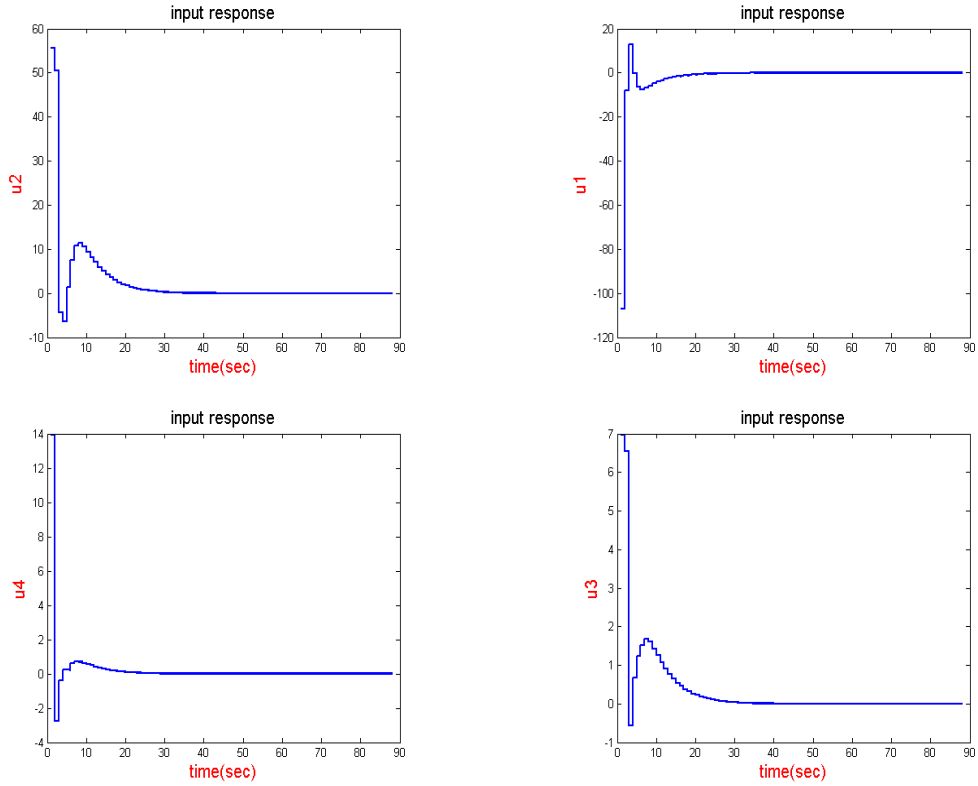


Figure 4: Input vectors to move your balance point

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Remarks on invertible subrings of the form $I + \mathbb{R}$

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Abstract

In this paper, we review the subrings of $C(X)$ in the form $I + \mathbb{R}$ where I is an ideal of $C(X)$ and we investigate $J + \mathbb{R}$ that J is an ideal of the subring S of $C(X)$, especially if S is an invertible subring of $C(X)$.

Keywords: Invertible subring, Lattice-ordered ring, *F-space, F_c-space.*

1 Introduction and preliminaries

Throughout this paper, it is assumed that all topological spaces are Hausdorff and completely regular. For a topological space X, the ring of real-valued continuous functions (resp., bounded continuous functions) on X is denoted by $C(X)$ (resp., $C^*(X)$). For $f \in C(X)$, the zero-set of f is denoted by $Z(f) = \{x \in X : f(x) = 0\}$ and the cozero-set of f by $Coz(f)$ which is the complement of $Z(f)$ with respect to X. For a subring R of $C(X)$, $Z(R)$ (resp., $Coz(R)$) denotes the set all of zero-sets (resp., cozero-sets) of elements of R as well as $Z_R(I) = \{Z(f) : f \in I\}$ and $Z_R^{-1}(\mathcal{F}) = \{f \in R : Z(f) \in \mathcal{F}\}$. A subring of $C(X)$ containing $C^*(X)$ is called an *intermediate subalgebra*, see [8]. A subring of $C(X)$ is called a *C-subalgebra* if it is isomorphic with $C(Y)$, for some Tychonoff space Y, see [8]. Moreover, a subring R of $C(X)$ is called invertible, if $f \in R$ and $Z(f) = \phi$ imply that f is invertible in R; i.e., there exists $g \in R$ such that $fg = 1$ [6, 8]. An ideal I of $C(X)$ is called a *z-ideal* if $M_f \subseteq I$, for every $f \in I$, in which M_f denotes the intersection of all the maximal ideals of $C(X)$ containing f. The Stone-Čech compactification of a topological space X is designated by βX and υX denotes the Hewitt-realcompactification of X. Every function $f \in C(X)$ has a Stone-extension $f^* : \beta X \longrightarrow R^*$, where R^* is the one-point compactification of R. For a subring R and for every $p \in \beta X$, $M^p(R) = \{f \in R : p \in cl_{\beta X} Z(f)\}$ (if $p \in X$ then $M^p(R) = M_p(R)$). Also $O^p(R) = \{f \in R : p \in int_{\beta X} cl_{\beta X} Z(f)\}$ (if $p \in X$ then $O^p(R) = O_p(R)$). For terms and notations here we follow the standard text [5]. A subalgebra $A(X)$ is called a *β-subalgebra* whenever the structure space of $A(X)$ is homeomorphic with βX, see [8]. A subring R is called an *SZ-subring* if for any two disjoint subsets A and B of X which are separated by $Z(R)$ implies A and B completely separated by R. We know that every invertible subalgebra and *β-subalgebra* is an *$SZ-$subring* [6]. The subring S is called PM^+-ring if the prime ideals of S containing a prime ideal, form a chain [3]. A space X is an F-space when the ideal O^p is prime ideal in $C(X)$ for every $p \in \beta X$, see [5].

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2 Main results

Subrings of the form $I + \mathbb{R}$ in which I be an ideal of $C(X)$ has been studied in [9, 3]. We remind that the prime ideals in these subrings are of the form $Q = P \cap I + \mathbb{R}$ that P is a prime ideal of $C(X)$. These subrings are PM^+-ring and the invertible subrings. In [8] was explained that $M^p + \mathbb{R}$ when $p \in \beta X \setminus X$ is a *β-subalgebra*. Also if I be an ideal in $C(X)$, then $I + \mathbb{R}$ is a lattice-ordered ring if and only if $|i| \in I$ for each $i \in I$, see [4] .The following statements are results in [3, 4].

Proposition 2.1. *For each ideal I in $C(X)$, the sum of every two prime ideals of the ring $I + \mathbb{R}$ is prime or all of $I + \mathbb{R}$ if and only if X is an F-space.*

Remark 2.2. Let I be an absolutely convex ideal of $C(X)$. Then the sum of every two z-ideals in $I + \mathbb{R}$ is a z-ideal in $I + \mathbb{R}$ or all of $I + \mathbb{R}$.

Since every z-ideal in $C(X)$ is an absolutely convex ideal, we have the result that a z-ideal in $I + \mathbb{R}$ containing a prime ideal of $I + \mathbb{R}$ is a prime ideal of $I + \mathbb{R}$ as a consequence of the above sentences. The following Theorem was explained in [4].

Theorem 2.3. *The following statements are equivalent.*

a. *For each ideal I in $C(X)$, the sum of every two z-ideals in $I + \mathbb{R}$ is a z-ideal in $I + \mathbb{R}$ or all of $I + \mathbb{R}$.*

b. *$I + \mathbb{R}$ is a lattice ordered ring for each ideal I in $C(X)$.*

c. *X is an F-space.*

In this part, we used S for an arbitrary subring(or subalgebra) of $C(X)$. Obviously $J + \mathbb{R}$ in which J be an ideal of S is a subalgebra of S. By [9, Lemma 3.1], the prime ideal of $J + \mathbb{R}$ is the like Q of form $Q = P \cap (J + \mathbb{R})$ that P is a prime ideal of S. In [3, Proposition 1.1] has been shown when I and J be two ideals of $C(X)$, then $J \cap (I + \mathbb{R}) = J \cap I$ if and only if $I + J$ is a proper ideal. We can see the following result.

Corollary 2.4. *Let J and K be ideals of the subalgebra S, then $J \cap (K + \mathbb{R}) = J \cap K$ if and only if $J + K$ is a proper ideal of S.*

Now suppose that the subring S is an invertible subring of $C(X)$. First, let's review some facts about invertible subrings of $C(X)$ and then turn to the main goal of the article, which study subrings of the form $J + \mathbb{R}$ that J is an ideal of the invertible subring S of $C(X)$. We remember that subring S of $C(X)$ is invertible if and only if each ideal I in S is a unit-free ideal (i.e. an ideal I in a subring S of $C(X)$ is said to be unit-free if it contains no invertible element of $C(X)$). The following statements are explained in [3, 7] and our recent work in 54^{nd} Annual Iranian Mathematics Conference.

Proposition 2.5. *Every maximal ideal of R is of the form $M^p(R)$ for some $p \in \beta X$ if and only if R is an invertible subalgebra.*

For every invertible subalgebra R of $C(X)$ and every ideal I of R, there exits $p \in \beta X$ such that $I \subseteq M^p(R)$.

Let R be an invertible subring of $C(X)$. Then $O^p(R)$, for every $p \in \beta X$, is a prime ideal if and only if $R_{M^p(R)}$is a domain.

Corollary 2.6. *Let R be invertible subring of $C(X)$, then following statments are equivalent.*

a. *R_M is a domain for all maximal ideal M of R.*

b. *There is a $Z \in Z(O^p(R))$, that all $f \in R$ dose not chang sign on Z for $p \in \beta X$ (This is equivalent by $O^p(R)$ is a prime ideal).*

c. Every prime ideal of R contains a unique minimal prime ideal.

d. R is locally uniform.

Proposition 2.7. *If S is an invetible subring of $C(X)$, then $J+\mathbb{R}$ where J be an ideal of S is an invertble subring of S.*

Proof. The proof resemble part 4 of [3, Proposition 1.1]. □

In [3, Proposition 1.7] has been shown when I and J be two ideals of $C(X)$, then $(I+\mathbb{R})\cap(J+\mathbb{R}) = (I\cap J)+\mathbb{R}$ if and only if $I+J\neq C(X)$. In the following Proposition, we show that this fact also holds for invertible subalgebra of $C(X)$.

Proposition 2.8. *If S is an invetible subralgebra of $C(X)$ whenever I and J are two ideals of S, then $(I+\mathbb{R})\cap(J+\mathbb{R}) = (I\cap J)+\mathbb{R}$ if and only if $I+J\neq S$*

Proof. According to [3, Proposition 1.7], the proof of this proposition is straightforward. □

We intend to conduct more investigations on such subring, $J+\mathbb{R}$ when J be an ideal of invertible subring S of $C(X)$, so the question arises: Do the results reviewed in Proposition 2.1 and Remark 2.2 hold for all invertible subrings? In this regard, we consider the invertible subring $C_c(X)$ (i.e., the subring of $C(X)$ that consists of all functions with countable image) of $C(X)$ and we state the following proposition that is the counterpart of Proposition 2.1. We remind that X is an F_c-space whenever every localization $C_c(X)_P$ of $C_c(X)$ at a prime ideal P is an integral domain. In [1, Theorem 5.9], some equivalences of F_c-space are stated. One of them, which is used in the proof of the following Proposition, is that the prime ideals of $C_c(X)$ contained in any given maximal ideal of $C_c(X)$ form a chain.

Proposition 2.9. *Let I be an ideal of $C_c(X)$, P and Q are two arbitrary prime ideals of $I+\mathbb{R}$, then $P+Q$ is a prime ideal or all of $I+\mathbb{R}$ if and only if X is an F_c-space.*

Proof. The proof is similar to [3, Theorem 2.4]. □

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Non-reduced configuration with a general fat point to obtain an unexpected curve

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Abstract

Studying the dimension of a given linear system of divisors is one of the most classical subject in algebraic geometry. The concept of the unexpected hypersurfaces have been in the focus of much recent research. In this note we present and study a new series of the so-called unexpected curves in projective space $\mathbb{P}^2$ which arising from a finite non-reduced zero dimensional subscheme and a general fat point, hence we show the existence and uniqueness of unexpected quintic curve for the non-reduced subscheme. Then present some unexpected curve of degrees $t > 5$ and show where we can find them. Most of the results obtained so far about unexpected curves have come from reduced configurations, whereas here we establish unexpected curves arising from non-reduced configurations.

Keywords: Configuration, Unexpected curve, Fat points, Hilbert function, Non-reduced scheme.

1 Introduction

The notion of an unexpected curve in the plane was introduced in 2018. The ground-breaking paper [1] defined the concept of unexpected curves and has proposed to give a new approach to solve the problem addressed by the SHGH conjecture. The SHGH conjecture established the problem of how many conditions a general union X of fat points imposes on the complete linear system of divisors in $\mathbb{P}^2$.

One of the very large problems in algebraic geometry is the characterize and then classify all triples (Z, X, j) where $Z = c_1Q_1 + ... + c_sQ_s$ for distinct points Q_i, and $X = m_1P_1 + ... + m_rP_r$ for general points P_i, such that X fails to impose the unexpected number of conditions on $V = [I_Z]_j$.[1]

If Z is the empty set, then $V = [R]_j$, so this addressed by the SHGH conjecture. If Z is reduced, $r = 1$ and $j = m_1 + 1$, this becomes the problem of deciding the existence of an unexpected curve of degree j. Motivated by an example described first by Di Gennaro, Ilardi, and Vallés in their article, they observe that there exist configurations Z of points in $\mathbb{P}^2$ such that imposing an additional point P of multiplicity m on the linear system of curves of degree d vanishing at all points of Z imposes less conditions than expected.

Let $\mathbb{K}$ be an algebraic closed field of characteristic 0, and given subscheme X of $\mathbb{P}^2_{\mathbb{K}}$, its defining saturated homogeneous ideal $I_X \subseteq R = \mathbb{K}[\mathbb{P}^2_{\mathbb{K}}] = \mathbb{K}[x_0, x_1, x_2]$. Consider a general point $P \subseteq \mathbb{P}^2_{\mathbb{K}}$. The fat point scheme $Q = mP$ is the scheme defined by the homogeneous ideal $I_Q = (I_P)^m \subset R$ where I_P is the ideal of general by all forms that vanish at P.

More generally we say that the subscheme $Z \subset \mathbb{P}^n$ with respect to Q admits an unexpected hypersurface of degree t if

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$$\dim[I_Z \cap I_Q]_t > \max\{0, \dim[I_Z]_t - \tbinom{m-1+n}{n}\}$$

That is, Z admits an unexpected hypersurface with respect to Q of degree t if the conditions imposed by Q on forms of degree t vanishing on Z are not independent.

In this note we will work in projective plane and often say that Z admits an unexpected curve with a general point P of multiplicity m.

Definition 1. *[4] Let M be a finitely generated graded R-module. The numerical function*

$$h_M : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$

defined as

$$h_M(t) := \dim_{\mathbb{K}}(M_t)$$

for all $t \in N$ is the Hilbert function of M.

According to Macaulay's theorem, let $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a numerical function. There exists a standard homogeneous k-algebra X with Hilbert function $h_X = h$ if and only if $h(0) = 1$ and $h(t+1) \leq h(t)^{<t>}$ for every $t \geq 1$.

Macaulay's theorem that for each t in the sense that any numerical function satisfying it can be realized as Hilbert function of a suitable homogeneous $k-$algebra.

Now we define three number associated to configuration of points X, multiplicity of general fat point m and degree of forms t.

$$\operatorname{adim}(X, t, m) = \dim[I_X \cap I_P^m]_t, \tag{1.1}$$

$$\operatorname{vdim}(X, t, m) = \dim[I_X]_t - \binom{n+m-1}{n}, \tag{1.2}$$

$$\operatorname{edim}(X, t, m) = \max\big\{0, \operatorname{vdim}(X, t, m)\big\}. \tag{1.3}$$

We say that X admits an unexpected curve of degree t vanishing at a general point P with multiplicity m whenever

$$\operatorname{adim}(X, t, m) > 0$$

and

$$\operatorname{adim}(X, t, m) > \operatorname{vdim}(X, t, m).$$

Now we consider the example that motivated the study of unexpectedness.

Example 1.1. Let $Z \subset \mathbb{P}^2$ be the reduced scheme consisting of the following 9 points.

$$[1:0:0], [0:1:0], [0:0:1], [1:1:1], [0:1:1], [1:0:1], [1:1:0], [-1:1:0], [1:1:2].$$

In $\mathbb{P}^2$ is a set of reduced points. Then the defining ideal of these points is:

$$\begin{aligned} I(X) = \big\langle & 2x^2z - 2y^2z - 2xz^2 + 2yz^2, 2x^3y - 2xy^3, \\ & 2xy^2z - 2xyz^2 - y^2z^2 + yz^3, 2y^3z - 3y^2z^2 + yz^3 \big\rangle. \end{aligned}$$

Namely start with 4 general points, shown in black. There are three singular conics that contain all four general points; their singular points are the points shown ass dotted circles. Then draw in the line through any two of these three singular points and take two points where this line intersects the singular conic whose singular points is third dotted circle. Let P is a general point(show as yellow circles). Then there is a unique quartic curve containing Z with a triple point at P, so we have

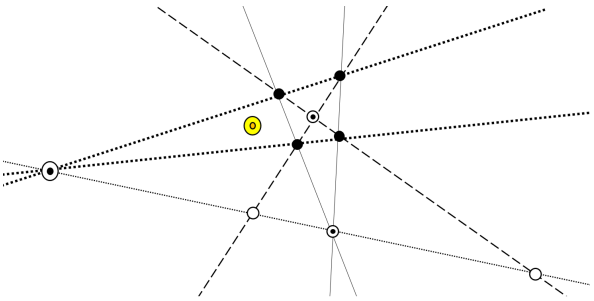


Figure 2: Reduced subscheme X

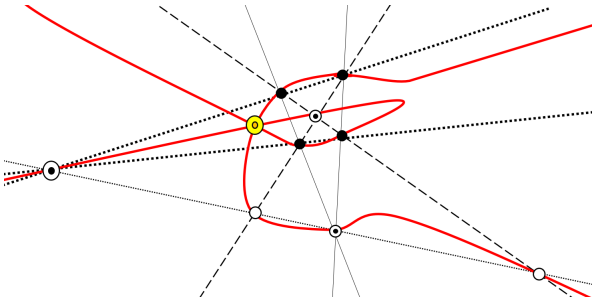


Figure 1: Unexpected curve

$$h_{Z+3P} = \{1, 3, 6, 10, 14, 15, 15, ...\}$$

and hence for $t = 4$ we have $\operatorname{adim}(X, 4, 3) = 1 > 0$ and

$$1 = \operatorname{adim}(X, 4, 3) > \operatorname{vdim}(X, t, m) = 0$$

Therefore there exist an unexpected curve of degree 4 arising from a reduced configuration of points with a general fat point of multiplicity 3, as can see it below:

$$\begin{aligned} I_4(X) = \langle & a_1x^3y - a_2x^3z - a_3x^2yz + a_4x^2z^2 - a_5xy^3 + a_6xy^2z \\ & - a_7xyz^2 - a_8xz^3 + a_9y^3z - a_{10}y^2z^2 + a_{11}yz^3 \rangle \end{aligned}$$

Theorem 1.2. *Up to projective equivalence, the configuration of points $X \subset \mathbb{P}^2$ in 2 is the only one which admits an quartic unexpected curve.[5]*

2 Main results

Now we can answer to some of questions about unexpectedness in projective plane. In theorem 1.2 shown that there exist a configuration of points that only for a specific multiplicity of a general fat point has a unique unexpected curve of degree 4.

Is there a configuration of points that for a specific multiplicity m of a general fat point has a unique unexpected curve of degree 5?

In the following theorem the positive answer is given.

Theorem 2.1. 1. *There exist a finite non-reduced zero dimensional subscheme $Z \subset \mathbb{P}^2$ such that just for the general fat point of multiplicity $m = 4$ admits an unexpected quintic curve.*

2. *There exists a non-reduced configuration of points such that for all fat point multiplicity $m \neq 4$ there is not any unexpected hypersurface of any degree.*

Proof. Let $Z = 2P_1 + P_2 + P_3 + ... + P_9$ be the non-reduced subscheme in projective plane such that support of its points described in 2. The saturated homogeneous ideal I_Z is defined as follow

$$\begin{aligned} I_Z = \langle & 2y^3z - 3y^2z^2 + yz^3, xy^2z - xyz^2 + y^3z - 2y^2z^2 + yz^3, \\ & x^2z^2 - xz^3 - y^2z^2 + yz^3, x^2yz - x^2z^2 + xy^2z - 2xyz^2 + xz^3, \\ & 4x^3y^2 - 4xy^4 + 2y^3z^2 - 3y^2z^3 + yz^4 \rangle \end{aligned}$$

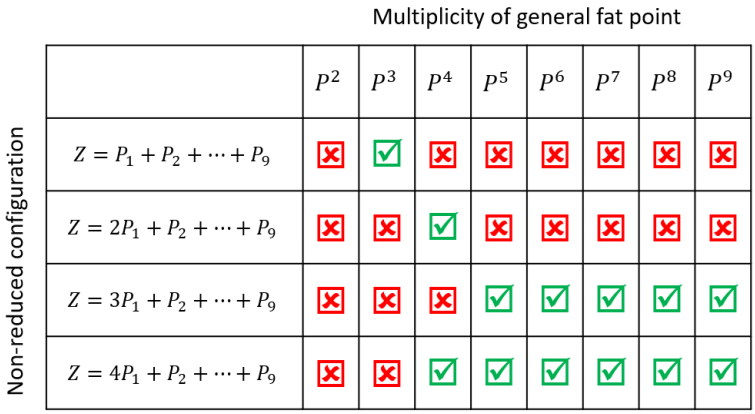


Figure 3: Unexpectedness with respect to multiplicity of a special and a general fat point

Let P be a general fat point of multiplicity $m = 4$ then $X = Z + 4P$ and hence

$$gin(I_X) = \{x^5, x^4y^2, x^4yz, x^4z^2, x^3y^3, x^3y^2z^2, x^3yz^3, x^2y^5,$$
$$x^2y^4z^2, x^2y^3z^3, x^2y^2z^5, xy^8, xy^7z^2, xy^6z^4, xy^5z^6, xy^4z^8, y^{14}\}$$

Where $gin(I_X)$ is the generic initial ideal of saturated homogeneous ideal I_X with respect to lexicographic order. So the initial degree of $I_X \subseteq \mathbb{K}[x_0 : x_1 : x_2]$ is $\alpha(I_X) = 5$.

The Hilbert function of ideal I_{Z+4P} is given as below

$$h_X = \{1, 3, 6, 10, 15, 20, 21, 21, ...\}$$

Hence the actual dimension of X when the forms of degree 5 in I_X vanished at a general fat point $4P$ is equal to:

$$\text{adim}(X, 5, 4) = 1$$

In otherwise, the dimension of linear system of the forms of degree 5 in I_X is equal to the expected number of conditions imposed by a general fat point $4P$, i.e.,

$$\text{vdim}(X, 5, 4) = 0$$

Therefore, X fails to impose independent conditions on linear system of divisors, i.e., there is an unexpected quintic curve with respect to m. We now show this unexpected curve

Consider that $Z = 2P_1 + P_2 + P_3 + ... + P_9$ be the non-reduced subscheme in $\mathbb{P}^2$ so the unexpected quintic curve arising from Z with respect to general fat point P of mult(P) $= 4$ is

$$I_5 = \langle a_1x^3y^2 - a_2x^3yz + a_3x^3z^2 - a_4x^2y^2z + a_5x^2yz^2 - a_6x^2z^3 - a_7xy^4 +$$
$$a_8xy^3z - a_9xy^2z^2 - a_{10}xyz^3 + a_{11}xz^4 + a_{12}y^4z - a_{13}y^3z^2 + a_{14}y^2z^3 - a_{15}yz^4\rangle$$

This configuration only for a general fat point of the specific multiplicity $m = 4$ can admits an unexpected curve of degree 5. In fact for all $m \neq 4$ configuration Z with respect to P of $mult(P) \neq 4$ impose the expected number of conditions on the linear system of degree t. □

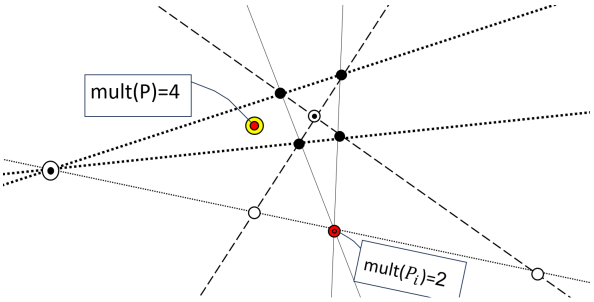


Figure 5: Non-reduced subscheme X

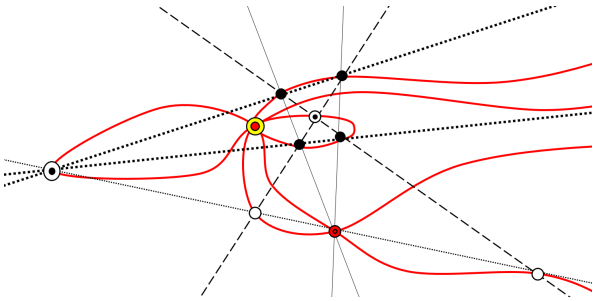


Figure 4: Unexpected curve

Theorem 2.2. *Up to projective equivalence, the non-reduced configuration $Z \subset \mathbb{P}^2$ in Example2 is the only one which admits an unexpected curve of degree 5.*

Proof. Given two points $A, B \in \mathbb{P}^2$, we denote AB the line joining them. We call a line *simple* if it contains only two points. For each $k \geq 3$, we say that a line is *k-rich* if it contains exactly k points of Z. By [[5], Proposition 2.1] let t be a positive integer , Z be a set of $2t+1$ distinct points, and $P \in \mathbb{P}^2$ a general point. If there are a *t-rich line* L_Q and a simple line L_R such that $L_Q \cap L_R \notin Z$, then $I(Z+(t-1)P)_t = 0$. It can shown by assuming contradiction that $I(Z+(t-1)P)_t$ and use the Bézout theorem and hence by this fact that every points imposes one condition and every doubled point imposes three conditions, when the configuration Z of points is non-reduced containing a double point, Z impose 11 conditions and hence $t = 5$. So with generalizing [[5], Proposition 3.1] if there are two $5-$rich lines L_Q, L_R such that $L_Q \cap L_R \notin Z$, then $I(Z+4P)_5 = 0$. So when $I(Z+4P)_5 \neq 0$ and if there are two $5-rich$ lines L_Q, L_R, then the configuration of the points of Z is the one described in Example2, i.e, the obtained configuration is projectively equivalent to the one described in Example2.

So the configuration of Example2 is the only non-reduced configuration of nine points contain a point of multiplicity 2 in $\mathbb{P}^2$ containing a $5-$rich line which admits an unexpected quintic curve. □

Special linear systems with multiplicities $m_1 = m_2 = ... = m_s = 2$ have been completely classified by Alexander and Hirschowitz. Now we investigate two variation of the non-reduced subshemes where $m_1 = 3$, $m_2 = m_3 = ... = m_9 = 1$ and $m_1 = 4$, $m_2 = m_3 = ... = m_9 = 1$ with respect to a general fat point of $2 \leq \text{mult}(P) \leq 9$ and show that when this configuration impose an unexpected curves. In particular, determining degree of each unexpected curves arising from it.

Theorem 2.3. *Let $Z = 3P_1 + P_2 + ... + P_9$ be the non-reduced subscheme with a special point of* $\text{mult}(P_i) = 3$ *for $1 \leq i \leq 9$ and P be a general fat point with* $\text{mult(P)} = \text{m}$. *Then*

1. *for $m \in 2, 3, 4$, X impose expected number of conditions on linear system and hence there is not any unexpected curve in these cases.*

2. *for all $m \geq 5$ there is an unexpected curve of degree $m+1$.*

Proof. Let $Z = 3P_1 + P_2 + P_3 + ... + P_9$ be the non-reduced subscheme in projective plane such that support of its points described in 2. The saturated homogeneous ideal I_Z is defined as follow

$$I_Z = \langle 2y^3z - 3y^2z^2 + yz^3, 2xy^2z - 2xyz^2 - y^2z^2 + yz^3, x^3y^3 - xy^5,$$
$$x^2z^3 - 2xy^2z^2 + 2xyz^3 - xz^4, x^2yz^2 - x^2z^3 + xy^2z^2 - 2xyz^3 + xz^4 \rangle$$

Let P be a general fat point of multiplicity $m = 5$ then $X = Z + 5P$ and hence its generic initial ideal with respect to lexicographic order

$$gin(I_X) = \{x^6, x^5y^2, x^5yz, x^5z^2, x^4y^3, x^4y^2z^2, x^4yz^3, x^3y^5, x^3y^4z, x^3y^3z^3,$$
$$x^3y^2z^4, x^2y^7, x^2y^6z^2, x^2y^5z^3, x^2y^4z^5, xy^{10}, xy^9z^2, xy^8z^4, xy^7z^6, xy^6z^8, y^{16}\}$$

So the initial degree of $I_X \subseteq \mathbb{K}[x_0 : x_1 : x_2]$ is $\alpha(I_X) = 6$. The Hilbert function of ideal I_{Z+4P} is given as below

$$h_X = \{1, 3, 6, 10, 15, 21, 27, 29, 29, ...\}$$

Hence the actual dimension of X when the forms of degree 6 in I_X vanished at a general fat point $5P$ is equal to:

$$\text{adim}(X, 6, 5) = 1$$

In otherwise, the virtual dimension is equal to

$$\text{vdim}(X, 6, 5) = -1$$

Therefore, X admits an unexpected curve of degree 6 with respect to $m = 5$. This unexpected curve is shown as below

$$I_6 = \langle a_1 3y^2z^2 - a_2x^3yz^3 + a_3x^3z^4 + a_4x^2y^4z - a_5x^2y^3z^2 + a_6x^2y^2z^3 - a_7x^2yz^4 +$$
$$a_8x^2z^5 - a_9xy^5z + a_{10}xy^4z^2 - a_{11}xy^3z^3 + a_{12}xy^2z^4 + a_{13}xyz^5 - a_{14}xz^6 - a_{15}y^6z$$
$$+ a_{17}y^5z^2 -_a 18y^4z^3 + a_{19}y^3z^4 - a_{20}y^2z^5 + a_{21}yz^6\rangle$$

This configuration for all general fat point of multiplicities $m \geq 5$ can admits an unexpected curve of degree $m + 1$. □

Theorem 2.4. *Let $Z = 4P_1 + P_2 + ... + P_9$ be the non-reduced subscheme with a special point of* $\text{mult}(P_i) = 4$ *for $1 \leq i \leq 9$ and P be a general fat point with* $\text{mult(P)} = \text{m}$.

1. *If $m \in \{2, 3\}$, then X impose expected number of conditions on linear system and hence there is not any unexpected curve in these cases.*

2. *If $m = 4$, then X admits an unexpected curve of degree 6.*

3. *If $m \in \{5, 6\}$, then X admits an unexpected curve of degree 7.*

4. *For all $m \geq 7$ the non-reduced configuration of points X admits an unexpected curve of degree $m + 1$.*

In theorem 2.15 [2] they have shown there exist a finite set of points $Z \subset \mathbb{P}^2$ admitting an unexpected curve of degree d and multiplicity m at a general point if and only if $t > m > 2$. According to supposing configuration as a non-reduced subscheme in projective plane we can improve this theorem as follows:

Theorem 2.5. *There exist a finite set of non-reduced points $Z \subset \mathbb{P}^2$ admitting an unexpected curve of degree t and multiplicity m at a general point if and only if $t > m \geqslant 2$.*

Non-reduced configuration	Multiplicity of general fat point							
	P^2	P^3	P^4	P^5	P^6	P^7	P^8	P^9
$Z = P_1 + P_2 + \cdots + P_9$	☒	4	☒	☒	☒	☒	☒	☒
$Z = 2P_1 + P_2 + \cdots + P_9$	☒	☒	5	☒	☒	☒	☒	☒
$Z = 3P_1 + P_2 + \cdots + P_9$	☒	☒	☒	6	7	8	9	10
$Z = 4P_1 + P_2 + \cdots + P_9$	☒	☒	6	7	7	8	9	10

Figure 6: Degree of unexpected curve with respect to multiplicity of a special point in non-reduced subscheme and multiplicity of general fat point

Proof. We first note that there is an unexpected curve of degree $m+1$ and $\text{mult}(P) = m$ at a general point for each $m \geq 3$ in projective plane $\mathbb{P}^2$. For $m = 2$ when $Z = 6P_1 + P_2 + ... + P_9$ be the non-reduced zero-dimensional subscheme with a general fat point of $\text{mult(P)} = 2$ we have $h_X = \{1, 3, 5, 10, 15, 21, 27, 31, 32, 32, ...\}$. Since $\text{adim}(X, 6, 2) = 1$ and $\text{vdim}(X, 6, 2) = 0$, there is an unexpected curve of degree 6 arising from this configuration and a general fat point with $\text{mult}(P) = 2$ as below

$$I_6 = \langle a_1 y^5 z - a_2 y^4 z^2 + a_3 y^3 z^3 - a_4 y^2 z^4 + a_5 y z^5 \rangle$$

□

Indeed, there was shown that this occurs in characteristic 2.

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Some New Results on Stochastic Comparisons of the largest order statistics from dependent random variables under Archimedean copula dependence

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Abstract

In this paper, we consider two parallel systems comprising heterogeneous dependent components, with an Archimedean copula. We then provide sufficient conditions on the distributions of components' lifetimes and the generator of the Archimedean copula and on the vector of parameters for comparing the lifetimes of two systems with respect to the usual stochastic order.

Keywords: Usual stochastic order, Archimedean copula, Majorization, Order statistics.

1 Introduction

Consider a system with n components and suppose this system fails if and only if at least k components fail. Such a system is said to be a $(n-k+1)$-out-of-n system. Let $X_1, \ldots, X_n$ be the component's lifetimes of a $(n-k+1)$-out-of-n system with corresponding oder statistics $X_{1:n} \leq \ldots \leq X_{n:n}$. In particular, $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. For a general theory of order statistics and their applications, interested readers may refer to [1] and [6].

Reliability and stochastic properties of series and parallel systems have been considered by various researchers based on different scenarios. For example [21] and [23] deal with the case of heterogeneous Weibull distributions, [7] and [10] deal with the case of heterogeneous exponentiated Weibull (EW) distributions and [19] deals with the case of heterogeneous extended generalized exponential distributions. A recent review on the topic can be also found in [3].

In recent years, attention has spread from specific distributions to the case of stochastic comparisons of order statistics when the populations belong to some semiparametric families; see [4], [8], [9] and [15].

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{X^*} = (X_1^*, \ldots, X_n^*)$ be two vectors of continuous dependent random variables from a general family of distributions assembled by Archimedean (distributional or survival) copulas with the respective generators ϕ_1 and ϕ_2 and let $X_i \sim F(x;\lambda_i)(X_i^* \sim F(x;\lambda_i^*))$ where $F(x;\lambda_i)(F(x;\lambda_i^*))$ denotes the distribution function of $X_i(X_i^*)$ and $\lambda_i(\lambda_i^*) > 0$ is the model parameter of $X_i(X_i^*)$, for $i = 1, \ldots, n$. Recently, [16] derived usual stochastic ordering results of the extreme order statistics from r.v.'s following these general family of distributions and linked them with Archimedean copulas. They assumed that if ϕ_1 or ϕ_2 is log-convex, $\psi_2 \circ \phi_1$ is super-additive, and for any x the function $\bar{F}(x;\lambda)(F(x;\lambda_i))$ is decreasing [increasing] and log-concave in $\boldsymbol{\lambda}$, then $(\lambda_1, \ldots, \lambda_n) \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\preceq}](\lambda_1^*, \ldots, \lambda_n^*)$ implies $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^* (X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*)$. [22] derived

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usual stochastic ordering results for coherent systems of dependent components with these general family of distributions with some other conditions like the survival function $\bar{F}(x;\lambda_i)$ is concave/convex in λ and with Schur-concave/convex copulas.

In this paper, the problem of stochastic comparisons between two parallel systems of dependent components in general models with dependence defined by Archimedean copulas based on majorization theory has been considered. Specifically, we study the usual stochastic order between the maximum from two sets of dependent and heterogeneous random variables when the marginal distributions are arbitrary depending on some parameter. In this case, we consider that the parameter vectors verify the weak majorization orders and show that the condition of log convexity of the Archimedean generator is not necessary and the results hold without it. The results proved here generalize some recent results in [16] and [22].

The organization of the rest of this article is laid out as follows. In Section 2, we recall some preliminary notions of copula models, stochastic orders and lemmas. Section 3 presents the general results on the stochastic comparisons of two parallel systems with dependent heterogeneous components in terms of the usual stochastic order. Finally, Sect. 4 concludes the paper.

To continue our discussion, we need definitions of some stochastic orders and the concept of majorization that are stated in the next section.

2 The basic definitions and some prerequisites

In this section, we briefly present definitions of some well-known concepts relating to stochastic orders, majorization orders, copulas and some useful lemmas, which are helpful for proving our main results. Throughout the article, we use the notations $\mathbb{R} = (-\infty, +\infty)$ and the term increasing means nondecreasing and decreasing means nonincreasing.

There are many ways in which a random variable X can be said to be smaller than another random variable Y. In the usual stochastic ordering case, a random variable X with survival function $\bar{F} = 1 - F$ is stochastically smaller than a random variable Y with survival function $\bar{G} = 1 - G$, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x. For a comprehensive discussion on various stochastic orderings, one may refer to [20] and [12].

The concept of copula plays an important role for dependent samples. In the literature, different copulas have been defined, e.g., Archimedean copula, FGM copula, etc. For a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with the joint distribution function F, joint survival function $\bar{F}$, univariate marginal distribution functions $F_1, \ldots, F_n$ and univariate survival functions $\bar{F}_1, \ldots, \bar{F}_n$, if there exists some $C : [0,1]^n \longrightarrow [0,1]$ and $\hat{C} : [0,1]^n \longrightarrow [0,1]$ such that, for all x_i , $i = 1, \ldots, n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),$$
$$\bar{F}(x_1, \ldots, x_n) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),$$

then C and $\hat{C}$ are called as the copula and survival copula of X, respectively. For the sake of completeness, we give the following definition of the Archimedean copula. A real function ϕ is n-monotone on $(a,b) \subseteq (-\infty, +\infty)$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in (a,b) and $(-1)^k \phi^{(k)}(x) \geq 0$ for all $x \in (a,b), k = 0, 1, \ldots, n-2$, in which $\phi^{(i)}(.)$ is the ith derivative of $\phi(.)$. For a n-monotone ($n \geq 2$) function $\phi : [0, +\infty) \longrightarrow [0,1]$ with $\phi(0) = 1$ and $\lim_{x\to+\infty} \phi(x) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse, then

$$C_\phi(u_1, \ldots, u_n) = \phi(\psi(u_1) + \ldots + \psi(u_n)), \quad \text{for all } u_i \in [0,1], i = 1, \ldots, n,$$

is called an Archimedean copula with the generator ϕ. Archimedean copulas cover a wide range of dependence structures including the independence copula with the generator $\phi(t) = e^{-t}$. For more on Archimedean copulas, readers may refer to [17] and [14].

Next we give the definitions of some majorization orders and Schur convexity of functions that will also be needed in this article. For encyclopedic information on the theory of these orders and their applications, one can see the book written by [13]. Let us recall that the notation $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ is used to denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$.

Definition 1. The vector $\boldsymbol{x}$ is said to be

(i) weakly submajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$) if $\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$ for all $j = 1, \ldots, n$,

(ii) weakly supermajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n$,

(iii) majorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n-1$.

Now, definitions of the Schur-convex and Schur-concave functions are presented.

Definition 2. A real valued function φ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathcal{A}$ if

$$\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y} \quad \text{on} \quad \mathcal{A} \Longrightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y}).$$

The below stated lemmas will be useful to establish the main results in this manuscript.

Lemma 2.1 ([13], Theorem 3.A.4)**.** Suppose $\mathbb{I} \subset \mathbb{R}$ is an open interval and $\Phi : \mathbb{I}^n \longrightarrow \mathbb{R}_+$ is continuously differentiable. Necessary and sufficient conditions for Φ to be Schur-convex (Schur-concave) on $\mathbb{I}^n$ are

(i) Φ is symmetric on $\mathbb{I}^n$,

(ii) for $i \neq j$ and all $z \in \mathbb{I}^n$,

$$(z_i - z_j)\left(\frac{\partial \Phi(z)}{\partial z_i} - \frac{\partial \Phi(z)}{\partial z_j}\right) \geq (\leq) 0,$$

where $\frac{\partial \Phi(z)}{\partial z_i}$ denotes the partial derivative of Φ with respect to its i-th argument.

Lemma 2.2 ([13], Theorem 3.A.8)**.** For a function l on $A \in \mathbb{R}^n$, $\boldsymbol{x} \preceq_{\mathrm{w}} (\overset{\mathrm{w}}{\preceq}) \boldsymbol{y}$ implies $l(\boldsymbol{x}) \leq l(\boldsymbol{y})$ if and only if it is increasing (decreasing) and Schur-convex on A.

The following lower orthant order on Archimedean copulas will also be utilized in what follows.

Lemma 2.3 ([11], Lemma A.1)**.** For two n-dimensional Archimedean copulas $C_{\phi_1}(\mathbf{u})$ and $C_{\phi_2}(\mathbf{u})$, if $\psi_2 \circ \phi_1$ is super-additive, then $C_{\phi_1}(\mathbf{u}) \leq C_{\phi_2}(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^n$.

3 Main results

This section compares the largest order statistic of two vectors of dependent and heterogeneous random variables having the general model with Archimedean copula in the sense of usual stochastic order. [16] obtained the following theorem for the comparison of parallel systems under the usual stochastic order.

Theorem 3.1. Let $\boldsymbol{X}$ and $\boldsymbol{X^*}$ be two vectors of continuous random variables with $X_i \sim F(x; \lambda_i)$ and $X_i^* \sim F(x; \lambda_i^*)$, $i = 1, \ldots, n$. Assume that their associated copulas are Archimedean with the respective generators ϕ_1 and ϕ_2. If ϕ_1 or ϕ_2 is log-convex, $\psi_2 \circ \phi_1$ is super-additive, and for any x the function $F(x; \lambda)$ is decreasing [increasing] and log-concave in $\boldsymbol{\lambda}$, then $(\lambda_1, \ldots, \lambda_n) \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\succeq}] (\lambda_1^*, \ldots, \lambda_n^*)$ implies $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$.

In the following theorem, we show that the condition of log convexity of the Archimedean generator is not necessary and the results hold without it.

Theorem 3.2. Let $\boldsymbol{X}$ and $\boldsymbol{X^*}$ be two vectors of continuous random variables with $X_i \sim F(x;\lambda_i)$ and $X_i^* \sim F(x;\lambda_i^*)$, $i=1,\ldots,n$. Assume that their associated copulas are Archimedean with the respective generators ϕ_1 and ϕ_2. If $\psi_2\circ\phi_1$ is super-additive, and for any x the function $F(x;\lambda)$ is decreasing (increasing) and concave in $\boldsymbol{\lambda}$, then $(\lambda_1,\ldots,\lambda_n) \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\preceq}](\lambda_1^*,\ldots,\lambda_n^*)$ implies $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$.

Proof. The distribution function of $X_{n:n}$ is

$$F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1) = \mathbb{P}(X_k < x, 1\leq k\leq n) = \phi_1\big(\sum_{i=1}^{n}\psi_1(F(x,\lambda_i))\big). \tag{3.1}$$

the partial derivatives of $F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)$ with respect to λ_i are

$$\frac{\partial F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)}{\partial\lambda_i} = \frac{\partial F(x,\lambda_i)}{\partial\lambda_i}\frac{\phi_1'\big(\sum_{i=1}^{n}\psi_1(F(x,\lambda_i))\big)}{\phi_1'\big(\psi_1(F(x,\lambda_i))\big)} \leq [\geq]0,$$

$$\text{for all} \quad x>0.$$

which the inequality holds due to the decreasingness [increasingness] property of $F(x,\lambda)$ in λ and decreasingness property of the generator of Archimedean copulas. Thus $F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)$ is decreasing [increasing] with respect to λ_i's. To prove its Schur-concavety, it follows from Lemma 2.1 that we have to show that for $i\neq j$,

$$(\lambda_i-\lambda_j)\left(\frac{\partial F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)}{\partial\lambda_i} - \frac{\partial F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)}{\partial\lambda_j}\right)\leq 0, \tag{3.2}$$

that is, for $i\neq j$,

$$(\lambda_i-\lambda_j)\phi_1'\big(\sum_{i=1}^{n}\psi_1(F(x,\lambda_i))\big)$$

$$\left(\frac{\dfrac{\partial F(x,\lambda_i)}{\partial\lambda_i}}{\phi_1'\big(\psi_1(F(x,\lambda_i))\big)} - \frac{\dfrac{\partial F(x,\lambda_j)}{\partial\lambda_j}}{\phi_1'\big(\psi_1(F(x,\lambda_j))\big)}\right)\leq 0.$$

Now, let us consider the function $g(\lambda) = \dfrac{\dfrac{\partial F(x,\lambda)}{\partial\lambda}}{\phi_1'\big(\psi_1(F(x,\lambda))\big)}$. Taking derivative with respect to λ, we get

$$g'(\lambda) = \frac{\dfrac{\partial^2 F(x,\lambda)}{\partial\lambda^2}\times\phi_1'\big(\psi_1(F(x,\lambda))\big) - (\dfrac{\partial F(x,\lambda)}{\partial\lambda})^2\dfrac{\phi_1''\big(\psi_1(F(x,\lambda))\big)}{\phi_1'\big(\psi_1(F(x,\lambda))\big)}}{(\phi_1'\big(\psi_1(F(x,\lambda))\big))^2}.$$

The function $F(x;\lambda)$ is concave in λ and ϕ_1 is decreasing and convex. Thus, $g(\lambda)$ is increasing with respect to λ, from which it follows that (3.2) holds. According to Lemma 2.2 $\boldsymbol{\lambda}\overset{\mathrm{w}}{\succeq}\boldsymbol{\lambda^*}$ implies $F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)\leq F_{X_{n:n}}(x,\boldsymbol{\lambda^*},\phi_1)$. On the other hand, since $\psi_2\circ\phi_1$ is super-additive by Lemma 2.3, we have $F_{X_{n:n}}(x,\boldsymbol{\lambda^*},\phi_1)\leq F_{X_{n:n}}(x,\boldsymbol{\lambda^*},\phi_2)$. So, it holds that

$$F_{X_{n:n}}(x,\boldsymbol{\lambda},\phi_1)\leq F_{X_{n:n}}(x,\boldsymbol{\lambda^*},\phi_1)\leq F_{X_{n:n}}(x,\boldsymbol{\lambda^*},\phi_2).$$

That is, $X_{n:n}\geq_{\mathrm{st}} X_{n:n}^*$. □

Remark 3.3. It is known that the majorization order implies both weak submajorization and weak supermajorization orders. Therefore, the results of Theorem 3.2 are also valid under the majorization order between parameters.

Remark 3.4. When $\psi_2 \circ \phi_1$ is sub-additive, analogously to the proof of Theorem 3.2, one can prove that $(\lambda_1, \ldots, \lambda_n) \preceq_{\mathrm{w}} \overset{\mathrm{w}}{[\preceq]}(\lambda_1^*, \ldots, \lambda_n^*)$ implies $X_{n:n} \leq_{\mathrm{st}} X_{n:n}^*$, whenever $F(x;\lambda)$ is decreasing (increasing) and concave in $\boldsymbol{\lambda}$.

In the next result, we consider the models with a common Archimedean copula. By following the same lines as Theorem 3.2, the next result can be established, so its proof is omitted.

Theorem 3.5. Let $\boldsymbol{X}$ and $\boldsymbol{X^*}$ be two vectors of continuous random variables with $X_i \sim F(x;\lambda_i)$ and $X_i^* \sim F(x;\lambda_i^*)$, $i = 1, \ldots, n$ and share a common Archimedean copula with generator ϕ. If for any x the function $F(x;\lambda)$ is decreasing [increasing] and concave in $\boldsymbol{\lambda}$, then $(\lambda_1, \ldots, \lambda_n) \succeq_w \overset{\mathrm{w}}{[\succeq]}(\lambda_1^*, \ldots, \lambda_n^*)$ implies $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$.

Remark 3.6. It should be noted that sub-additive or super-additive assumption of $\psi_2 \circ \phi_1$ is satisfied by many members of well-known Archimedean copulas. For example, Archimedean (survival) copula with generators

- $\phi_1(x) = e^{1-(1+x)^{\frac{1}{\beta}}}$ and $\phi_2(x) = \dfrac{\beta}{\log(e^\beta + x)}$, where $0 < \beta \leq 1$,
- $\phi_1(x) = \dfrac{\beta}{\log(e^\beta + x)}$ and $\phi_2(x) = \log(e + x)^{\frac{-1}{\beta}}$, where $\beta > 1$,
- $\phi_1(x) = e^{1-(1+x)^{\frac{1}{\beta_1}}}$ and $\phi_2(x) = e^{1-(1+x)^{\frac{1}{\beta_2}}}$, where $\beta_2 \geq \beta_1 \geq 1$,

satisfy super-additivity and

- $\phi_1(x) = e^{-x^{\frac{-1}{\beta_1}}}$ and $\phi_2(x) = e^{-x^{\frac{-1}{\beta_2}}}$ where $\beta_1 \geq \beta_2 \geq 1$,
- $e^{\frac{1-e^x}{\beta_1}}$ and $e^{\frac{1-e^x}{\beta_2}}$ where $0 < \beta_1 \leq \beta_2 \leq 1$

satisfy sub-additivity.

In the following, we highlight some potential applications of the established results. [16] stated many semi-parametric models that satisfy the conditions of theorem 3.1. Some of these semi-parametric models are: transmuted-G (TG) model, Marshall–Olkin family, Marshall-Olkin family which the baseline distribution corresponds to the proportional hazard rate model, and TG model corresponds to the proportional hazard rate model. According to the theorems proved in this article, it is possible to have usual stochastic order between the lifetimes of the systems with lifetime components of these distributions without the condition of log convexity of the Archimedean generator.

Discussion and conclusions

In this work, we have studied stochastic comparisons in the usual stochastic order sense between largest order statistics from dependent random variables arising from general marginal distribution functions depending

on parameters. The established results here generalize many existing results in the literature. An example was given to show the results. As the results established here are for the usual stochastic order, further research could be conducted to generalize the results of this work to some other stochastic orders such as hazard rate, reversed hazard rate, likelihood ratio, and dispersive orders.

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The Bogomolov multiplier of Lie superalgebras

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Abstract

In this paper, we extend the concept of the Bogomolov multiplier and the commutativity-preserving extension to Lie superalgebras. Additionally, we compute the Bogomolov multiplier of some Lie superalgebras.

Keywords: Commutativity preserving exterior product, $\tilde{B}_0$-pairing, Curly exterior product, Bogomolov multiplier.

1 Introduction

In [7], Kac introduced the theory of Lie superalgebras, which are commonly referred to as $\mathbb{Z}_2$-graded Lie algebras. Subsequently, this theory, akin to Lie's original work, has been developed to establish a connection between Lie superalgebras and Lie supergroups. Here, we introduce the non-abelian commutativity-preserving exterior product and the Bogomolov multiplier for Lie superalgebras. The Bogomolov multiplier was initially defined for groups by Fedro Bogomolov [4] and serves as a group-theoretical invariant introduced as an obstruction to a problem in algebraic geometry known as the rationality problem or Noether's problem. Recently, in [33, 2], we extended this concept to Lie algebras and established its connection with the Bogomolov multiplier of groups via the Lazard correspondence. Throughout this paper, all modules and algebras are defined over the unital commutative ring $\mathbb{K}$. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be a field, and we define $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. A $\mathbb{Z}_2$-graded module (or supermodule) M is $M = M_{\bar{0}} \oplus M_{\bar{1}}$, whose elements are called even and odd, respectively. Non-zero elements of $M_{\bar{0}} \cup M_{\bar{1}}$ are said to be homogeneous. For a homogeneous element $m \in M_{\bar{\alpha}}$ with $\alpha \in \mathbb{Z}_2$, $|m| = \bar{\alpha}$ is the degree of m. Therefore, whenever we have the notation $|m|$, m will be a homogeneous element. A submodule N of M is called a $\mathbb{Z}_2$-graded submodule (or sub supermodule) if $N = N_{\bar{0}} \oplus N_{\bar{1}}$ where $N_{\bar{0}} = N \cap M_{\bar{0}}$ and $N_{\bar{1}} = N \cap M_{\bar{1}}$.

Definition 1. *A Lie superalgebra is a superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a multiplication denoted by $[,]$, called the super bracket operation, satisfying the following identities:*

(i) $[x, y] = -(-1)^{|x||y|}[y, x]$,

(ii) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$,

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(iii) $[l_{\bar{0}}, l_{\bar{0}}] = 0$

for all homogeneous elements $x, y, z \in L$ *and* $l_{\bar{0}} \in L_{\bar{0}}$.

Utilizing the aforementioned identities, it becomes evident that for a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, the even component $L_{\bar{0}}$ possesses the structure of a Lie algebra, while the odd component $L_{\bar{1}}$ serves as a module over $L_{\bar{0}}$. Consequently, if $L_{\bar{1}} = 0$, the superalgebra L is a Lie algebra. Conversely, if $L_{\bar{0}} = 0$, L transforms into an abelian Lie superalgebra, characterized by the property that $[x, y] = 0$ for all $x, y \in L$. However, it is important to note that in general, a Lie superalgebra does not necessarily possess the structure of a Lie algebra. The sub-superalgebra K of L is a $\mathbb{Z}_2$-graded vector subspace that is closed under the bracket operation. Consider $[L, L]$, which represents a graded subalgebra of L and is denoted as L^2. A $\mathbb{Z}_2$-graded subspace I is referred to as a graded ideal of L if $[I, L] \subseteq I$ and for all $x \in L$, the ideal $Z(L) = \{z \in L \mid [z, x] = 0\}$ is also a graded ideal and is known as the center of L. If I is an ideal of L, the quotient Lie superalgebra L/I inherits a canonical Lie superalgebra structure, such that the natural projection map transforms into a homomorphism. The notions of epimorphisms, isomorphisms, and automorphisms are straightforwardly defined. According to the super dimension structure of Lie superalgebras over a field, we denote $L = L_{\bar{0}} \oplus L_{\bar{1}}$ as an $(m|n)$ Lie superalgebra if $\dim L_{\bar{0}} = m$ and $\dim L_{\bar{1}} = n$. Throughout, $A(m|n)$ represents an abelian Lie superalgebra with dimension $(m|n)$. The descending central sequence of a Lie superalgebra L is defined recursively as $L^1 = L$ and $L^{c+1} = [L^c, L]$ for all $c \geq 1$. If for some positive integer c, $L^{c+1} = 0$ and $L^c \neq 0$, L is characterized as nilpotent, with the nilpotency class denoted as c. Furthermore, it is worth noting that $|[m, n]| = |m| + |n|$.

Definition 2. *Let M and N be two Lie superalgebras. A linear map $f : M \to N$ is called a homomorphism of degree $|f| \in \mathbb{Z}_2$ (or Lie super homomorphism), if $f(M_{\bar{\alpha}}) \subseteq N_{\bar{\alpha}+|f|}$, for every $x, y \in M$ and $\bar{\alpha} \in \mathbb{Z}_2$.*

Especially if $|f| = \bar{0}$, then the homomorphism f will be called an even linear map (or even grade). A Lie superalgebra homomorphism $f : M \to N$ is a homogeneous linear map of even degree such that $f([x, y]) = [f(x), f(y)]$ holds for all $x, y \in M$.

2 Main results

Definition 3. *Let L be a Lie superalgebra and M and N be graded ideals of L. A function $\rho : M \times N \to L$, is called a Lie super $\tilde{B}_0$-pairing, if the following holds.*

(i) $\rho(\lambda m, n) = \rho(m, \lambda n) = \lambda\rho(m, n)$,

(ii) $\rho(m + m', n) = \rho(m, n) + \rho(m', n)$,
where m, m' have the same grading,

(iii) $\rho(m, n + n') = \rho(m, n) + \rho(m, n')$,
where n, n' have the same grading,

(iv) $\rho([m, m'], n) = \rho(m, [m', n]) - (-1)^{|m||m'|}\rho(m', [m, n])$,

(v) $\rho(m, [n, n']) = (-1)^{|n'|(|m|+|n|)}\rho([n', m], n) - (-1)^{|m||n|}\rho([n, m], n')$,

(vi) $[\rho(m, n), \rho(m', n')] = -(-1)^{|m||n|}\rho([n, m], [m', n'])$,

(vii) *If* $[m, n] + (-1)^{|m'||n'|}[m', n'] = 0$, *then* $\rho(m, n) + (-1)^{|m'||n'|}\rho(m', n') = 0$,
If $[m_{\bar{0}}, n_{\bar{0}}] = 0$, *then* $\rho(m_{\bar{0}}, n_{\bar{0}}) = 0$,

for all $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ *and* $n_{\bar{0}} \in N_{\bar{0}}$.

Definition 4. *A Lie super $\tilde{B}_0$-pairing $\rho : M \times N \to L$ is called universal, if for any Lie super $\tilde{B}_0$-pairing $\rho' : M \times N \to L'$, there is a unique Lie homomorphism $\theta : L \to L'$ such that $\theta\rho = \rho'$.*

Definition 5. *Let L be a Lie superalgebra and M and N be two graded ideals of L. The CP exterior product $M \curlywedge N$ is the Lie superalgebra generated by all symbols $m \curlywedge n$ subject to the following relations*

(i) $\lambda(m \curlywedge n) = \lambda m \curlywedge n = m \curlywedge \lambda n$,

(ii) $(m + m') \curlywedge n = m \curlywedge n + m' \curlywedge n$,
where m, m' have the same graded,

(iii) $m \curlywedge (n + n') = m \curlywedge n + m \curlywedge n'$,
where n, n' have the same graded,

(iv) $[m, m'] \curlywedge n = m \curlywedge [m', n] - (-1)^{|m||m'|} m' \curlywedge [m, n]$,

(v) $m \curlywedge [n, n'] = (-1)^{|n'|(|m|+|n|)}[n', m] \curlywedge n - (-1)^{|m||n|}[n, m] \curlywedge n'$,

(vi) $[(m \curlywedge n), (m' \curlywedge n')] = -(-1)^{|m||n|}[n, m] \curlywedge [m', n']$,

(vii) *If* $[m, n] + (-1)^{|m'||n'|}[m', n'] = 0$, *then* $m \curlywedge n + (-1)^{|m'||n'|} m' \curlywedge n' = 0$,
If $[m_{\bar{0}}, n_{\bar{0}}] = 0$, *then* $m_{\bar{0}} \curlywedge n_{\bar{0}} = 0$,

for all $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ and $n_{\bar{0}} \in N_{\bar{0}}$.

In the case $M = N = L = L_{\bar{0}} \oplus L_{\bar{1}}$, we call $L \curlywedge L$ the curly exterior product of L and for any $x, y \in L_{\bar{0}} \cup L_{\bar{1}}$ and $x_{\bar{0}} \in L_{\bar{0}}$, since $[x, y]+(-1)^{|x||y|}[y, x] = 0$ and $[x_{\bar{0}}, x_{\bar{0}}] = 0$, we have $x \curlywedge y+(-1)^{|x||y|} y \curlywedge x = 0 \quad , \quad x_{\bar{0}} \curlywedge x_{\bar{0}} = 0$. If $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$ then the $M \curlywedge N$ is called the non-abelian CP exterior product of Lie algebras which is introduced in [33].

Lemma 2.1. *The function $\rho : M \times N \to M \curlywedge N$ given by $(m, n) \longmapsto m \curlywedge n$ is a universal Lie super $\tilde{B}_0$-pairing.*

Theorem 2.2. *Let L be a Lie superalgebra and M and N be two ideals of L. Then we have $M \curlywedge N \cong \frac{M \wedge N}{\mathcal{M}_0(M,N)}$. where $\mathcal{M}_0(M, N)$ is the graded submodule of $M \wedge N$ generated by the elements*

(i) $m \wedge n + (-1)^{|m'||n'|} m' \wedge n'$, *where* $[m, n] + (-1)^{|m'||n'|}[m', n'] = 0$,

(ii) $m_{\bar{0}} \wedge n_{\bar{0}}$, *where* $[m_{\bar{0}}, n_{\bar{0}}] = 0$,

with $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ and $n_{\bar{0}} \in N_{\bar{0}}$.

Similar to the theory of Lie algebras, we denote $\mathcal{M}(M, N)/\mathcal{M}_0(M, N)$ by $\tilde{B}_0(M, N)$, and we call it the Bogomolov multiplier of the pair of Lie superalgebras (M, N). Therefore, we have an exact sequence $0 \to \tilde{B}_0(M, N) \to M \curlywedge N \to [M, N] \to 0$. In the case $M = N = L$, we denote $\mathcal{M}_0(L, L)$ by $\mathcal{M}_0(L)$. Also, we denote $\mathcal{M}(L)/\mathcal{M}_0(L)$ by $\tilde{B}_0(L)$, and we call it the Bogomolov multiplier of the Lie superalgebra L. So we have an exact sequence $0 \to \tilde{B}_0(L) \to L \curlywedge L \to L^2 \to 0$.

Proposition 2.3. *Let M and N be two graded ideals of a Lie superalgebra L. Then $(M \oplus N) \curlywedge (M \oplus N) \cong (M \curlywedge M) \oplus (N \curlywedge N)$.*

Proposition 2.4. *Let L be a Lie superalgebra with the free presentation $L \cong F/R$, then $\tilde{B}_0(L) \cong (R \cap F^2)/ < K(F) \cap R >$, where $K(F) = \{[x, y] \mid x, y \in F\}$.*

Proposition 2.5. *Let L be a Lie superalgebra with the free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$. If T is an ideal in F with $M = T/R$, then the following sequences are exact.*

(i) $\tilde{B}_0(L) \to \tilde{B}_0(L/M) \to M/ < K(L) \cap M > \to L/L^2 \to L/M/(L/M)^2 \to 0$,

(ii) $0 \to \dfrac{R \cap < K(F) \cap T >}{< K(F) \cap R >} \to \tilde{B}_0(L) \to \tilde{B}_0(\dfrac{L}{M}) \to \dfrac{M \cap L^2}{< K(L) \cap M >} \to 0.$

Definition 6. *Let L, M, and C be Lie superalgebras. An exact sequence of Lie superalgebras $0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is called a commutativity preserving extension (CP extension) of M by L, if commuting pairs of elements of L have commuting lifts in C. A special type of CP extension with the central kernel is named a central CP extension.*

Proposition 2.6. *Let $e : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ be a central extension. Then e is a CP extension if and only if $\chi(M) \cap K(C) = 0$.*

Definition 7. *An abelian ideal M of a Lie superalgebra L is called a CP Lie sub superalgebra of L if the extension $0 \to M \to L \to L/M \to 0$ is a CP extension.*

Example 2.7. Let $0 \to R \to F \to L \to 0$ be a free presentation of L. Then the extension $0 \to \frac{R \cap F^2}{<K(F) \cap R>} \to \frac{R}{<K(F) \cap R>} \to \frac{R}{R \cap F^2} \to 0$ is a CP extension. So, the abelian ideal $\frac{R \cap F^2}{<K(F) \cap R>}$ of $\frac{R}{<K(F) \cap R>}$ is a CP Lie sub-superalgebra.

Example 2.8. We know in general that a Lie algebra is a Lie superalgebra. Thus, when L is a cyclic Lie algebra and C is a non-abelian Lie algebra, then $e : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is a CP extension that is not central.

Proposition 2.9. *Let L be a Lie superalgebra with the free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$. If T is an ideal in F with $N = T/R$, then the following are equivalent.*

1. *N is a CP sub Lie superalgebra of L,*
2. *The canonical map $\psi : \mathcal{M}_0(L) \to \mathcal{M}_0(L/N)$ is surjective.,*
3. *The canonical map $\varphi : L \curlywedge L \to L/N \curlywedge L/N$ is an isomorphism..*

Proposition 2.10. *Let L_1 and L_2 be Lie superalgebras. Then $\tilde{B}_0(L_1 \oplus L_2) \cong \tilde{B}_0(L_1) \oplus \tilde{B}_0(L_2)$.*

The Heisenberg Lie superalgebra is a Lie superalgebra $H = H_{\bar{0}} \oplus H_{\bar{1}}$ such that its first derived ideal is equal to its 1-dimensional homogeneous center, i.e. $Z(H) = H^2$ and $\dim H^2 = 1$.

Theorem 2.11. *Every Heisenberg Lie superalgebra with odd (or even) center has a trivial Bogomolov multiplier.*

Theorem 2.12. *Let L be a real (trivial or nontrivial) Lie superalgebra of dimension at most 4. Then $\tilde{B}_0(L) = 0$.*

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Modeling Tumor Growth with Cancer Stem Cells Using Fractional Order Equations

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Abstract

We extend the integer-order tumor growth model to the fractional-order domain. The proposed model consists of a system of nonlinear integro-differential equations that describe tumor growth and spatial spread over time. We then prove the existence, uniqueness, and boundedness of solutions.

Keywords: Cancer stem cells, Integro-differential equations, fractional derivative.

1 Introduction

Cancer stem cells ($CSCs$), also known as tumor-initiating or tumor-propagating cells, represent a distinct subpopulation of cancer cells with the capacity for self-renewal and differentiation. These cells possess stem cell-like properties, enabling them to give rise to both new ($CSCs$) and non-stem cancer cells (CCs). ($CSCs$)have been identified in a wide range of malignancies, including breast, brain, colon, prostate, pancreatic, and ovarian cancers. Due to their pivotal role in tumor initiation, recurrence, metastasis, and resistance to conventional treatments, ($CSCs$) are regarded as promising targets for the development of more effective cancer therapies [5, 2].

In 2009, Enderling developed a cellular automaton model to simulate non-stem cancer cell (CCs) behavior. His model showed that when the death rate of non-stem cells increases, it creates space for cancer stem cells ($CSCs$)to proliferate, leading to larger tumors. This unexpected effect is known as the tumor growth paradox [3].

In 2013, Hillen used geometric singular perturbation theory to study diseases where cancer stem cells ($CSCs$) and non-stem cancer cells (CCs) coexist. Hillen transformed Enderling's cellular automaton model into a system of integro-partial differential equations ($iPDEs$)to describe the continuous population densities of ($CSCs$) and (CCs) [4].

Considering the integro-differential system of $PDEs$ in the following model

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = D_u \triangle u(x,t) + \delta\gamma \displaystyle\int_\Omega k(x,y,p(x,t))u(y,t)dy, \\ \dfrac{\partial v(x,t)}{\partial t} = D_v \triangle v(x,t) + (1-\delta)\gamma \displaystyle\int_\Omega k(x,y,p(x,t))u(y,t)dy, \\ \qquad\qquad + \rho \displaystyle\int_\Omega k(x,y,p(x,t))v(y,t)dy - \alpha v(x,t) \end{cases} \tag{1.1}$$

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Borsi analyzes Hillen's model under the assumption of a zero diffusion coefficient and proves key mathematical properties of the model, including positivity, boundedness, existence, and uniqueness [2].

In recent years, there has been increased attention on using fractional-order equations to model infectious diseases. These models are more precise than traditional integer-order ones, which has led to a rise in their adoption. This study introduces a model that effectively captures both time-dependent and spatially nonlocal effects. A major benefit of fractional-order models is their ability to reduce errors from neglected parameters in natural phenomena. We use time-fractional derivatives to simulate tumor growth in the presence of cancer stem cells, expanding on the integer-order model previously analyzed by Hillen [4]. The goal of this research is to examine how varying mortality rates of (CCs) influence the cancer growth paradox, demonstrating that the paradox occurs under different parameter conditions in the proposed model [1].

In Section 2, we introduce fractional-order models, and in Section 3, we establish the existence, uniqueness, and boundedness of their solutions.

2 The Model with Fractional Derivatives

Assuming no diffusion term, as in previous work [4], there are various interpretations of fractional calculus, though most are abstract and lack physical insight. The interpretation we present focuses on the concept of memory, which plays a central role in fractional calculus. In memoryless systems, the output depends solely on the current input at time t, while in memory-based systems, the output depends on past inputs. Fractional derivatives, which incorporate long-term memory, provide a better fit for modeling cellular responses. Additionally, we introduce a positive coefficient $\frac{1}{\lambda^{1-\beta}}$ where λ acts as an auxiliary parameter to ensure dimensional consistency. This leads to a simplified form of the system

$$\frac{1}{\lambda^{1-\beta}}\left(\frac{\partial^\beta u(x,t)}{\partial t^\beta}\right) = \delta\gamma\int_\Omega k(x,y,p(x,t))u(y,t)\,dy \tag{2.1}$$

$$\frac{1}{\lambda^{1-\beta}}\left(\frac{\partial^\beta v(x,t)}{\partial t^\beta}\right) = (1-\delta)\gamma\int_\Omega k(x,y,p(x,t))u(y,t)\,dy + \rho\int_\Omega k(x,y,p(x,t))v(y,t)\,dy - \alpha v(x,t). \tag{2.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad u_0(x) + v_0(x) \geq 0 \tag{2.3}$$

Where

β (range: $(0,1]$) represents the fractional order of the time derivative; δ (range: $[0,1]$) denotes the symmetric division fraction for cancer stem cells $(CSCs)$, serving as their average symmetric division ratio; α (range: $(0,\infty)$) models the mortality rate of non-stem cancer cells (CCs); D_u and D_v (both ranging in $[0,\infty)$) are the diffusion coefficients for (CCs) and $(CSCs)$, respectively, describing their spatial spread; ρ (range: $(0,\infty)$) represents the cell cycle rate (number of divisions per unit time) of (CCs), while γ (range: $(0,\infty)$) describes the cell cycle rate of $(CSCs)$; and λ (range: $(0,\infty)$) is an auxiliary parameter included for additional model flexibility. In this system, the densities of cancer stem cells $(CSCs)$ and non-stem cancer cells (CCs) are represented by the functions $u(x,t)$ and $v(x,t)$, respectively, at spatial position x and time t, with the total tumor density given by $p(x,t) = u(x,t) + v(x,t)$.

The spatial domain is denoted by Ω, and $k(x,y,p(x,t))$ represents the transition probability density for a cell at location y to generate a cell at x.

3 Analysis of the Existence and Uniqueness of Solutions

The following analysis is dedicated to proving that the formulated tumor growth model has unique and bounded solutions, both locally and globally in time.

Lemma 3.1. *Assumption* (2.3) *guarantees that all solutions to equations* (2.1) *and* (2.2) *are subject to an a priori bound.*

Proof. To derive prior estimates for the bounds of u and v, we introduce the following norm definitions

$$\|u\|_t := \max_{\bar{\Omega}\times[0,t]} |u(x,\tau)|, \tag{3.1}$$

$$\|v\|_t := \max_{\bar{\Omega}\times[0,t]} |v(x,\tau)|. \tag{3.2}$$

In the following steps, we take the RL fractional integral operator of order β on both sides of (2.1)

$$I^{\beta}\left[\frac{1}{\lambda^{1-\beta}}\left(D^{\beta}u(x,t)\right)\right] = I^{\beta}\left[\delta\gamma\int_{\Omega}k(x,y,p(x,t))u(y,t)\,dy\right],$$

By simplifying and applying the rules of fractional calculus, along with the separability property of the kernel and taking the norm with respect to t, we obtain

$$\|u(x,t)\|_t \leq \max_{\bar{\Omega}}|u_0| + \frac{\delta\gamma}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|u\|_s ds,$$

Finally, by applying Gronwall's Lemma, the following equation is derived

$$\|u(x,t)\|_t \leq \max_{\bar{\Omega}}|u_0|E_{\beta}(\gamma\delta t^{\beta}) \equiv U(t). \tag{3.3}$$

Next, with a similar method to the previous process, to obtain the relationship (2.1), we conclude

$$\|v(x,t)\|_t \leq \left(\max_{x\in\bar{\Omega}}|v_0| + \frac{t^{\beta}}{\beta\Gamma(\beta)}\gamma(-\delta+1)\max_{\bar{\Omega}}|u_0|\,E\left(\gamma\delta t^{\beta}\right)\right)E_{\beta}\left((\rho+\alpha)t^{\beta}\right) \equiv V(t). \tag{3.4}$$

In the following, we present the following result about the continuous dependences on the initial conditions. □

Theorem 3.2. *Let* $u^{(i)}$ *and* $v^{(i)}$ *solve* (2.1) *and* (2.2) *with data* ${u_0}^{(i)}, {v_0}^{(i)}, i = 1, 2, \ldots$. *Denote by* $\Delta u, \cdots$ *the differences* $u^{(1)} - u^{(2)}$, *etc. Then*

$$\|\Delta u\|_t + \|\Delta v\|_t \leq Ge^{gt}\max[\Delta u_0 + \Delta v_0],$$

where G, g *are known constants.*

Proof. We assume that

$$\left[\frac{1}{\lambda^{1-\beta}}\left(D^{\beta}u^1\right)\right] = \delta\gamma\int_{\Omega}k(x,y)F(p^1)u^1(y,t)dy,$$

$$\left[\frac{1}{\lambda^{1-\beta}}\left(D^{\beta}u^2\right)\right] = \delta\gamma\int_{\Omega}k(x,y)F(p^2)u^2(y,t)dy.$$

Then, by subtracting one relation from the other, we obtain

$$D^{\beta}(u^1 - u^2) = \delta\gamma \int_{\Omega} k(x,y)(F(p^1)u^1(y,t) - F(p^2)u^2(y,t))dy$$

In the following, we add and subtract $F(p^1)u^1(y,t)$ to the above equation. The following equation is derived

$$D^{\beta}\Delta u = \delta\gamma \int_{\Omega} k(x,y)[u(y,t)L\Delta p + \Delta u(y,t)]dy,$$

Thus, , by employing the Lipschitz property of the function and taking norm with respect to t on both side, we have

$$D^{\beta}\Delta u \leq \delta\gamma U(t)L\|\Delta p\|_t + \delta\gamma\|\Delta u(y,t)\|_t,$$

Finally, by applying the fractional RL integral operator to both sides of above equation, we obtain

$$\|\Delta u\|_t \leq \max_{\Omega}|\Delta u_0| + \frac{\gamma\delta}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|\Delta u\|_s \,\mathrm{d}s + \frac{\gamma\delta LU(t)}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|\Delta p\|_s \,\mathrm{d}s, \tag{3.5}$$

In a similar procedure for the function v we have

$$\|\Delta v\|_t \leq \max_{\Omega}|\Delta v_0| + \frac{L(\delta(1-\delta)U(t) + \rho V(t))}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|\Delta p\|_s \,ds + \frac{(1-\delta)\gamma}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|\Delta u\|_s \,ds, \tag{3.6}$$

Let $U(t)$ and $V(t)$ denote the upper bounds of $u^{(i)}$ and $v^{(i)}$ for $i = 1, 2$, as established in the preceding lemma. The constant L represents the Lipschitz constant associated with the function $F(p)$. By merging equations (3.6) and(3.6) , and employing Grönwall's inequality, we obtain the following bound for the expression $\|\Delta u\|_t + \|\Delta v\|_t$

$$\|\Delta u\|_t + \|\Delta v\|_t \leq \max_{\Omega}|\Delta u_0 + \Delta v_0| E_{\beta}\left([L(\gamma U(t) + \rho V(t)) + \alpha + \rho + \gamma]t^{\beta}\right),$$

which implies

$$\|\Delta p\|_t \leq \max_{\Omega}|\Delta u_0 + \Delta v_0| \, E_{\beta}\left(G(t)t^{\beta}\right),$$

for the known function G. We obtain the upper bound for $\|\Delta p\|_t$ and the theorem is proved.

□

As a result, the uniqueness is proved.

Corollary 3.3. *The system* (2.1) *and* (2.2) *with data* (2.3)*, has at most one solution.*

Sharper estimates on the bounds of u and v are summarized in the following lemma.

Lemma 3.4. *Let u and v solve equations* (2.1) *and* (2.2) *in* $[\Omega \times [(t_1, t_1 + h)]]^2$ *for some* $0 \leq t_1 \leq t_1 + h$, $h \neq 0$ *and* $p = u + v$

1. *If* $u(x,t_1) \geq 0$, $v(x,t_1) \geq 0$ *for all* $x \in \Omega$, *then*

$$u(x,t) \geq 0, \quad v(x,t) \geq 0,$$

in $\Omega \times [t_1, t_1+h]$.

2. *Assume that* $1 \geq u(x,t_1) \geq 0$, $1 \geq v(x,t_1) \geq 0$ *for all* $x \in \Omega$, *then*

   i) if $u(\hat{x},t_1) = p(\hat{x},t_1) = 1$, *for some* $\hat{x} \in \Omega$, *then*

$$u(\hat{x},t) = 1, \quad v(\hat{x},t) = 1,$$

   for $t \in [t_1, t_1+h]$.

   ii) if $u(\hat{x},t) < 1$, *for some* $\hat{x} \in \Omega$, *then*

$$u(\hat{x},t) < 1,$$

   for $t \in [t_1, t_1+h]$.

3. *If* $p(\hat{x},t_1) < 1$, *for some* $\hat{x} \in \Omega$, *then* $p(\hat{x},t) < 1$, *for* $t \in (t_1, t_1+h)$.

4. *If* $v(\hat{x},t_1) < 1$, *for some* $\hat{x} \in \Omega$, *then* $v(\hat{x},t) < 1$, *for* $t \in (t_1, t_1+h)$.

In the following, we will introduce a theorem to show the existence of the required solution.

Theorem 3.5. *Assuming the conditions associated with the integral equations* (2.1) *and* (2.2) *are satisfied, the system admits a unique solution that exists globally.*

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A numerical solution for an optimal control problem with parabolic PDE constraints

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Abstract

This paper addresses the solution of an optimal control problem with parabolic PDE constraints for the temperature distribution in a solid body. The main objective is to minimize the difference between the final and the desired temperature, subject to boundary controls. The existence and uniqueness of weak solutions are established, and the necessary optimality conditions are derived. A gradient descent method is used to solve the problem through control updates.
Keywords: Optimal control problem, Optimality conditions, Gradient descent method.

1 Introduction

In many physical and engineering systems, maintaining an optimal temperature distribution is essential for ensuring both performance and safety. Applications include thermal regulation in industrial furnaces[33], cooling mechanisms in electronic devices[2], and temperature control in medical treatments such as hyperthermia[3]. These processes are typically governed by the heat equation. In practical scenarios, control is often exerted at the boundary, for example by adjusting the heat flux applied to the surface. This leads naturally to boundary control problems, where the goal is to influence the system's evolution using inputs applied only at its boundary. In this study, we consider an optimal boundary control problem for the one-dimensional heat equation, where the aim is to drive the temperature distribution to a desired profile at a prescribed final time T while minimizing a cost functional. The boundary control is assumed to be bounded, reflecting physical constraints on the applied heat flux.

Consider $(0,l) \subset \mathbb{R}$, and define the space-time domain $(0,T) \times (0,l)$, with boundary $(0,T) \times \{0,l\}$. Let the temperature distribution be modeled by a function $u : [0,T] \times (0,l) \to \mathbb{R}$, with initial and desired final temperatures denoted by $u_0 \in L^2(0,l)$ and $u_d \in L^2(0,l)$, respectively. To determine a control function $v : (0,T) \times \{0,l\} \to \mathbb{R}$ that minimizes the deviation between the actual temperature $u(T,\cdot)$ at time T and

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the desired temperature u_d, we introduce the following optimization problem:

$$\begin{aligned} \min\ & J(u,v) := \frac{1}{2}\int_0^l (u(T,x)-u_d(x))^2\,dx + \frac{\alpha}{2}\int_0^T \left[v(t,0)^2 + v(t,l)^2\right] dt \\ \text{subject to:}\quad & u_t(t,x) - u_{xx}(t,x) = 0 \quad \text{in } (0,T)\times(0,l), \\ & \frac{\partial u}{\partial \eta}(t,x) = v(t,x) \quad \text{on } (0,T)\times\{0,l\}, \\ & u(0,x) = u_0(x) \quad \text{in } (0,l), \\ & a \le v \le b \quad \text{on } (0,T)\times(0,l), \end{aligned} \tag{1.1}$$

where $a, b \in L^2((0,T)\times(0,l))$ with $a < b$, and $\alpha > 0$ is a regularization parameter. The function u_t denotes the time derivative and u_{xx} the second spatial derivative.

2 Existence and Uniqueness Results

In this section, we discuss the existence and uniqueness of solutions to the optimal control problem (1.1). To this end, we first address the existence and uniqueness of the weak solution to the state equation, governed by the one-dimensional heat equation with boundary control. We then formulate the abstract framework for the analysis.

2.1 Existence and Uniqueness of the State Equation

To study the weak formulation of the state equation, we begin with the following definition:

Definition 1 (Weak solution of the parabolic PDE)**.** *[4] Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. Consider the Hilbert spaces*

$$H := L^2(\Omega), \quad V := H^1(\Omega),$$

which form the Gelfand triple

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*,$$

that is,

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

Let $v \in L^2(0,T;V^*)$ and $u_0 \in L^2(\Omega)$. A function $u \in W(0,T;H,V) := \{u \in L^2(0,T;V) \,|\, u_t \in L^2(0,T;V^*)\}$ is called a weak solution of the initial-boundary value problem associated with (1.1) if u satisfies the variational equation

$$\int_0^T \langle u_t(t), \varphi(t)\rangle_{V^*,V}\,dt + \int_0^T a(u(t),\varphi(t);t)\,dt = \int_0^T \langle v(t),\varphi(t)\rangle_{V^*,V}\,dt, \quad \forall \varphi \in L^2(0,T;V), \tag{2.1}$$

together with the initial condition $u(0) = u_0$. Here, the bilinear form $a(\cdot,\cdot;t) : V \times V \to \mathbb{R}$ is defined by

$$a(u,\varphi;t) := \int_0^l u_x(t)\cdot\varphi_x dx, \quad \forall u,\varphi \in H^1(\Omega).$$

We now impose the following standard assumptions:
Assumption 1[4]

(i) The embedding $V \hookrightarrow H \hookrightarrow V^*$ forms a Gelfand triple, with H and V separable Hilbert spaces.

(ii) The bilinear form $a(\cdot,\cdot;t)$ satisfies the following properties for almost every $t \in (0,T)$:

- Boundedness: There exists a constant $\alpha > 0$ such that

$$|a(\varphi,\psi;t)| \leq \alpha \|\varphi\|_V \|\psi\|_V, \quad \forall \varphi,\psi \in V.$$

- Coercivity: There exist constants $\beta > 0$ and $\gamma \geq 0$ such that

$$a(\varphi,\varphi;t) + \gamma\|\varphi\|_H^2 \geq \beta \|\varphi\|_V^2, \quad \forall \varphi \in V.$$

- Measurability: For all $\varphi,\psi \in V$, the mapping $t \mapsto a(\varphi,\psi;t)$ is measurable.

(iii) The initial condition satisfies $u_0 \in H$, and the control input satisfies $v \in L^2(0,T;V^*)$.

Theorem 2.1. *[4] Let Assumption 1 hold. Then the abstract parabolic problem (1.4) admits a unique weak solution $u \in W(0,T;H,V)$. Furthermore, this solution satisfies the following energy estimate for all $t \in (0,T]$:*

$$\|u(t)\|_H^2 + \|u\|_{L^2(0,t;V)}^2 + \|u_t\|_{L^2(0,t;V^*)}^2 \leq C\left(\|u_0\|_H^2 + \|v\|_{L^2(0,t;V^*)}^2\right), \tag{2.2}$$

where $C > 0$ is a constant depending only on the constants α,β,γ in Assumption 1.

2.2 Existence of Optimal Controls

To examine the existence of control term v, we consider the linear-quadratic optimization problem of the form as follows[4]:

$$\min_{(u,v)\in Y\times U} J(u,v) := \frac{1}{2}\|Qu - q_d\|_H^2 + \frac{\alpha}{2}\|v\|_U^2 \tag{2.3}$$

subject to

$$Au + Bv = g, \quad v \in U_{ad}, \quad u \in Y_{ad} \tag{2.4}$$

where H, U are Hilbert spaces, Y, Z are Banach spaces, and

$$q_d \in H, \quad g \in Z, \quad A \in \mathcal{L}(Y,Z), \quad B \in \mathcal{L}(U,Z), \quad Q \in \mathcal{L}(Y,H).$$

We now state the assumptions under which the optimization problem admits a solution:
Assumption 2[4]

1. $\alpha \geq 0$, $U_{ad} \subset U$ is convex, closed, and in the case $\alpha = 0$, also bounded.
2. $Y_{ad} \subset Y$ is convex and closed, and problem (2.4) admits at least one feasible point.
3. The operator $A \in \mathcal{L}(Y,Z)$ is invertible and has a bounded inverse.

Definition 2 (Optimal state-control pair)**.** *[4] A pair $(\bar{u},\bar{v}) \in Y_{ad} \times U_{ad}$ is called an optimal solution of problem (2.4) if*

$$A\bar{u} + B\bar{v} = g \quad \text{and} \quad J(\bar{u},\bar{v}) \leq J(u,v) \quad \forall (u,v) \in Y_{ad} \times U_{ad}.$$

Theorem 2.2 (Existence of an optimal control)**.** *[4] Let Assumption 2 hold. Then the optimization problem (2.4) admits at least one optimal solution $(\bar{u},\bar{v})$. Moreover, if $\alpha > 0$, then the optimal solution is unique.*

3 Optimality Conditions

In this section, we derive the first-order optimality conditions for the control problem (1.1). Hence, define the spaces

$$U = L^2((0,T)\times\{0,l\}), \quad Y = W(0,T;L^2(0,l),H^1(0,l)),$$
$$Z = L^2(0,T;H^1(0,l)^*)\times L^2(0,l).$$

It is straightforward to verify that the spaces satisfy the assumptions required for well-posedness.

Furthermore, the weak formulation of the state equation defines a bounded affine-linear operator, which admits a unique solution, and its partial derivative with respect to u is boundedly invertible [4]. Moreover, using the embedding $Y \hookrightarrow C([0,T];L^2(0,l))$, it follows that the objective functional $J : Y \times U \to \mathbb{R}$ is continuously Fréchet differentiable. Therefore, Assumption 2 is satisfied. We now state the optimality conditions in terms of the optimal state $\bar{u}$, optimal control $\bar{v}$, and the adjoint state $\bar{p}$.

Theorem 3.1. *Let $(\bar{u},\bar{v})$ be an optimal solution of (1.1). Then there exist $\bar{p}\in Y$ and multipliers $\bar{\lambda}_a,\bar{\lambda}_b \in L^2((0,T)\times\{0,l\})$ such that the following optimality system holds in the weak sense:*

$$\begin{aligned} &\bar{u}_t - \bar{u}_{xx} = 0 \quad \text{in}(0,T)\times(0,l),\\ &\tfrac{\partial \bar{u}}{\partial \eta} = \bar{v} \quad \text{on}(0,T)\times\{0,l\}\\ &\bar{u}(0,x) = u_0(x) \quad \text{in}(0,l), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &-\bar{p}_t - \bar{p}_{xx} = 0 \quad \text{in}(0,T)\times(0,l),\\ &\tfrac{\partial \bar{p}}{\partial \eta} = 0 \quad \text{on}(0,T)\times\{0,l\},\\ &\bar{p}(T,x) = -(\bar{u}(T,x)-u_d(x)) \quad \text{in}(0,l), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\alpha\bar{v} - \bar{p} + \bar{\lambda}_b - \bar{\lambda}_a = 0 \quad \textit{on } (0,T)\times\{0,l\},\\ &\bar{v}\geq a, \quad \bar{\lambda}_a \geq 0, \quad \bar{\lambda}_a(\bar{v}-a) = 0 \quad \textit{on } (0,T)\times\{0,l\},\\ &\bar{v}\leq b, \quad \bar{\lambda}_b \geq 0, \quad \bar{\lambda}_b(b-\bar{v}) = 0 \quad \textit{on } (0,T)\times\{0,l\}. \end{aligned} \tag{3.3}$$

4 Gradient Descent for Optimal Control

Theorem 2.1 provides the necessary optimality conditions for the boundary control problem (1.1). These conditions include the state equation (3.1), the adjoint equation (3.2), and the optimality condition (2.1), which together form a coupled system that must be solved iteratively to determine the optimal control v.

To numerically compute v, we employ a gradient descent method, which iteratively updates the control in a direction that reduces the objective functional.

Gradient Descent Algorithm:

- **Initialize:** Choose an initial guess $v^{(0)} \in L^2(\Sigma_T)$ and compute the corresponding state $u^{(0)}$.

1. Solve the state equation to obtain $u^{(k)}$.
2. Solve the adjoint equation to obtain $p^{(k)}$.
3. Compute the gradient of the Lagrangian with respect to the control:

$$\nabla_v L = \alpha v^{(k)} - p^{(k)} + \lambda_b^{(k)} - \lambda_a^{(k)}.$$

4. Update the control using a step size $\eta > 0$:

$$v^{(k+1)} = v^{(k)} - \eta \nabla_v L.$$

5. Apply the control constraints pointwise:

$$v^{(k+1)} = \min\big(\max(v^{(k+1)}, a), b\big).$$

6. Terminate the iteration if $\|\nabla_v L\|$ falls below a given tolerance.

5 Numerical Implementation

To obtain the weak form of (1.1) in the one-dimensional case, we choose an arbitrary test function $\varphi \in L^2(0,T;H^1(0,l))$, multiply the entire equation by φ, and integrate over $\{0,l\} \times (0,T)$. Applying integration by parts in space (Gauss-Green theorem), we arrive at the following weak form:

$$\int_0^T \int_0^l (u_t\varphi + u_x\varphi_x)\, dx\, dt = \int_{\Sigma_T} \frac{\partial u}{\partial \eta}\varphi\, dt.$$

By substituting the boundary condition from the state equation (1.1), we obtain:

$$\int_0^T \int_0^l (u_t\varphi + u_x\varphi_x)\, dx\, dt = \int_0^T [v(0)\varphi(0) + v(l)\varphi(l)]dt$$

We now discretize the above weak form. For the temporal discretization, we use the forward finite difference method; for the spatial discretization, we use the radial basis function (RBF) method. First, we discretize in time. Let

$$u_t(t,x) \approx \frac{u(t_{n+1},x) - u(t_n,x)}{h},$$

so the semi-discrete form becomes

$$\int_0^T \int_0^l \left(\frac{u(t_{n+1},x) - u(t_n,x)}{h}\varphi(x) + u_x(t_{n+1},x)\varphi_x(x)\right) dx\, dt = \int_0^T [v(t_{n+1},0)\varphi(0) + v(t_{n+1},l)\varphi(l)]dt$$

Next, we apply the Gaussian radial basis function method. We consider the following basis functions:

$$\phi_i(x) = \exp\left(-\alpha r^2\right), \qquad r = \|x - x_i\|_2,$$

where $\|\cdot\|_2$ is the Euclidean norm. We approximate

$$u(t,x) \approx \sum_{j=0}^{N} c_j(t)\xi_j(x), \qquad \varphi(x) \approx \xi_i(x),$$

so the time-discrete weak form becomes:

$$\sum_{j=0}^{N} \int_0^T c_j(t_{n+1})(M_{ij} + hK_{ij})\, dt = \sum_{j=0}^{N} \int_0^T c_j(t_n)M_{ij}\, dt + h\int_0^T f_i^{n+1}\, dt,$$

where

$$\begin{cases} M_{ij} = \int_0^l \xi_j(x)\xi_i(x)\,dx, \\ K_{ij} = \int_0^l \xi_j'(x)\xi_i'(x)\,dx, \\ f_i^{n+1} = [v(t_{n+1},0)\varphi(0) + v(t_{n+1},l)\varphi(l)] \end{cases}$$

Therefore, the final discrete form of the state equation with its initial condition is:

$$(M + hK)c^{n+1} = Mc^n + hf^{n+1},$$

$$u_0(x) = \sum_{j=0}^{N} c_j(0)\xi_j(x),$$

where M is the mass matrix, K is the stiffness matrix, and f^{n+1} is the boundary source vector.

Adjoint Form

To derive the adjoint equation, we define the Lagrangian:

$$\begin{aligned} L(u,v,p,q) = \frac{1}{2}\int_0^l (u(T,x) - u_d(x))^2 dx + \frac{\alpha}{2}\int_0^T \left[v(t,0)^2 + v(t,l)^2\right] dt \\ + \int_0^T \int_0^l u_t(t,x)p(t,x)\,dx\,dt + \int_0^T \int_0^l u_x(t,x)p_x(t,x)\,dx\,dt \\ - \int_0^T [v(0)p(t,0) + v(l)p(t,l)]dt + \int_0^l (u(0,x) - u_0(x))q(x)\,dx, \end{aligned}$$

with the constraint $q = p(0,x)$.

Taking the derivative of L with respect to u, we obtain:

$$\frac{\partial L}{\partial u} = u(T,x) - u_d(x) - \partial_t p(t,x) + p(T,x) - p(0,x) - \partial_{xx} p(t,x) + q = 0.$$

Rearranging terms yields the adjoint problem:

$$\begin{aligned} -p_t - p_{xx} &= 0 && \text{in } (0,T)\times(0,l), \\ \frac{\partial p}{\partial \eta} &= 0 && \text{on } (0,T)\times\{0,l\}, \\ p(T,x) &= -(u(T,x) - u_d(x)) && \text{in } (0,l). \end{aligned}$$

The weak form of the adjoint equation is:

$$\int_0^T \int_0^l (-p_t\varphi - p_x\varphi_x)\,dx\,dt = 0, \qquad p(T,x) = -(u(T,x) - u_d(x)).$$

Following the same time and space discretization procedure, we arrive at the linear system:

$$(M + hK)c^{n+1} = Mc^n,$$

with final condition:

$$p(T,x) = -(u(T,x) - u_d(x)).$$

Here, M and K are again the mass and stiffness matrices, respectively.

Table 1: Optimal Control Values

Time	$v(t,0)$	$v(t,1)$
0.00	0.4112	0.4115
0.25	0.4110	0.4112
0.50	0.4032	0.4035
0.75	0.2768	0.2032
1.00	0.1924	-0.9687

Table 2: Final State at $T = 1.0$

Position	$u(T,x)$	$u_d(x)$	Error
0.00	0.4358	0.0000	0.4358
0.25	0.5682	0.7071	0.1389
0.50	0.6541	1.0000	0.3459
0.75	0.5682	0.7071	0.1389
1.00	0.4358	0.0000	0.4358

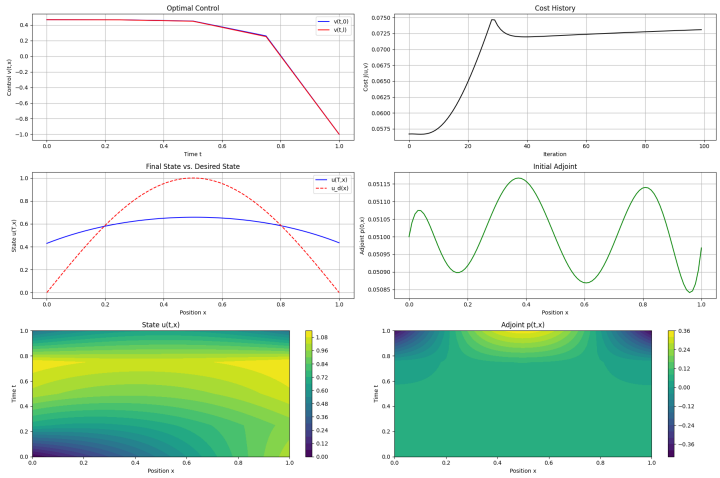


Figure 1: Boundary Control Optimization for Heat Equation.

5.1 Numerical Results

Example 5.1. Consider the boundary control problem (1.1), where the desired state is given by $u_d(x) = \sin(\pi x)$, the control v is constrained within the bounds $[a, b] = [-1, 1]$, and the regularization parameter is $\alpha = 0.1$. The spatial domain is discretized using Gaussian radial basis functions (RBFs) in the interval $[0, 1]$, and the RBF shape parameter is set to $\sigma = 1.0$. The temporal discretization employs a time step of $h = 0.25$, resulting in five time points including the initial time $t = 0$. For the optimization procedure, a fixed step size of $\eta = 0.1$ is used with a maximum of 100 iterations. The algorithm converged after 100 iterations with a final cost value of $J(u, v) = 0.0731$ and a root-mean-square (RMS) error given by

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}|u(T,x_i) - u_d(x_i)|^2} = 0.0826$$

and the results are obtained as follows in Tables 1 and 2 and Figure 1.

6 Conclusion

This paper has established the mathematical framework for optimal boundary control of temperature distribution governed by parabolic PDEs. We proved existence and uniqueness of solutions, derived necessary optimality conditions, and used a gradient descent algorithm for numerical implementation. Our approach combines forward finite difference time discretization with radial basis function spatial approximation.

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solving the shortest path problem on network with intuitionistic fuzzy arc length

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Abstract

In this paper, the Shortest Path Problem (SPP) has been considered in an uncertain environment. The cost parameters for traveling each arc have been considered as trapezoidal Intuitionistic Fuzzy Numbers (TIFNs) which are the more generalized form of fuzzy numbers involving a degree of acceptance and a degree of rejection. We have ranked TIFNs using the centroid concept. Then, we have presented a new approach for finding the shortest path from the source node to the destination node. Finally, an illustrative example is also included to demonstrate our proposed algorithm.
Keywords: Shortest path problem, Trapezoidal intuitionistic fuzzy number, Centroid of an intuitionistic fuzzy number.

1 Introduction

Shortest path problem (SPP) is a fundamental and well-known combinatorial optimization problem in the area of graph theory. In real-life scenarios, the arc weighs in a shortest path of a network/graph have the several parameters which are very hard to define exactly (i.e., capacity, cost, demand, traffic frequency, time, etc.). We can incorporate the fuzziness into a graph to handle this type of uncertain situation. Shortest Path Problem with fuzzy arc lengths have been solved by different methods by many authors. However, it seems that in the literature there is no investigation on SPP with data in the form of intuitionistic fuzzy numbers which is a more generalized form of fuzzy number. The major objective of this paper is to bridge this gap, by posing and proposing a methodology for solving the SPP with IF arc parameters.
In 2007, Karunambigai et al. refined an approach found on dynamic programming to solve SPP using intuitionistic fuzzy graphs (IFGs) [5]. Mukherje pre-owned an interesting methodology to solve IFSPP using the idea of Dijikstra's algorithm and intuitionistic fuzzy hybrid Geometric (IFHG) operator [7]. Rangasamy et al. proposed score-based methodology to find the shortest hyper paths for a given network where hyper edges are characterized by IF weights without describing similarity measure and Euclidean distance [11]. Babacioru conferred an algorithm to find the minimum arc length of an IF hyper path using MAPLE [3]. In [4], Jayagowri and GeethaRamani solved SPP on a network with the use of Trapezoidal Intuitionistic Fuzzy Numbers (TpFNs).In 2014, Porchelvi and Sudha recommended a minimum path labeling algorithm to solve

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SPP using triangular IF number (TIFN) [9]. Also, they proposed a new and different methodology to solve SPP with TIFNs, where the authors found the minimal edge using IF distance by applying graded mean integration and examined SPP from a particular vertex to all other ones in a network [9]. In 2015, Kumar et al. suggested a design to identify the SP and shortest distance in an IVIF graph where the nodes are taken as crisp numbers and edge weights are assigned by IVITpFNs (Interval Valued Intuitionistic Trapezoidal Fuzzy Numbers) [6].
This paper organized as follows: Some basic knowledge concepts on intuitionistic fuzzy set theory used throughout this paper are introduced in Section 2. Section 3 includes the existing method for finding shortest path and shortest path length from source node to destination node. In Section 4, a numerical example have been given to reveal the effectiveness of existing model. Finally some conclusions have been drawn in the last section.

2 Preliminaries

In this section, we review the fundamental notions of IFS theory, used throughout this paper.

Definition 1. *[2] An intuitionistic fuzzy set $\tilde{A}$ assigns to each element x of the universe X a membership degree $\mu_{\tilde{A}}(x) \in [0,1]$ and a non-membership degree $\nu_{\tilde{A}}(x) \in [0,1]$ such that $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$. An IFS $\tilde{A}$ is mathematically represented as*

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle | x \in X\}.$$

The membership function $\mu_{\tilde{A}}(x)$ (non-membership function $\nu_{\tilde{A}}(x)$) represents the degree of belongingness (non-belongingness) of the element $x \in X$ in the IFS $\tilde{A}$. The value $\pi_{\tilde{A}}(x) = 1 - (\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x))$ represents the degree of hesitation for the element x being in $\tilde{A}$. A fuzzy set is a special case of IFS in which $\nu(x) = 1 - \mu(x)\ \forall x \in X$, i.e. the hesitation degrees are 0.

Definition 2. *[2] An intuitionistic fuzzy set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle | x \in X\}$ is said to be intuitionistic fuzzy normal if there exist at least two points $x_0, x_1 \in X$ such that $\mu_{\tilde{A}}(x_0) = 1,\ \nu_{\tilde{A}}(x_1) = 1$.*

Definition 3. *[8] An intuitionistic fuzzy set $\tilde{A}$ is said to be intuitionistic fuzzy number, if it is*

i) Intuitionistic fuzzy normal.

ii) Convex for the non-membership function $\mu_{\tilde{A}}(x)$, i.e., $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \max(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$ for every $x_1, x_2 \in \mathbb{R},\ \lambda \in [0,1]$.

iii) Concave for the non-membership function $\nu_{\tilde{A}}(x)$, i.e., $\nu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \leq \max(\nu_{\tilde{A}}(x_1), \nu_{\tilde{A}}(x_2))$ for every $x_1, x_2 \in \mathbb{R},\ \lambda \in [0,1]$.

Definition 4. *[8] A trapezoidal intuitionistic fuzzy number (TIFN) $\tilde{A}$ is an IFN with the membership function $\mu_{\tilde{A}}$ and non-membership function $\nu_{\tilde{A}}$ given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a_2 \\ \frac{x-a_2}{a_4-a_2} & a_2 \leq x < a_4 \\ 1 & a_4 < x \leq a_5 \\ \frac{x-a_7}{a_5-a_7} & a_5 < x \leq a_7 \\ 0 & x \geq a_7, \end{cases}$$

$$\nu_{\tilde{A}}(x) = \begin{cases} 0 & x < a_1 \\ \frac{x-a_1}{a_3-a_1} & a_1 \leq x < a_3 \\ 1 & a_3 < x \leq a_6 \\ \frac{x-a_8}{a_6-a_8} & a_6 < x \leq a_8 \\ 0 & x \geq a_8, \end{cases}$$

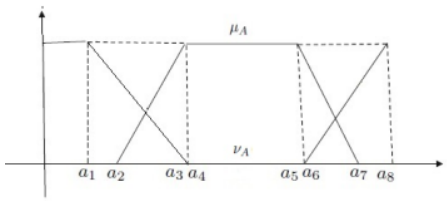


Figure 1: Trapezoidal intuitionistic fuzzy number.

where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8$ *(see Fig. 1). This TIFN is denoted by* $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$. *In the above definition, if we let* $a_3 = a_4 = a_5 = a_6$, *then we will get a triangular intuitionistic fuzzy number and denoted by* $(a_1, a_2, a_3, a_7, a_8)$.

Definition 5. *[1] Consider a TIFN* $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$, *Prakash et al. defined the centeroid point* $(x_\mu(\tilde{A}), y_\mu(\tilde{A}))$ *for the membership function and* $(x_\nu(\tilde{A}), y_\nu(\tilde{A}))$ *for the non-membership function as follows:*

$$x_\mu(\tilde{A}) = \frac{1}{3}[\frac{a_5^2 + a_7^2 - a_2^2 - a_4^2 - a_2a_4 + a_5a_7}{a_7 + a_5 - a_4 - a_2}] \tag{2.1}$$

$$y_\mu(\tilde{A}) = \frac{1}{3}[\frac{a_2 + 2a_4 - 2a_5 - a_7}{a_2 + a_4 - a_5 - a_7}] \tag{2.2}$$

$$x_\nu(\tilde{A}) = \frac{1}{3}[\frac{2a_8^2 - 2a_1^2 - 2a_3^2 + 2a_6^2 + a_1a_3 - a_6a_8}{a_6 + a_8 - a_1 - a_3}] \tag{2.3}$$

$$y_\nu(\tilde{A}) = \frac{1}{3}[\frac{2a_1 + a_3 - a_6 - 2a_8}{a_1 + a_3 - a_6 - a_8}] \tag{2.4}$$

Then, using the Euclidean metric, they expressed the ranking function between TIFNs as follows:

$$R(\tilde{A}) = \sqrt{\frac{1}{2}([x_\mu(\tilde{A}) - y_\mu(\tilde{A})]^2 + [x_\nu(\tilde{A}) - y_\nu(\tilde{A})]^2)} \tag{2.5}$$

Theorem 2.1. *[1] Let* $\tilde{A}$ *and* $\tilde{B}$ *be two TIFNs, then* $R(\tilde{A} + \tilde{B}) = R(\tilde{A}) + R(\tilde{B})$

3 Proposed algorithm to find the SPP on network with TIFNs

Let $G(X, \tilde{Y})$ be an acyclic discrete directed fuzzy network consisting of n number of vertices symbolized as $X = \{1, \cdots, n\}$ and of m number of directed edges $\tilde{Y} \subseteq X \times X$. Each edge is represented by a simple ordered pair (a, b); here $a \neq b$ and $a, b \in X$. We have considered in this network that there exists only one directed edge (a, b) among the two vertices a and b. A path is a consequence of finite of alternating vertices and edges between the two nodes in which the starting vertex of each edge is same as the ending vertex of preceding edge in the sequence. The source vertex of this fuzzy network is s and ending vertex is t. Here, we consider a trapezoidal Intuitionistic Fuzzy Number $\tilde{d}_{ij}$ to represent the fuzzy cost between the nodes i and j.

Now, we present an algorithmic approach to determine the optimal paths within a source node s and every other nodes in the fuzzy network. The proposed algorithm is presented as follows:

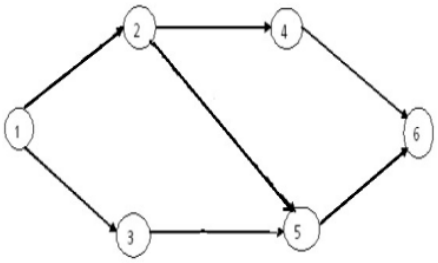


Figure 2: Network with eight vertices

Table 1: List of Arc lengths

Arc(i,j)	Trapezoidal intuitionistic fuzzy arc length	$R(\tilde{d}_{ij})$
(1,2)	(4.4,4.5,5.5,6,6.2,6.4,6.8)	5.0052
(1,3)	(2,2.5,2.7,3,3.2,3.5,3.8,4)	2.602729
(2,5)	(5,5.3,5.5,5.7,5.9,6.2,6.4,6.5)	5.328941
(3,5)	(3,3.1,3.2,3.5,3.7,3.9,4,4.1)	3.095802
(2,4)	(2.5,2.7,3,4,4.2,4.4,4.5,5)	3.305153
(4,6)	(1.5,2,2.2,2.5,2.8,3,3.2,3.4)	2.069045
(5,6)	(1,1.2,1.4,1.8,2,2.5,3,3.2)	1.704334

Table 2: Possible Shortest path From Origin to Destination

path name	Possible shortest path	ranking function
s_1	1-2-4-6	10.379398
s_2	1-2-5-6	12.038475
s_3	1-3-5-6	7.402865

- **Step1**. Construct an acyclic network $G(V, \tilde{E})$, where V is the set of vertices and E is the set of edges. Let $\tilde{d}_{ij} = (a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}, a^5_{ij}, a^6_{ij}, a^7_{ij}, a^8_{ij})$ denote the trapezoidal intuitionistic fuzzy numbers associated with the edge (i, j), that is the length necessary to traverse from node i to node j. Each length corresponds to the cost, time etc., in practical problems.
- **Step2**. Form the possible paths from starting node to destination node and compute the corresponding path lengths S_i for possible m paths.
- **Step3**. Compute the ranking function for all possible path lengths this is defined as $R(\tilde{S}_i)$ using the relation (2.5)
- **Step4**. Decide the shortest path length and corresponding path with the minimum rank.

4 Numerical example

To show the effectiveness of the existing model on TIFNs, we consider an example with six vertices and each of which are connected at least to one of the vertexes on the given graph below. Compute the maximum numbers of possible paths from source to destination and their ranking function are as follows: Path length S_3 is having the minimum rank, therefore the shortest path length is $1 - 3 - 5 - 6$.

5 Conclusion

In this paper, we used the ranking function based on centroid concept to find the shortest path and shortest path length on trapezoidal intuitionistic fuzzy number from source node to destination node. Thus it can be concluded that the current method is more effective and versatile to solve problems by dealing with uncertainty and ambiguity and consequently making the required decision as it has been derived from intuitionistic fuzzy numbers.

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Mean ergodic shadowing in non-autonomous systems

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Abstract

In this paper, we extend the mean-ergodic shadowing property to non-autonomous discrete dynamical systems (NDDS). It is shown that the mean-ergodic shadowing property is a topological property and is invariant under iterations.

Keywords: non-autonomous discrete dynamical system, mean-ergodic shadowing, shadowing property.

1 Introduction

The shadowing theory is an important subject in theory of dynamical sysytems. In practice, to find the orbits of a dynamical system one has to accomplish the numerical computation. These numerical methods when implemented in computers have unavoidable errors such as round-off errors. So, indeed, we get pseudo orbits. The pseudo orbit tracing property (POTP, for short) means that if a map possesses this property, then the orbits obtained by implication of numerical methods reflect the real dynamical behavior of that map.

The notion of nonautonomous discrete dynamical system (NDDS) was introduced by Kolyada and Snoha [6]. Unlike to autonomous dynamical systems, which are defined by a singhle map f, a nonautonomous discrete dynamical system is defined by a sequence of maps that are composed in a special way. Thus the study of NDDSs is more complex than that of autonomous systems.

Recently, in [5], the authors introduced the notion of mean-ergodic shadowing in autonomous dynamical system as a type of average shadowing property and proved that a minimal system with the shadowing property can not have mean-ergodic shadowing. In additions, they proved that the mean-ergodic shadowing is weaker than the average shadowing property.

This note aims to extend the mean-ergodic shadowing property to NDDSs.

2 Preliminaries

Throughout this note, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, and (X, d) denotes a compact metric space. Suppose $f_n : X \to X$ be a sequence of self maps, $n = 1, 2, \dots$. The orbit of a point $x_0 \in X$ is defined by the sequence $\{x_n\}_{n\geq 0}$, where $x_{n+1} = f_{n+1}(x_n)$, $n = 0, 1, 2, \dots$ and denoted by $O(x_0)$. We denote $f_{1,\infty} := \{f_n\}_{n\geq 1}$, and for any $i \leq j$ in $\mathbb{Z}^+$ define $f_i^j = f_j \circ f_{j-1} \circ \cdots \circ f_i$ and for $i > j$, we define $f_i^j = Id$. For any $k > 0$, we define the kth-iterate of $f_{1,\infty}$ by $F^k = \{f_{(n-1)k+1}^{nk}\}_{n\geq 1}$. The pair $(X, f_{1,\infty})$ is called a *NDDS* on X.

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We say that $(X, f_{1,\infty})$ is *uniformly equicontinuous* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for any two points $x, y \in X$

$$d(x,y) < \delta \implies d\big(f_n(x), f_n(y)\big) < \epsilon, \ \forall n > 0.$$

For $\delta > 0$ given, the sequence $\{x_i\}_{i\geq 0}$ in X is said to be a *δ-pseudo-orbit* for $(X, f_{1,\infty})$ if $d(f_{i+1}(x_i), x_{i+1}) < \delta$ for all $i \geq 0$. $(X, f_{1,\infty})$ is said to have *shadowing property (POTP)* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-traced by some point $z \in X$, that is,

$$d\big(f_1^i(z), x_i\big) < \epsilon, \ \forall i \geq 0.$$

For $A \subset \mathbb{Z}^+$, the *upper density* of A is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n-1\}|.$$

If lim sup is replaced by lim inf in the above definition, then the *lower density* of A, $\underline{d}(A)$, is obtained. In case of $\underline{d}(A) = \overline{d}(A) = d(A)$, we say that A has density $d(A)$.

In the following, we extend the mean-ergodic shadowing to NDDS.

3 Mean-Ergodic Shadowing in NDDSs

Definition 1. *Let $\delta > 0$ be given and $\xi = \{x_i\}_{i\geq 0} \subset X$. We say that ξ is a δ-ergodic pseudo-orbit for NDDS $(X, f_{1,\infty})$ if*

$$d\big(\{i \in \mathbb{Z}^+ : d\big(f_{i+1}(x_i), x_{i+1}\big) < \delta\}\big) = 1.$$

It is said that $(X, f_{1,\infty})$ has mean-ergodic shadowing property if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for every δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+} \subset X$ there is some $z \in X$ that ϵ-shadows $\{x_i\}$ in average. That is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\big(f_1^i(z), x_i\big) < \epsilon.$$

First of all, we show that the definition of mean-ergodic shadowing is independent of choosing a metric if X is compact. To do this, let d_1 and d_2 be two equivalent metrics on a compact metric space X. Let $(X, f_{1,\infty})$ has mean-ergodic shadowing with metric d_1 and $\epsilon > 0$ is arbitrary. Since d_1 and d_2 are equivalent there exits an $0 < \epsilon_1 < \epsilon$ such that for any $x \in X$, $N_{d_1}(x, \epsilon_1) \subset N_{d_1}(x, \epsilon)$, where $N_{d_i}(x, \delta)$ denotes open ball centered at x and of radius δ with metric d_i, $i = 1, 2$. Since X is compact, ϵ_1 depends only on ϵ and not on x. Let $\delta_1 > 0$ be ϵ_1-corresponding in the definition of mean-ergodic shadowing for $(X, f_{1,\infty})$ with respect to d_1. Again by the equivalence of d_1 and d_2 there is a $\delta > 0$ so that $N_{d_2}(x, \delta) \subset N_{d_1}(x, \delta_1)$ for any $x \in X$. Now if $\{x_i\}_{i\in\mathbb{Z}^+}$ is a δ-ergodic pseudo-orbit for $(X, f_{1,\infty})$ then by choosing δ it is also a δ-ergodic pseudo-orbit with metric d_1. So, there exists $z \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_1\big(f_1^i(z), x_i\big) < \epsilon_1.$$

By the Lemma 3.2, the set $E_1 = \{i \in \mathbb{Z}^+ : d_1\big(f_1^i(z), x_i\big) \geq \epsilon_1\}$ has upper density less that ϵ_1. Since the set $E_2 = \{i \in \mathbb{Z}^+ : d_2\big(f_1^i(z), x_i\big) \geq \epsilon\} \subset E_1$ so $\overline{d}(E_2) < \epsilon_1 < \epsilon$.

Proposition 3.1. *Suppose that $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ are two conjugate NDDS. Then $(Y, g_{1,\infty})$ has mean-ergodic shadowing property if and only if so is $(X, f_{1,\infty})$.*

Proof. Suppose that $h : X \to Y$ is a homeomorphism where conjugates $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$. That is, $h \circ f_n = g_n \circ h$, for $n \geq 1$. Let $\epsilon > 0$ be arbitrary, by uniformly continuity of h there exists $0 < \epsilon_1 < \epsilon$ such that $d_X(x, y) < \epsilon_1$ implies $d_Y\big(h(x), h(y)\big) < \epsilon$. Let $\delta_1 > 0$ be corresponding to ϵ_1 in the definition of mean-ergodic shadowing property for $(X, f_{1,\infty})$. By uniformly continuity of h^{-1} there is a $\delta > 0$ such that $d_X\big(h^{-1}(x), h^{-1}(y)\big) < \delta_1$ whenever $d_Y(x, y) < \delta$. Now suppose that $\{x_i\}_{i=0}^{\infty}$ is a δ-ergodic pseudo-orbit for NDDS $(Y, g_{1,\infty})$ i.e., $\{i \in \mathbb{Z}^+ : d_Y\big(g_{i+1}(x_i), x_{i+1}\big) < \delta\}$ has lower density one. By choosing δ we deduce that

$$\{i \in \mathbb{Z}^+ : d_X\big(h^{-1}(g_{i+1}(x_i)), h^{-1}x_{i+1}\big) < \delta_1\} = \{i \in \mathbb{Z}^+ : d_X\big(f_{i+1}h^{-1}(x_i), h^{-1}x_{i+1}\big) < \delta_1\}$$

also has lower density one. So, the sequence $\{h^{-1}x_i\}_{i=0}^{\infty}$ is a δ_1-ergodic pseudo-orbit for $(X, f_{1,\infty})$. Therefore, there exists $z \in X$ such that $\overline{d}(E_1) < \epsilon_1$, where $E_1 = \{i \in \mathbb{Z}^+ : d_X\big(f_1^i(z), h^{-1}x_i\big) \geq \epsilon_1\}$. Let $E = \{i \in \mathbb{Z}^+ : d_Y\big(hf_1^i(z), hh^{-1}x_i\big) \geq \epsilon\} = \{i \in \mathbb{Z}^+ : d_Y\big(g_1^i(h(z)), x_i\big) \geq \epsilon\}$. By choosing ϵ_1 we have $E \subset E_1$ so

$$\overline{d}(\{i \in \mathbb{Z}^+ : d_Y\big(g_1^i(h(z)), x_i\big) \geq \epsilon\}) < \epsilon_1 < \epsilon.$$

□

Lemma 3.2. *$(X, f_{1,\infty})$ has mean-ergodic shadowing property if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+}$ there is $z \in X$ such that $d\big(f_1^i(z), x_i\big) < \epsilon$ for all $i \in \mathbb{Z}^+$, except for a set of upper density less than ϵ.*

Proof. Suppose that $(X, f_{1,\infty})$ has mean-ergodic shadowing property and let $\epsilon > 0$ be given. So, there is a $\delta > 0$ such that for every δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+}$ there is $z \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\big(f_1^i(z), x_i\big) < \epsilon^2.$$

If we set $E = \{i \in \mathbb{Z}^+ : d\big(f_1^i(z), x_i\big) \geq \epsilon\}$ then

$$\epsilon^2 > \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\big(f_1^i(z), x_i\big) \geq \limsup_{n\to\infty} \frac{1}{n} |E \cap [0, n-1)| \ \epsilon = \epsilon \overline{d}(E).$$

Therefore, $\overline{d}(E) < \epsilon$.

Conversely, suppose that $\epsilon > 0$ is arbitrary. Choose $0 < \eta < \frac{\epsilon}{diam(X)+1}$, and take $\delta > 0$ such that every δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+}$ can be η-shadowed on $\mathbb{Z}^+$ except for a subset with upper density less than η. Let $E = \{i \in \mathbb{Z}^+ : d(f_1^i(z), x_i) \geq \eta\}$ then $\overline{d}(E) < \eta$. So, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_1^i(z), x_i) &\leq \limsup_{n\to\infty} \frac{1}{n} (diam(X) \ |E \cap \{0, 1, \ldots, n-1\}| + n\eta) \\ &= diam(X)\overline{d}(E) + \eta < \eta \ diam(X) + \eta < \epsilon.\end{aligned}$$

□

In the following, we show that the mean-ergodic shadowing property is invariant of iterations.

Proposition 3.3. *If $(X, f_{1,\infty})$ has mean-ergodic shadowing property, then so does $F^k = \{f_{ik+1}^{ik+k}\}_{i=0}^{\infty}$, for each $k > 0$.*

Proof. Suppose that $\epsilon > 0$ is given and let $\delta > 0$ be ϵ-correspond in the definition of mean-ergodic shadowing for $(X, f_{1,\infty})$. Let $\{y_i\}_{i\in\mathbb{Z}^+}$ be a δ-ergodic pseudo-orbit for $F^k = \{f_{ik+1}^{ik+k}\}_{i=0}^{\infty}$. Then by inserting the points $f_{ki+1}(y_i), f_{ki+1}^{ki+2}(y_i), \ldots, f_{ki+1}^{k(i+1)-1}(y_i)$ between y_i and y_{i+1} we get a δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+}$ for $(X, f_{1,\infty})$. Note that in this way we have $x_{ik+j} = f_{ik+1}^{ik+j}(y_i)$, for every $i \geq 0$ and $0 \leq j < k$. By assumption, there exists $z \in X$ such that $\overline{d}(\{i \in \mathbb{Z}^+ : d\big(f_1^i(z), x_i\big) \geq \epsilon\}) < \epsilon$. So, clearly $\overline{d}(\{i \in \mathbb{Z}^+ : d(f_1^{ik}(z), x_{ik}) \geq \epsilon\}) < \epsilon$. This completes the proof. □

In what follows, we consider the converse case.

Proposition 3.4. *If for some integer $k > 0$, $F^k = \{f_{ik+1}^{ik+k}\}_{i=0}^{\infty}$ has mean-ergodic shadowing property and $f_{1,\infty}$ is an equicontinuous family of self maps, then $f_{1,\infty}$ has mean-ergodic shadowing property.*

Proof. Let $\epsilon > 0$ be arbitrary. By equicontinuity of $f_{1,\infty}$ there exists $0 < \eta < \epsilon$ such that $d(x, y) < \eta$ implies $d\big(f_j^{j+i}(x), f_j^{j+i}(y)\big) < \frac{\epsilon}{2}$, for every $j \geq 1$ and $0 \leq i < k$. By mean-ergodic shadowing property for $F^k = \{f_{ik+1}^{ik+k}\}_{i=0}^{\infty}$ there exists $0 < \delta < \frac{\eta}{2}$ such that every δ-ergodic pseudo-orbit can be η-shadowed in average by some true orbit. Again, By equicontinuity there exists $0 < \gamma < \frac{\delta}{k}$ such that $d(x, y) < \gamma$ implies $d\big(f_j^{j+i}(x), f_j^{j+i}(y)\big) < \frac{\delta}{k}$, for every $j \geq 1$ and $0 \leq i < k$. Now, suppose $\xi = \{x_i\}_{i\geq 0}$ is a γ-ergodic pseudo-orbit for $f_{1,\infty}$ we claim that $\zeta = \{x_{ik}\}_{i\geq 0}$ is a δ-ergodic pseudo-orbit for $F^k = \{f_{ik+1}^{ik+k}\}_{i=0}^{\infty}$. Set

$$I = \{i \in \mathbb{Z}^+ : d\big(f_{i+1}(x_i), x_{i+1}\big) < \gamma\},$$
$$E = \{i \in \mathbb{Z}^+ : \{ik, ik+1, \ldots, ik+k\} \cap I^C \neq \emptyset\}$$

For $i \in E^C$, we have

$$d\big(f_{ik+j+1}(x_{ik+j}), x_{ik+j+1}\big) < \gamma, \quad 0 < j < k.$$

Then by property of γ we get

$$d\big(f_{ik+j+1}^{ik+k}(x_{ik+j}), f_{ik+j+2}^{ik+k}(x_{ik+j+1})\big) < \frac{\delta}{k}, \quad 0 \leq j < k.$$

Hence,

$$\begin{aligned} d\big(f_{ik+1}^{ik+k}(x_{ik}), x_{ik+k}\big) &< \sum_{j=0}^{k-1} d\big(f_{ik+j+1}^{ik+k}(x_{ik+j}), f_{ik+j+2}^{ik+k}(x_{ik+j+1})\big) \\ &< \sum_{j=0}^{k-1} \frac{\delta}{k} = \delta \end{aligned}$$

So, the sequence $\zeta = \{x_{ik}\}_{i\geq 0}$ is a δ-ergodic pseudo-orbit for $F^k = \{f_{ik+1}^{ik+k}\}_{i=0}^{\infty}$. Therefore, there exists $z \in X$ such that the set

$$E_1 = \{i \in \mathbb{Z}^+ : d\big(f_1^{ik}(z), x_{ik}\big) < \eta\}$$

has $\overline{d}({E_1}^C) < \eta$. For each $i \in E_1$ by the property of η we have

$$d\big(f_{ik+1}^{ik+j}(f_1^{ik}(z)), f_{ik+1}^{ik+j}(x_{ik})\big) < \frac{\epsilon}{2} \tag{3.1}$$

On the other hand, for $0 \leq j < k$ we have

$$\begin{aligned} d\big(f_{ik+1}^{ik+j}(x_{ik}), x_{ik+j}\big) &\leq d\big(f_{ik+1}^{ik+j}(x_{ik}), f_{ik+2}^{ik+j}(x_{ik+1})\big) \\ &\quad + d\big(f_{ik+2}^{ik+j}(x_{ik+1}), f_{ik+23}^{ik+j}(x_{ik+2})\big) \\ &\quad + \\ &\quad \vdots \\ &\quad + d\big(f_{ik+j}(x_{ik+j-1}), x_{ik+j}\big) \\ &\quad < \underbrace{\frac{\delta}{k} + \frac{\delta}{k} + \cdots + \frac{\delta}{k}}_{j-1} + \gamma < j\frac{\delta}{k} \leq \delta \end{aligned}$$

So, we have

$$d\big(f_{ik+1}^{ik+j}(x_{ik}), x_{ik+j}\big) < \delta, \quad 0 \leq j < k \tag{3.2}$$

Therefore, by relations (1.1) and (1.4), for each $i \in E_1$ we obtain that

$$d\big(f_1^{ik+j}(x_{ik+j})(z), x_{ik+j}\big) < \frac{\epsilon}{2} + \delta < \epsilon, \quad 0 \leq j < k$$

Now, define $E_2 = \{i \in \mathbb{Z}^+ : d\big(f_1^i(z), x_i\big) \geq \epsilon\}$ then

$$\begin{aligned} \overline{d}(E_2) &= \limsup_{n\to\infty} \frac{1}{n}|E_2 \cap [0, n-1)| \\ &\leq \limsup_{n\to\infty}\ k\ \frac{[\frac{n}{k}] - |E_1 \cap \{0, 1, \ldots, [\frac{n}{k}] - 1\}|}{n} \\ &\leq \limsup_{n\to\infty}\ k\ \frac{[\frac{n}{k}] - |E_1 \cap \{0, 1, \ldots, [\frac{n}{k}] - 1\}|}{[\frac{n}{k}]} \\ &= 1 - \liminf_{n\to\infty}\ k\ \frac{|E_1 \cap \{0, 1, \ldots, [\frac{n}{k}] - 1\}|}{[\frac{n}{k}]} \\ &\leq 1 - \underline{d}(E_1) = \overline{d}({E_1}^C) < \eta < \epsilon. \end{aligned}$$

This proves that $f_{1,\infty}$ has mean-ergodic shadowing property. □

In the following, we prove that the product of NDDSs has the mean-ergodic shadowing property.

Proposition 3.5. *If $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ are two NDDS, then $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ have mean-ergodic shadowing property if and only if $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ have mean-ergodic shadowing property.*

Proof. Suppose that $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ have mean-ergodic shadowing property and let $\epsilon > 0$ be arbitrary. By assumption on mean-ergodic shadowing property there are $\delta_1 > 0$ and $\delta_2 > 0$ corresponding to $\frac{\epsilon}{2}$ for $f_{1,\infty}$ and $g_{1,\infty}$, respectively. Take $\delta = \min\{\delta_1, \delta_2\}$. Now let $\{(x_i, y_i)\}_{i\in\mathbb{Z}^+}$ be a δ-ergodic pseudo-orbit for $(f_{1,\infty} \times g_{1,\infty})$. Then we have

$$d\big((f_{i+1}, g_{i+1})(x_i, y_i), (x_{i+1}, y_{i+1})\big) < \delta$$

So, by choosing δ we get

$$\begin{aligned} d_X\big(f_{i+1}(x_i), x_{i+1}\big) < \delta < \delta_1 \\ d_Y\big(g_{i+1}(y_i), y_{i+1}\big) < \delta < \delta_2 \end{aligned}$$

Therefore, the sequences $\{x_i\}_{i\in\mathbb{Z}^+}$ and $\{y_i\}_{i\in\mathbb{Z}^+}$ are δ_1-ergodic and δ_2-ergodic pseudo-orbits , respectively for $f_{1,\infty}$ and $g_{1,\infty}$. Hence, there exist $p\in X$ and $q\in X$ such that if we set $A=\{i\in\mathbb{Z}^+ : d_X\big(f_1^i(p),x_i\big)\geq\frac{\epsilon}{2}\}$ and $B=\{i\in\mathbb{Z}^+ : d_Y\big(g_1^i(q),y_i\big)\geq\frac{\epsilon}{2}\}$. Therefore, we have $\overline{d_X}(A)<\frac{\epsilon}{2}$ and $\overline{d_Y}(B)<\frac{\epsilon}{2}$. We claim that the point (p,q) ϵ-shadows the sequence $\{(x_i,y_i)\}_{i\in\mathbb{Z}^+}$ in average. In fact, if we let $C=\{i\in\mathbb{Z}^+ : d((f\times g)_1^i(p,q),(x_i,y_i))\geq\epsilon\}$ then $C\subset A\cup B$. Therefore, $\overline{d}(C)\leq\overline{d}(A)+\overline{d}(B)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$.

Conversely, assume that $(X\times Y, f_{1,\infty}\times g_{1,\infty})$ has mean-ergodic shadowing property and let $\epsilon>$ be arbitrary. There is δ corresponding to ϵ in the definition of mean-ergodic shadowing. Take $\{x_i\}_{i\in\mathbb{Z}^+}$ and $\{y_i\}_{i\in\mathbb{Z}^+}$ as two δ-ergodic pseudo-orbits, respectively for $f_{1,\infty}$ and $g_{1,\infty}$. So, the sequence $\{(x_i,y_i)\}_{i\in\mathbb{Z}^+}$ is a δ-ergodic pseudo-orbit for $(f_{1,\infty}\times g_{1,\infty})$, because if we take $A=\{i\in\mathbb{Z}^+ : d_X\big(f_{i+1}(x_i),x_{i+1}\big)<\delta\}$, $B=\{i\in\mathbb{Z}^+ : d_Y\big(g_{i+1}(y_i),y_{i+1}\big)<\delta\}$ and $C=\{i\in\mathbb{Z}^+ : d\big((f_{i+1},g_{i+1})(x_i,y_i),$
$(x_{i+1},y_{i+1})\big)<\delta\}$, then we have $C=A\cap B$. Besides, it is known that the intersection of two sets with density one is also from density one. Therefore $\{(x_i,y_i)\}_{i\in\mathbb{Z}^+}$ is a δ-ergodic pseudo-orbit for $(f_{1,\infty}\times g_{1,\infty})$. So, there exists point $(p,q)\in X\times Y$ such that ϵ-shadows $\{(x_i,y_i)\}_{i\in\mathbb{Z}^+}$ in average. Hence,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_X\big(f_1^i(p),x_i\big)<\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d\big((f\times g)_1^i(p,q),(x_i,y_i)\big)<\epsilon$$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_Y\big(g_1^i(q),y_i\big)<\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d\big((f\times g)_1^i(p,q),(x_i,y_i)\big)<\epsilon$$

This completes the proof. □

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Construction and Description of Identification of Binary Matroid

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Abstract

Eulerian and bipartite matroids have been discussed by Welsh [6]. Andras Recski in the matroid theory and its applications proved more theorems and applications [33]. Applications of binary matroid have been studied by Shikare and analyzed more examples in his papers. In this paper we study the converse of splitting of matroid. Let M be a binary matroid. We consider to find matroid M' by splitting a pair of elements of M we call such M' the identification of M. We also show that M and M' aren't necessarily of the same type.

Keywords: Description, Equivalent, Binary Matroid, Identification, Splitting

1 Introduction

Here we provide same definitions [2, 3].

Definition 1. *A matroid is a pair (S, I) where S a finite set and I is a collection of subset of S with the following properties:*

1. $\phi \in I$
2. *If $X \in I$ and $Y \subseteq X$ then $Y \in I$*
3. *If $X \in I$ and $Y \in X$ and $|X| > |Y|$ then there exists an elements $x \in X - Y$ such that $Y \cup \{x\} \in I$*

S is called the underlying set of M whereas members of I are called the independent set of M.

Definition 2. *A matroid M is also defined as the pair (S, C) where S is a finite set and C is a collection of non-empty subset of S satisfying the following axioms:*

1. $\phi \notin C$
2. *No member of C contains another member properly*
3. *If $c_1, c_2 \in C$, $c_1 \neq c_2$ and $e \in c_1 \cap c_2$ then there exists $c_3 \in C$ such that $c_3 \subseteq (c_1 \cup c_2) - e$*

Members of C are called circuits of M.

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We say that M is simple if every circuit of it has at least three elements.

Definition 3. *A matroid M on a set S is defined to be binary if it is representable over the Galois field GF(2).*

Definition 4. *[4] Let $M = (S, C)$ be a binary matroid and $x, y \in S$. Let*

$$C^{xy} = C_0^{xy} \cup C_1^{xy} \tag{1.1}$$

such that:

1. $C_0^{xy} = \{c \in C \mid x, y \in C \text{ or } x, y \notin C\}$
2. $C_1^{xy} = \{c_1 \cup c_2 \mid c_1, c_2 \in C, c_1 \cap c_2 = \phi, x \in c_1, y \in c_2, \nexists c \in C_0^{xy} \text{ such that } c \subseteq c_1 \cup c_2\}$

The matroid $M_{xy} = (S, C^{xy})$ is called splitting of the matroid $M = (S, C)$ by the pair x, y of elements of S.

Example 1.1. Consider the matroid $M = (S, C)$ where $S = \{a, b, c, d, e, f, g\}$ and

$$C = \{\{a, f, g\}, \{b, c, d\}, \{a, b, d, e\}, \{d, e, f, g\}, \{a, b, c, d, f\}\}. \tag{1.2}$$

1. Consider the pair a, b of elements in S then $M_{ab} = (S, C^{ab})$ where
$$C^{ab} = C_0^{ab} \cup C_1^{ab} = \{\{a, b, d, e\}, \{d, e, f, g\}, \{a, b, c, d, f, g\}\}. \tag{1.3}$$
2. Splitting of M by the pair a, b of elements is the matroid $M_{ab} = (S, C^{ab})$ where
$$C^{ab} = C_0^{ab} \cup C_1^{ab} = \{\{a, b, d, e\}, \{a, b, c, d, f\}\}. \tag{1.4}$$
3. Splitting of the matroid M by the pair is same as the matroid M.

We can do the splitting of matroid M twice where $(M_{ab})_{xy} = (M_{xy})_{ab}$.

2 Main Results

Let D and D' be two circuits of matroid M such that D is a circuit in a matroid M which does not contain the pair x, y of elements and D' is a circuit in a matroid M which containing both the elements x and y [5, 7, 8].

Definition 5. *Let $M = (S, I)$ be a binary matroid with an equivalent pair x, y of elements. Partition C into two subsets C_1 and C_2 which contain x and y respectively.*

Let S_0 be a subset of $S - C$ satisfying the following two conditions:

1. *$C_1 \cup S_0$ and $C_2 \cup S_0$ do not contain any of the circuits D.*
2. *For every collection $D_1, D_2, \ldots, D_L$ of disjoint circuits containing none of x and y, we must have a circuit $D_j^n \subseteq \bigcup_{i=1}^{L} D_i$ such that*

$$((C_n \cup S_0) \cap (\bigcup_{i=1}^{L} D_i)) \subseteq ((C_n \cup S_0) \cap D_j^n) \quad \text{for } n = 1, 2. \tag{2.1}$$

Let C^{id} denote the collection of subset of S of the following type:

1. *All* $D \in C$
2. $C_1 \cup S_0$, $C_2 \cup S_0$
3. $(C_n \cup S_0)\Delta D_j$ *for* $n = 1, 2$ *such that* $(C_n \cup S_0)\Delta D_j$ *do not contain sets of type (1) or (2)*
4. *All* $D' \in C$ *where* D' *do not contain any sets of type (2) or (3)*

Theorem 2.1. *$M^{id} = (S, C^{id})$ is a binary matroid.*

Theorem 2.2. *$(M^{id})_{xy} = M$. That is splitting of M^{id} by the pair x and y is M.*

Proof. We have to prove $(C^{id})^{xy} = C$. □

Definition 6. *The matroid $M^{id} = (S, C^{id})$ is called an identification of the matroid $M = (S, C)$.*

Theorem 2.3. *Let $M = (S, C)$ be a binary matroid and x, y pair of equivalent elements in M. Suppose $M_1^{id} = (S, C_1^{id})$ and $M_2^{id} = (S, C_2^{id})$ are two identification of M generated respective by $C_1 \cup S_0$, $C_2 \cup S_0$ and by $C_1' \cup S_0'$, $C_2' \cup S_0'$.*

If there is a circuit D_k' in M such that

$$C_1' = D_k' \cap (C_1 \cup S_0), \quad C_2' = D_k' - C_1' \quad \text{and} \quad S_0' = S_0 \cap (S - D_k) \tag{2.2}$$

then $M_1^{id} = M_2^{id}$.

3 Results

1. An identification of an Eulerian matroid is Eulerian and conversely.
2. If the identification of a matroid is bipartite then the matroid must be bipartite.
3. Identification of a bipartite matroid may not be bipartite.
4. If a matroid does not have a pair of equivalent elements then it has no identification.
5. Identification of a graphic matroid need not be graphic.
6. If $M^{id} = (S, C^{id})$ is an identification of the matroid $M = (S, C)$ and r denotes the rank of M and r^{id} denotes the rank of M^{id}, then from rank theorem it follows that $r^{id}(M^{id}) = r(M) - 1$.

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The Minimal Displacement Problem and Optimal Retraction Constant in Banach Spaces

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Abstract

In this paper, for some Banach spaces, we determine or give a good estimate of the minimal displacement characteristic $\psi_X(k)$. We also give lower and upper bounds for the optimal retraction constant of some Banach spaces.
Keywords: Lipschitzian mapping, Minimal displacement, Retraction constant, Optimal retraction.

1 Introduction

The Minimal Displacement Characteristic and Optimal Retraction Constant

The study of the **minimal displacement problem** began in 1973 by Goebel. The minimal displacement value in an infinite-dimensional space is defined as:

$$d_T = \inf\{\|x - Tx\| : x \in B_X\},$$

where B_X is the **closed unit ball**. Goebel showed that for Lipschitzian mappings T, the value d_T can be positive. Subsequently, he proved that for $k \geq 0$ and $d \leq 1 - \frac{1}{k}$, there exist sets and spaces such that $d = 1 - \frac{1}{k}$. These spaces are called **Freudenthal spaces** (or spaces with property P). Furthermore, for every Banach space X, the **minimal displacement characteristic function** $\psi_X : [1, \infty) \longrightarrow [0, 1)$ is defined as:

$$\psi_X(k) = \sup_{T \in \mathcal{L}(k)} \inf_{x \in B_X} \|x - Tx\| = \sup_{T \in \mathcal{L}(k)} d_T \quad \text{for } k \geq 1.$$

We know that for every space X, $\psi_X(k) \leq 1 - \frac{1}{k}$. For Freudenthal spaces such as $BC(\mathbb{R})$, C_0, C, $C^n[a, b]$, and ℓ^∞, the equality $\psi_X(k) = 1 - \frac{1}{k}$ holds. Lin and Sternfeld showed that for $k > 1$, $\psi(k) > 0$. We obtain

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lower bounds for the minimal displacement characteristic in several spaces. The determination of the lower bound for the minimal displacement characteristic in the space ℓ^∞ remains an open problem. The purpose of this function is to study the minimal displacement problem. In 1983, Benyamini and Sternfeld showed that in every infinite-dimensional Banach space, the unit sphere S_X is a Lipschitz retract of the unit ball B_X. Thus, corresponding to every Banach space X, the constant

$$k_0(X) = \inf \{k \mid \exists R : B_X \longrightarrow S_X, \|Rx - Ry\| \leqslant k\|x - y\|, \forall x, y \in X, R|_{S_X} = \mathrm{id}_{S_X}\}$$

is defined. The exact value of the constant $k_0(X)$ is not known for any Banach space. Various estimates for this constant have been obtained so far. We know that for every Banach space X, $k_0(X) > 3$, but this inequality can be improved for some Banach spaces. For example, $k_0(H) \geq 4/55$ and $k_0(\ell^1) \geq 4$. For every Freudenthal Banach space X, $k_0(X) \leq 30/84$. This approximation can also be improved: $k_0(C[0,1] \leq 23/31$ and $k_0(H) \leq 28/99$. Casini and Anzalone showed that the best approximation belongs to the space ℓ^1, and $k_0(\ell^1) \leq 8$. The objective is to find a suitable approximation for $k_0(X)$, which has not been achieved so far.

2 Banach Spaces

Let $\mathbb{F}$ denote the field of real numbers or the field of complex numbers. A vector space X over the scalar field $\mathbb{F}$ is called a **normed space** if to each $x \in X$ a real number $\|x\|$ is assigned such that

(a) $\|x\| \geqslant 0$ and $\|x\| = 0 \Longleftrightarrow x = 0$,

(b) For every $x \in X$ and every scalar α, $\|\alpha x\| = |\alpha|\|x\|$,

(c) For every $x, y \in X$, $\|x + y\| \leqslant \|x\| + \|y\|$.

The normed space $(X, \|.\|)$ is called a **Banach space** if every Cauchy sequence in X converges. If X is a normed space

$$B_X = \{x \in X : \ \|x\| \leqslant 1\}$$

and

$$S_X = \{x \in X : \ \|x\| = 1\}$$

are called the **closed unit ball** and the **unit sphere** of X, respectively. The metric space (X, d) and the metric space (Y, ρ) are called **isometric** if there exists a one-to-one and onto mapping $f : X \longrightarrow Y$ such that

$$\rho(fx, fy) = d(x, y), \quad \forall x, y \in X.$$

Theorem 2.1. [14] *Let H be an arbitrary Hilbert space. Then*

(a) If H is separable, then $H \simeq \ell^2 \simeq L^2[0,1]$.

(b) If H is not separable, there exists a set I such that $H \simeq \ell^2(I)$.

Theorem 2.2. *$L^2[0,1]$ can be embedded in the space $\ell^2(\xi)$, where ξ is an uncountable cardinal number.*

Theorem 2.3. *The nearest point projection map in a Hilbert space, $P_C : H \longrightarrow C$ is non-expansive. That is, P_C satisfies the inequality:*

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Theorem 2.4. *(**Nearest Point Projection Map**)* [1] *Let H be a Hilbert space and $C \subseteq H$ convex, closed and non-empty, and $x \in H$. Then there exists a unique point $y \in C$ such that*

$$d(x, C) = \inf\{\|x - u\| : \ u \in C\} = \|x - y\|.$$

y is denoted by $P_C(x)$. In fact $P_C : H \longrightarrow C$ is a map such that $P_C|_C = I$ and, in particular, $P_C(x)$ is characterized by the inequality:

$$\forall u \in C, \langle x - P_C(x), u - P_C(x)\rangle \leq 0 \Leftrightarrow d(x, C) = \|x - P_C(x)\|.$$

Let $(X, \|.\|)$ be a normed space. A map $T : X \longrightarrow X$ is a **Lipschitz map** if there exists a non-negative constant k such that

$$\|Tx - Ty\| \leqslant k\|x - y\|, \qquad \forall x, y \in X.$$

Theorem 2.5. [8] *Let A be an arbitrary subset of a Hilbert space H and assume that $T : A \longrightarrow H$ is a k-Lipschitz map. Then there exists a k-Lipschitz extension $\widetilde{T} : H \longrightarrow Conv(T(A))$ of T.*

Let X be a Banach space and $C \subseteq X$ a closed and convex subset of X, and $T : C \longrightarrow C$ a map. T is said to be **uniformly Lipschitz** if there exists a constant $k \geqslant 0$ such that

$$\|T^n x - T^n y\| \leqslant k\|x - y\|, \qquad \forall x, y \in C, \forall n \in \mathbb{N}.$$

Definition 1. *If C is a closed, bounded, and convex subset of a Banach space X. For each $x \in C$ the **Chebyshev radius** of C is defined as*

$$r_C(x) = \inf_{x \in X} \sup_{y \in X} \{\|x - y\| : \ y \in C\}$$

Theorem 2.6. [5] *(**Banach Contraction Principle**). Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a map that satisfies the condition $d(Tx, Ty) \leqslant kd(x, y)$, for all $x, y \in X$ and $0 < k \leq 1$. Then T has a unique fixed point.*

Theorem 2.7. *(**Brouwer Fixed Point Theorem**)* [4]. *The closed unit ball B in $\mathbb{R}^n$ has the topological fixed point property. That is, every continuous map $T : B \longrightarrow B$ has a fixed point.*

Theorem 2.8. [5] *A constant k exists such that every Banach space X has a retraction $R : B_X \longrightarrow S_X$ of class $\mathscr{L}(k)$.*

Theorem 2.9. [11] *If X is an arbitrary Banach space and $\ell^\infty \subseteq X$ then there exists a projection map $P : X \longrightarrow \ell^\infty$ with norm 1.*

3 The Minimal Displacement Problem

Definition 2. *Let X be a real infinite-dimensional Banach space and $T : B_X \longrightarrow B_X$ a Lipschitz map of class $\mathscr{L}(k)$. The **minimal displacement** of the map T is defined as:*

$$d_T = \inf\{\|x - Tx\| : \ x \in B_X\}.$$

Lemma 3.1. *In any Banach space X and for every $k \geqslant 1$, $\psi_X(k) \leqslant 1 - \dfrac{1}{k}$.*

Theorem 3.2. [6] *If the Banach space X is any of the spaces c_0 , $C[0, 1]$, or $BC(\mathbb{R})$ (all continuous and bounded functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ equipped with the norm $\|.\|_\infty$), then for every $k \geq 1$, $\psi_X(k) = 1 - \dfrac{1}{k}$.*

Remark 3.3. Spaces where, for every $k \geq 1$, $\psi_X(k) = 1 - \dfrac{1}{k}$, are called **Fernique spaces**.

4 The Optimal Retraction Problem (Continued)

For every space X, the constant $k_0(X)$ is defined as:

$$k_0(X) = \inf\{k \mid \exists R : B_X \longrightarrow S_X, \|Rx - Ry\| \leqslant k\|x - y\| \quad \forall x, y \in X, R|_{S_X} = \mathrm{id}_{S_X}\}$$

This value is called the **Optimal Retraction Constant** of the space X.

Theorem 4.1. *For every Banach space we have $k_0(X) \geqslant 3$.*

Lemma 4.2. *For uniformly convex spaces, $k > 3$.*

Remark 4.3. Currently, the exact value of $k_0(X)$ is unknown for any Banach space.

Theorem 4.4. *The retraction constant $k_0(X)$ is the same for isometric spaces.*

We now derive the relationship between the function $\psi_X(k)$ and the value of the constant $k_0(X)$. For any map T, the minimal Lipschitz constant $k(T)$ is:

$$k(T) = \sup_{x \neq y} \frac{\|Tx - Ty\|}{\|x - y\|}$$

Let P be the radial projection from X onto B_X:

$$Px = \begin{cases} x & \|x\| \leq 1, \\ \dfrac{x}{\|x\|} & \|x\| > 1. \end{cases}$$

We know that for all spaces $1 \leq k(P) \leq 2$, for Hilbert spaces $k(H) = 1$ and for uniformly non-square spaces $k(P) < 2$. For x and y outside the ball $B_r = rB_X$ we have:

$$\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| = \|P(\frac{x}{r}) - P(\frac{y}{r})\| \leq \frac{k(P)}{r}\|x - y\|.$$

Consider a Lipschitzian retraction $R : B_X \longrightarrow S_X$ and $0 < \epsilon < 1$. We define the map $T_\epsilon : B_X \longrightarrow B_X$ as follows:

$$T_\epsilon x = \begin{cases} -R(\dfrac{x}{\epsilon}) & \|x\| \leq \epsilon, \\ -\dfrac{x}{\|x\|} & \|x\| > \epsilon. \end{cases}$$

If $\|x\|, \|y\| \leq \epsilon$ then we have:

$$\|T_\epsilon x - T_\epsilon y\| = \|R(\frac{x}{\epsilon}) - R(\frac{y}{\epsilon})\| \leq \frac{k(R)}{\epsilon}\|x - y\|,$$

and if $\|x\| > \epsilon$, $\|y\| > \epsilon$ then

$$\begin{aligned} \|T_\epsilon x - T_\epsilon y\| &= \| - \frac{x}{\|x\|} + \frac{y}{\|y\|}\| \\ &\leq \frac{2}{\min\{\|x\|, \|y\|\}}\|x - y\| \\ &\leq \frac{2}{\epsilon}\|x - y\| \\ &\leq \frac{k_0(R)}{\epsilon}\|x - y\| \end{aligned}$$

Therefore $k(T_\epsilon) \leq \frac{k(R)}{\epsilon}$. Now we show that for every $x \in B_X$, $\|x - T_\epsilon x\| \geq 1 - \epsilon$. If $\|x\| \leq \epsilon$,

$$\|x - T_\epsilon x\| = \|x + R(\frac{x}{\epsilon})\| \geq \|R(\frac{x}{\epsilon})\| - \|x\| \geq 1 - \epsilon,$$

If $\|x\| > \epsilon$,

$$\|x - T_\epsilon x\| = \|x + \frac{x}{\|x\|}\| = 1 + \|x\| \geq 1 + \epsilon \geq 1 - \epsilon.$$

We set $k = \frac{k(R)}{\epsilon}$ and obtain the following relation:

$$\psi_X(k) \geq 1 - \frac{k(R)}{k}$$

Hence

$$\psi_X(k) \geq 1 - \frac{k_0(X)}{k}.$$

4.1 The Optimal Retraction Constant in Hilbert Spaces

Assume H is a Hilbert space. In this case, the radial projection $P : H \longrightarrow B_X$ is in $\mathscr{L}(1)$, and for every $x \in H \setminus B_X$ and $y \in B_X$, considering the inequality $(x - Px, Px - y) \geq 0$, it follows that $\|x - y\| \geq \|Px - y\|$. For every $r > 0$, and for the radial projection P_r onto the ball $B_r = rB_X$, the condition is preserved. Let $T_0 : B_X \longrightarrow B_X$ be a mapping in $\mathscr{L}(k)$ with $d_{T_0} = d > 0$. Now we show that if $\|x\| \geq \sqrt{1 - d^2}$, then $(T_0 x - x, x) \leq 0$. Considering the unit ball, we obtain the mapping $T : B_X \longrightarrow B_X$ as follows:

$$Tx = \frac{k}{k+1} T_4(\frac{k+1}{k} x), \tag{4.1}$$

which is in $\mathscr{L}(k)$ such that $T(S_X) = \{0\}$ and $d_T \geq (\frac{k}{k+1} d)$. We obtain the retraction map:

$$Rx = \frac{x - Tx}{\|x - Tx\|} = P(\frac{x - Tx}{d_T}) = P(\frac{k+1}{k} \frac{x - Tx}{d}), \tag{4.2}$$

where $P \in \mathscr{L}(1)$ and T_0 can be chosen such that d is close to $\psi_H(k)$. Since $R \in \mathscr{L}(\frac{(k+1)^2}{kd})$, we arrive at the following approximation:

$$k_0(H) \leq \frac{(k+1)^2}{k\psi_H(k)}, \quad \forall\ k > 1. \tag{4.3}$$

Therefore, considering property (g) of the function ψ,

$$k_0(H)\psi'_H(1) \leq \min_{k>1} \frac{(k+1)^2}{k-1} = 8, \tag{4.4}$$

which is minimized at the point $k = 3$. Based on relation (4.3), the following result is obtained:

$$k_0(H) \frac{k\psi_H(k)}{k-1} \leq \frac{(k+1)^2}{k-1}, \tag{4.5}$$

and taking the optimal value of k at $k = 3$, we obtain

$$k_0(H)\psi_H(3) \leq \frac{16}{3}. \tag{4.6}$$

From a previous chapter, we have the lower bound for ψ in a Hilbert space. Hence

$$k_0(H) \leq \frac{16}{3} \times \frac{10}{2} = \frac{80}{3} = 26.6\cdots .$$

5 Results

The problem of determining $\psi_X(k)$ remains open for many Banach spaces, including ℓ_∞ and $L_1[0,1]$.

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Linguistic Variables

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Abstract

Natural languages, by their very nature, are rich in terms that possess semantic flexibility. Among these are linguistic variables and linguistic descriptors, which are components present in many propositions generated in natural languages. Therefore, such propositions cannot be categorized solely based on truth or falsehood values. In this paper, we employ a mathematical approach that allows for the assessment of the truthfulness of sentences produced in natural languages. To clarify the proposed approach, an example will also be provided.
Keywords: Prioritization, Criterion weight, Nasta'liq

1 Introduction

In Aristotelian logic, every proposition has a truth value of either true or false. For example, the proposition "4 is an even number" is considered true, while the proposition "the sun revolves around the Earth" is regarded as false. However, when we encounter the evaluation of truth and falsehood in natural language propositions where some components of the proposition are ambiguous, imprecise, or graded, attributing truth or falsehood essentially sacrifices accuracy for the sake of convenience, and correlating truth or falsehood to such propositions leads to inaccuracy. This is where fuzzy logic plays a role instead of Aristotelian logic. Therefore, within such a framework, it would not be surprising if a proposition holds any value between completely true and completely false. For example, consider the following statement: Many of Hafez's poems are influenced by Saadi's poems As we know, this statement is neither completely true nor completely false, and to assess its accuracy, in addition to examining the poems of these two poets, we must deeply evaluate two concepts, namely "very much" and "under the influence." it is important to note that in natural languages, many propositions, like the one above, are constructed using linguistic variables and modifiers. Linguistic variables are those whose values are expressed through words, phrases, or sentences in natural language. For instance, if we express a person's age in years, we encounter a numerical variable; however, if we describe age using terms like infant, child, adolescent, young, middle-aged, old, elderly, etc., age becomes a linguistic variable. As another example, if a person's height is expressed in centimeters, then height is considered a numerical variable. However, if we use terms like short or tall instead, height is regarded as a linguistic variable. Furthermore, linguistic descriptors are essentially modifiers that create a continium of a

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characteristic or attribute. For example, very, extremely, relatively, completely, fairly somewhat, and slightly are modifiers that, when added to the beginning of the word "young", create a spectrum of diverse states for the concept of "young". What we intend to do in this article is to introduce a mathematical approach that can be used to assess the accuracy of statements produced in natural languages using flexible concepts and linguistic descriptors. Although for convenience, we will discuss the simplest case of such statements, which can easily be extended to more complex cases. We will now introduce the necessary premises in more detail, followed by an example to clarify this approach.

2 Fuzzy Logic

One of the successful attempts to reduce the gap between formal languages and natural languages is the introducing the fuzzy theory and, consequently, fuzzy logic. Fuzzy logic is actually a complement to Aristotelian logic, and in this logic (unlike Aristotelian logic, which only includes true and false propositions), propositions can also have intermediate truth values. Although fuzzy logic was officially introduced in 1965 by Lotfizadeh, an Iranian-born mathematician, its origins go back thousands of years, so much so that even signs of it can be found in the teachings of Buddha. However, in the modern world, the emergence of Russell's paradoxes in mathematical logic and the discovery of Heisenberg's uncertainty principle in quantum physics can be considered two major motivations for the need of fuzzy theory. Perhaps the most important feature of fuzzy logic is the ability to process concepts that are flexible and spectral in nature, and this feature has led to the increasing connection of natural languages with this theory. Because natural languages have a multitude of such concepts. One of the central concepts in fuzzy logic is the fuzzy set, which is actually a generalization of the concept of Aristotelian sets, in which the membership of an element can be a degree between full membership (i.e. 1) and full non-membership (i.e. 0). Therefore, fuzzy sets are always accompanied by a function called the membership function. So a fuzzy set is defined as a collection of members along with a membership function such as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X)\}$$

in which X is the universal set and $\mu_{\tilde{A}}(x)$ is the percentage of x belonging to X. In fuzzy theory, one of the most important roles pertains to fuzzy numbers, which are essentially numerical fuzzy sets with convex graphs. Among these, left-right fuzzy numbers hold greater significance and are introduced in the form of a function as

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{if} \quad 0 \leq \frac{m-x}{\alpha} \leq 1 \\ R(\frac{m-x}{\beta}) & \text{if} \quad 0 \leq \frac{x-m)}{\beta} \leq 1 \\ 0 & o.w \end{cases}$$

in which $R, L : \mathbb{R}^+ \to [0, 1]$ are descending functions with

1. $L(0) = R(0) = 1$
2. $L(x), R(x) < 1$ for all $x > 0$
3. $L(x), R(x) > 0$ for all $x < 1$
4. $L(x) = R(x) = 1$ or $L(x) > 0$ for all $x, \lim_{x\to\infty}(x) = 0$

In a specific case, if we have $L(x) = R(x) = 1-x$ the fuzzy number in question is referred to as a triangular fuzzy number(T.F.N), which can also be represented by the symbol $(m, \alpha, \beta)_{LR}$ or $(m, m-\alpha, m+\beta)$.

In fuzzy logic, to address linguistic variables and descriptors, appropriate leftright fuzzy numbers can be utilized. Some linguistic modifiers are introduced with well-known membership functions due to their commonality. For instance, if the truth of a proposition is evaluated with respect to $\mu_{true\tilde{A}}(x))$, we will have

$$\mu_{verytrue\tilde{A}}(x)) = \mu^2_{true\tilde{A}}(x))$$

$$\mu_{fairlytrue\tilde{A}}(x)) = \mu^{0.5}_{true\tilde{A}}(x))$$

In many cases, it is necessary to extract precise numbers from fuzzy numbers. In such situations, methods known as defuzzification methods can be employed. An example of such methods is the CAM approach, which extracts the crisp number

$$CAM(\tilde{A})) = \frac{\int_{Supp(\tilde{A})} x.\mu_{\tilde{A}} dx}{\int_{Supp(\tilde{A})} \mu_{\tilde{A}}(x) dx}$$

from a fuzzy set like $\tilde{A}$ with a membership function $\mu_{\tilde{A}}(x)$. If the fuzzy number is a triangular fuzzy number like $\tilde{A} = (m, \alpha, \beta)_{LR}$, the result of the $CAM(\tilde{A})$ approach will be equal to $\frac{3m-\alpha+\beta}{3}$. [5]

3 OWA operator

Multi-Attribute Decision-Making (MADM) problems are problems in which a limited number of alternatives are to be prioritized according to a limited number of criteria. Although such problems have a long history and humans have always faced decision-making on a daily basis, it can be said that serious theorizing in this area has a history of less than a hundred years. In this regard, there are various methods in the literature, among which we can mention methods such as AHP, ANP, TOPSIS, VIKOR, LINMAP,...[2, 5]. To apply decision-making methods, information related to alternatives in relation to different criteria is usually organized in the form of a matrix called the decision matrix. Such a matrix, considering m alternatives and n criteria, is in the form $D = [\xi_{ij}]_{m\times n}$, where x_{ij} represents the status of alternative i concerning criterion j. [2] One of the multi-criteria decision-making methods is based on an operator called Yager's operator[1], in which, considering n criteria with weights $w_1, ..., w_n$(which can be calculated using various methods, including the Shannon entrophy approach [3] or by utilizing pairwise comparison matrices [2]) the it'h alternative is assigned the number

$$OWA(A_i) = \sum_{i=1}^{n} \xi_{i_1} w_t, \quad \xi_{i_1} \geq ... \geq \xi_{i_n}$$

Where xi_j is the state of alternative i with respect to criterion j and $\xi_{i_1} \geq ... \geq \xi_{i_n}$ in is a permutation of $\xi_{i_1}, ..., \xi_{i_n}$. and we refer to it as the OWA operator on these data and introduce it as the truth value. Now, let us assume Q is a linguistic descriptor, $\tilde{A}$ is a linguistic variable and $\Delta = \{\theta_1, ..., \theta_n\}$ In this case, the degree of the truth of the statement

$$Q \text{ members of } \Delta \text{ are } \tilde{A}$$

is as

$$\sum_{i=1}^{n} \tilde{A}(\theta_i(Q(\frac{i}{n}) - Q(\frac{i-1}{n}))$$

Example 3.1. Let's assume the linguistic variable "youth" is defined in terms of age, and the linguistic descriptor "most" is defined in terms of percentage, represented by the following membership functions

respectively:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{if } x \leq 22 \\ \frac{30-x}{8} & \text{if } 22 \leq x \leq 30 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{Q}}(x) = \begin{cases} 1 & \text{if } x \geq 0.8 \\ \frac{x-0.2}{0.6} & \text{if } 0.2 \leq x \leq 0.8 \\ 0 & o.w \end{cases}$$

Additionally, let's assume $X = \{24, 40, 17, 18, 21, 21, 19, 46\}$ represents the ages of 8 individuals in years. We want to evaluate the degree of truth of the statement

"most of these individuals are young"

To do this, we will first calculate the descending order of the truth values of the concept of youth concerning the members of the set and the weights based on the Yager operator, as follows:

$\mu_{\tilde{A}}(17) = \mu_{\tilde{A}}(18) = \mu_{\tilde{A}}(19) = \mu_{\tilde{A}}(21) = 1, \mu_{\tilde{A}}(24) = 0.75, \mu_{\tilde{A}}(40) = \mu_{\tilde{A}}(46) = 0$

$$w_1 = Q(0.125) - Q(0.000) = 0.000$$

$$w_2 = Q(0.250) - Q(0.125) = 0.083$$

$$w_3 = Q(0.375) - Q(0.250) = 0.208$$

$$w_4 = Q(0.500) - Q(0.375) = 0.208$$

$$w_5 = Q(0.625) - Q(0.500) = 0.208$$

$$w_6 = Q(0.750) - Q(0.625) = 0.208$$

$$w_7 = Q(0.875) - Q(0.750) = 0.083$$

$$w_8 = Q(1.000) - Q(0.875) = 0.000$$

Therefore, the desired value will be equal to

$$\sum_{i=1}^{n} \tilde{A}(\theta_i(Q(\frac{i}{n}) - Q(\frac{i-1}{n})) \approx 0.863$$

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A mathematical approach for prioritizing calligraphic works

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Abstract

One of the most challenging issues in calligraphy festivals (especially concerning Nasta'liq) is the comparison, evaluation, and judging of the artworks. Such a process is typically carried out by a panel of experts. However, there is always the challenge that judges may excessively incorporate their personal opinions (which may also be conflicting) into this process. This paper presents a mathematical approach and designs an algorithm to systematize the judging process, reduce conflicts, and minimize the influence of judges' personal opinions in the evaluation process.

Keywords: Prioritization, Criterion weight, Nasta'liq

1 Introduction

Nasta'liq, recognized as one of the most prominent cultural and artistic heritages of Iranians, relies on exceptionally beautiful individual and combined forms and incorporates golden ratios. Consequently, it creates unparalleled and limitless opportunities for calligraphers to produce magnificent and exquisite works of calligraphy. On one hand, the aesthetic characteristics and stunning geometry of Nasta'liq, and on the other hand, its diverse and extensive functions, ranging from daily uses, writing documents, inscribing inscriptions in religious and historical sites, creating decorative works, transcribing literary and religious treasures, to various signboard writings, have endowed Nasta'liq with extraordinary value and significance among Iranians and, more broadly, among Islamic scripts as far as due to it's flawless grace, harmony, and aesthetic appeal, Nasta'liq is referred to as the "bride of Islamic calligraphy". [2]

In this context, numerous calligraphy festivals are regularly held, particularly in Iran, the birthplace of Nasta'liq, where the works of calligraphers compete against one another. In such festivals, a number of expert calligraphers typically judge the works based on their knowledge, experiences, and preferences, thereby identifying the selected individuals.

However, it is evident that in these festivals, the judges' perspectives may differ or even conflict with one another, making the establishment of consensus among expert judges a challenge. Furthermore, participants in calligraphy festivals may perceive the competition results as excessively influenced by the personal tastes of the judges, leading to potential objections regarding these outcomes.

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On the other hand, a multi-criteria decision-making issue involves the process of prioritizing several alternatives while considering various criteria. Thus, from a broader perspective, the evaluation and comparison process of artistic works in art festivals (especially in relation to calligraphy) can be regarded as a multi-criteria decision-making problem, where the alternatives consist of the works presented by artists, which are evaluated and prioritized based on the criteria deemed important by the judges.

This modeling indicates that there are multiple challenges in such a process, arising from the diversity and conflict among the experiences and tastes of expert judges, which can be summarized as follows:

1. Diversity of comparative criteria from the perspective of expert judges.
2. Disagreement among expert judges regarding the importance of the criteria.
3. Over-involvement of judges' preferences in evaluating criteria weights
4. Uncertainty in judgments.
5. Differences in the preferences of expert judges in evaluating works based on various criteria.
6. Aggregation of judges
7. Qualitative nature of some criteria
8. Selection of an appropriate method for the final prioritization of works.

In this paper, we aim to present an approach that addresses the first three(and to some extent, the fourth) issues and systematically reduces the existing discrepancies and conflicts among expert judges. Additionally, in this approach, the influence of the judges' personal tastes in assessing the importance of the criteria will be minimized, resulting in greater acceptance of the outcomes by the participants in the festivals. The researchers of this paper also intend to address issues 4, 5, 6, 7 and 8 through suitable approaches that will be presented in a series of future articles.

Although the approach discussed in this paper is centered on the evaluation of Nasta'liq works, the underlying idea can easily be generalized to any similar competition where a number of expert judges must evaluate the works. It should be noted that this approach can also be developed and utilized in the form of an algorithm and software.

Although there are traditional criteria for evaluating the quality of a calligraphic work (such as Proportion(Nesbat), Composition(Tarkib), Seating(Korsi), Strength(Ghovvat), Weakness(Za'f) and others which are known as the twelve principles of Nasta'liq in the literature and are discussed in booklets titled 'Adab al-Mashq'), the reality is that in a calligraphy competition, other criteria may also be considered by experts for the evaluation and prioritization of the works. For instance, criteria such as the size of the works, the size of the pen chosen for writing, the color of the ink, the tonal quality of the color, the content of the works, creativity, and others may also be considered by judges for the evaluation and prioritization of the works. What we aim to accomplish in this paper requires the introduction of certain concepts, methods, and the undertaking of steps that will be elaborated upon in the following sections.

(i) Fuzzy Theory: It can be said that the roots of this theory, which aims to complement and, in some cases, contradict Aristotelian logic, date back thousands of years, even to the teachings of Buddha. However, two significant events that played a fundamental role in the formal establishment of this theory are Heisenberg's uncertainty principle in quantum physics and Russell's paradoxes in mathematical logic, which revealed the limitations of Aristotelian logic in analyzing phenomena that do not possess complete certainty. Following these developments, logics known as multi-valued logics (as opposed to Aristotelian binary logic) were designed by mathematicians such as Lukasiewicz, Bochvar, Heyting, and others, which had greater capacities than Aristotelian logic for analyzing natural phenomena. Continuing along this path,

in 1965, fuzzy logic was officially introduced by Lotfi Zadeh, an Iranian-born scholar, in the form of a paper in which multi-valued logics were replaced by an infinite-valued logic. This logic is based on the concept of fuzzy sets, where membership in a set can be considered on a spectrum between 0 and 1.

A fuzzy set is defined as a collection of members along with a membership function such as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X)\}$$

in which X is the universal set and $\mu_{\tilde{A}}(x)$ is the percentage of x belonging to X. In fuzzy theory, one of the most important roles pertains to fuzzy numbers, which are essentially numerical fuzzy sets with convex graphs. Among these, left-right fuzzy numbers hold greater significance and are introduced in the form of a function as

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{if} \quad 0 \leq \frac{m-x}{\alpha} \leq 1 \\ R(\frac{m-x}{\beta}) & \text{if} \quad 0 \leq \frac{x-m)}{\beta} \leq 1 \\ 0 & o.w \end{cases}$$

in which $R, L : \mathbb{R}^+ \to [0, 1]$ are descending functions with

1. $L(0) = R(0) = 1$
2. $L(x), R(x) < 1$ for all $x > 0$
3. $L(x), R(x) > 0$ for all $x < 1$
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In a specific case, if we have $L(x) = R(x) = 1-x$ the fuzzy number in question is referred to as a triangular fuzzy number(T.F.N), which can also be represented by the symbol $(m, \alpha, \beta)_{LR}$ or $(m, m-\alpha, m+\beta)$.

The use of fuzzy numbers allows us to model a portion of the uncertainties that may exist in judgments for any reason. On the other hand, in many cases, it is necessary to extract precise numbers from fuzzy numbers. In such situations, methods known as defuzzification methods can be employed. An example of such methods is the CAM approach, which extracts the crisp number

$$CAM(\tilde{A})) = \frac{\int_{Supp(\tilde{A})} x.\mu_{\tilde{A}} dx}{\int_{Supp(\tilde{A})} \mu_{\tilde{A}}(x) dx}$$

from a fuzzy set like $\tilde{A}$ with a membership function $\mu_{\tilde{A}}(x)$.

If the fuzzy number is a triangular fuzzy number like $\tilde{A} = (m, \alpha, \beta)_{LR}$, the result of the $CAM(\tilde{A})$ approach will be equal to $\frac{3m-\alpha+\beta}{3}$. [5]

(ii) MADM: A Multi Attribute Decision Making Problem is the prioritization of a limited number of alternatives based on a limited number of criteria. Although such an issue has a very long history, the scientific approach to decision making and development of mathematical techniques related to this field, has seen significant growth, especially in the last 70 years. Typically, the information regarding the alternatives in relation to the criteria is organized in the form of a matrix, such as $D = [d_{ij}]_{m\times n}$, known as the decision matrix, where d_{ij} represents the status of alternative i concerning criterion j. [6]

(iii) PCM: By considering n criteria, a pairwise comparison matrix is a matrix like $A = [a_{ij}]_{1\leq i,j\leq n}$, where the entries range from 1/9 to 9, and for each $1 \leq i, j \leq n$, we have $a_{ij}a_{ji} = 1$. In fact, in these matrices, which are designed to evaluate the priority of decision criteria relative to one another, the strongest priority is represented by the number 9, while the weakest priority corresponds to the number 1/9 and these priorities must be determined by expert judges. It should be noted that inconsistencies may arise between the judgments of an expert, for which the inconsistency index

$$C.R = \frac{\lambda_{max} - n}{n-1}(R.I)^{-1}$$

is used, where λ_{max} is the largest eigenvalue of A, and $R.I$ is a random index related to a large number of similar matrixes and if $C.R$ exceeds the standard threshold ($C.R \leq .01$), the judgments must be reconsidered.

After the pairwise comparison matrices have been compiled by expert decision-makers based on $\frac{n(n-1)}{2}$ comparisons, it is necessary to aggregate them. This is done using the geometric mean of the pairwise comparison matrices, where considering k experts who compared n criteria pairwise, we have

$$G_{Ave} = [(\prod \frac{\int_{Supp(\tilde{a}_{ijt})} x.\mu_{\tilde{a}_{ijt}}(x)dx}{\int_{Supp(\tilde{a_{ijt}})} \mu_{\tilde{a}_{ijt}}(x)dx})^{\frac{1}{\sum_{t=1}^{k} \xi_t}}]_{1\leq i,j\leq n, 1\leq t\leq k}$$

in which $\tilde{a}_{ijt}$ represents the fuzzy judgment of the t-th expert regarding the priority of criterion i over criterion j and ξ_t is the weight of the expert t. [3]

(iv) Evaluation of Criteria Weights Using Pairwise Comparisons: There are various methods for evaluating the weights of decision criteria, some of which are based on pairwise comparisons, including AHP, ANP, SWARA, BWM, and others.

When using pairwise comparison matrices A to extract the weights of the criteria, one can utilize the relation

$$A[w_1, ..., w_n]^T = \lambda_{max}[w_1, ..., w_n]^T or\ [w_1, ..., w_n] = \lim_{k\longrightarrow\infty} \frac{A^k.e^T}{e.A^k.e^T}$$

in which $e^T = [1, ..., 1]_{n\times 1}$.[1, 3]

(v) Evaluation of Criteria Weights Based on Entropy: considering the decision matrix D, the weights of the criteria based on entropy are extracted as follows.

(**(a)** : For every $1 \leq j \leq n$, convert the data of criterion j into a probability distribution using the following transformation

$$d_{ij} \longrightarrow \bar{d}_{ij} = \frac{d_{ij}}{\sum_{i=1}^{m} d_{ij}}$$

(**b**) For every $1 \leq j \leq n$, calculate the entropy related to criterion j as follows:

$$E_j = -\frac{\sum_{i=1}^{m} \bar{d}_{ij} \ln \bar{d}_{ij}}{\ln m}$$

(**c**) : For every $1 \leq j \leq n$, calculate the initial weight of criterion j as $w_j = 1 - E_j$

(**d**) : If the decision-maker has an initial judgment regarding criterion j say η_j, calculate the adjusted weight of criterion j as $w_j^{'} = \eta_j w_j$

(**e**) : For every $1 \leq j \leq n$, calculate the normalized weight of criterion j as follows:

$$\bar{w^{'}}_j = \frac{w_j^{'}}{\sum_{j=1}^{n} w_j^{'}} \quad [4]$$

Of course, in the absence of or overlooking the judgments of expert decision-makers, one can solely consider the entropy approach.

(vi). Veto Criteria: A set of criteria for which the weakness of an alternative cannot be compensated by strengths in other criteria is referred to as veto criteria. Such criteria cannot be utilized in the process of pairwise comparison matrices or entropy methods. [3]

Now, based on the aforementioned concepts, the following steps can be employed for the systematic evaluation of criteria weights and as a preliminary measure for prioritizing artistic works:

step 1 . Designing the Criteria List: In the first step, judges must reach an agreement on the list of criteria (regardless of their importance) so that the final judgments are all based on a uniform set of agreed-upon criteria. In this regard, some inherently identical criteria may be interpreted differently by the judges, necessitating that through necessary modifications, the judges arrive at a common understanding. Additionally, a criterion that one judge considers important may not be accepted by another judge. Nevertheless, such a criterion should still be included in the final criteria list. However, it is preferable to reach a consensus through discussion regarding the inclusion or exclusion of such criteria.

step 2 . Determining Veto Criteria: In this step, judges must agree on the veto criteria. The presence of even one veto criterion will prevent the evaluation of the weights of the criteria. For example, failure to adhere to the specified dimensions and sizes for producing artistic works, or works whose content falls outside the framework introduced by the festival, may serve as veto criteria. Veto criteria may also have their own prioritization and may not hold equal importance. In this step, alternatives that are not sufficiently desirable can be eliminated through initial filtering techniques such as lexicographic, semi-lexicographic, or conductive methods.

step 3 . Pairwise comparisons: After eliminating the veto criteria, each judge is asked to evaluate the priority of the remaining criteria relative to one another in the form of a pairwise comparison matrix using either fuzzy or crisp numbers. Although calculations can be performed with fuzzy data, if fuzzy numbers are used, it is advisable to consider the results obtained from defuzzification, for example, through the CAM approach, before proceeding with the calculations.

step 4 . Consistency: For each of the pairwise comparison matrices, the inconsistency index C.R must be calculated, and if it falls outside the standard range, the judgments should be reviewed.

step 5 . Geometric average: In this step, the geometric mean of the pairwise comparison matrices must be calculated.

step 6 . Entropy: In this step, the weights obtained from entropy must be calculated within the averaged decision matrix.

step 7 . Final weights: In this step, the final weights of the criteria are computed by combining the results from Steps 5 and 6.

By following the above steps, it becomes possible to utilize multi-criteria decision-making methods for prioritizing calligraphic works, which the authors of this paper intend to address in a separate independent study. It should also be noted that if this process is applied in multiple festivals, the results will gradually conform to the normal statistical distribution and achieve stability.

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A Mathematical modelling for prioritizing projects in the mineral industries sector

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Abstract

Natural languages, by their very nature, are rich in terms that possess semantic flexibility. Among these are linguistic variables and linguistic descriptors, which are components present in many propositions generated in natural languages. Therefore, such propositions cannot be categorized solely based on truth or falsehood values. In this paper, we employ a mathematical approach that allows for the assessment of the truthfulness of sentences produced in natural languages. To clarify the proposed approach, an example will also be provided.

Keywords: Decision making, Weight, Minerals, prioritization

1 Introduction

In Aristotelian logic, every proposition has a truth value of either true or false. For example, the proposition "4 is an even number" is considered true, while the proposition "the sun revolves around the Earth" is regarded as false. However, when we encounter the evaluation of truth and falsehood in natural language propositions where some components of the proposition are ambiguous, imprecise, or graded, attributing truth or falsehood essentially sacrifices accuracy for the sake of convenience, and correlating truth or falsehood to such propositions leads to inaccuracy. This is where fuzzy logic plays a role instead of Aristotelian logic. Therefore, within such a framework, it would not be surprising if a proposition holds any value between completely true and completely false. For example, consider the following statement: Many of Hafez's poems are influenced by Saadi's poems As we know, this statement is neither completely true nor completely false, and to assess its accuracy, in addition to examining the poems of these two poets, we must deeply evaluate two concepts, namely "very much" and "under the influence." it is important to note that in natural languages, many propositions, like the one above, are constructed using linguistic variables and modifiers. Linguistic variables are those whose values are expressed through words, phrases, or sentences in natural language. For instance, if we express a person's age in years, we encounter a numerical variable; however, if we describe age using terms like infant, child, adolescent, young, middle-aged, old, elderly, etc., age becomes a linguistic variable. As another example, if a person's height is expressed in centimeters, then height is

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considered a numerical variable. However, if we use terms like short or tall instead, height is regarded as a linguistic variable. Furthermore, linguistic descriptors are essentially modifiers that create a continium of a characteristic or attribute. For example, very, extremely, relatively, completely, fairly somewhat, and slightly are modifiers that, when added to the beginning of the word "young", create a spectrum of diverse states for the concept of "young". What we intend to do in this article is to introduce a mathematical approach that can be used to assess the accuracy of statements produced in natural languages using flexible concepts and linguistic descriptors. Although for convenience, we will discuss the simplest case of such statements, which can easily be extended to more complex cases. We will now introduce the necessary premises in more detail, followed by an example to clarify this approach. **********

2 fuzzy logic

As mentioned, there are numerous sensitivities in prioritizing major projects in the mining and mineral industries because these projects are usually very costly, and any negligence in this regard can lead to incorrect prioritization and significant waste of resources. Therefore, to ensure accuracy in such prioritization, a comprehensive and diverse set of criteria must be considered [1,5]. Such criteria typically have economic, technical, environmental, political, or social natures. To clarify this point, we will introduce a sample of criteria and sub-criteria that were considered in a real study conducted in 2024 in Iranian mines, Mining Industries Development, Renovation Organization for prioritizing several major projects in the mineral industries:

Example 2.1. List of criteria and sub-criteria used for prioritizing 245 alternatives in the field of mineral industries:

C.1 : Compliance with Strategies

C.11 : Alignment with organization's vision, mission and objectives

C.12 : Executive Preferences (The degree of acceptability in C-level executive)

C.13 : Preferences of off-organization officials and benefactors

C.14 : Alignment with provincial/state investment priorities

C.2 : Cost Factors

C.21 : Capital expenditure

C.22 : Working capital expenditure

C.23 : Foreign expenditure share in CAPEX

C.24 : Foreign expenditure share in OPEX

C.25 :Financial progress

C.3 : Market Factors

C.31 : Local demand

C.32 : Import demand of the region countries

C.33 : Rivals in local market

C.34 : Rivals in region countries

C.35 : International trade share of the relative global market

C.36 : Vertical integration and well-distributed supply through the chain

C.37 : The opportunity to create local and regional market

C.38 : Meeting major industries' demands and guarantee to purchase

C.4 : Economic and Profitability Factors

C.41 : Return of capital(including construction duration)

C.42 : Profit Margin

C.43 : Export sale

C.44 : Meeting import demand

C.45 : Access to sufficient and inexpensive transportation facilities

C.5 : Technical and Operating Factors

C.51 : Skilled labor force

C.52 : Similar experiences in local market

C.53 : Technology Accessibility

C.54 : Energy and Fuel Accessibility

C.55 : Water demand

C.56 : Ease of water supply

C.57 : Self-sufficiency in supplying localized equipment

C.58 : Physical progress

C.59 : Contribution to developing other projects

C.6 : Risk Factors

C.61 : Substitution Risk

C.62 : Risk of technology and equipment import

C.63 : Price volatility risk

C.64 : Local community risk(social responsibility and local protestors)

C.65 : Environment risks

C.66 : Raw material supply dependency to neighboring towns and cities

C.67 : Reserve sustainability risk

C.68 : Investment participation risk

C.69 : Natural disasters risk

C.7 : Social Responsibility Factors

C.71 : employment

C.72 : Impact on local economy

C.73 : Alignment with national needs

C.74 : Severity of deprivation in the community

C.75 : Pollution Intensity and irreversible environmental effects

C.8 : Ease of Finance Factors

C.81 : Ability to finance the project via modern methods such as BOT and BOO

C.82 : Ease of finance by stakeholders resources and major off-organizational supporters

C.83 : Ability to finance by organizational resources

C.84 : Ease of finance by financial institutions and funds' facilities and mining investment hedge funds

Lemma 2.2. *Considering a decision-making matrix such as* $[A]_{m\times n}$ *(where the number of criteria preferably is a large number), the weight obtained from the entropy of each criterion is derived based on the following steps:*

step 1 *For every* $1 \leq j \leq n$*, convert the data of criterion* j *into a probability distribution using the following transformation*

$$a_{ij} \longrightarrow \bar{a}_{ij} = \frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}}$$

step 2 *For every* $1 \leq j \leq n$*, calculate the entropy related to criterion* j *as follows:*

$$E_j = -\frac{\sum_{i=1}^{m} \bar{a}_{ij} \ln \bar{a}_{ij}}{\ln m}$$

step 3 *For every* $1 \leq j \leq n$*, calculate the initial weight of criterion* j *as* $w_j = 1 - E_j$

step 4 *If the decision-maker has an initial judgment regarding criterion* j *say* λ_j*, calculate the adjusted weight of criterion* j *as* $w_j^{'} = \lambda_j w_j$

step 5 *For every* $1 \leq j \leq n$*, calculate the normalized weight of criterion* j *as follows:*

$$\bar{w^{'}}_j = \frac{w_j^{'}}{\sum_{j=1}^{n} w_j^{'}}$$

It's noteworthy that in this process, it is preferable to first calculate the weights through entropy. This way, criteria with entropy values close to 1 can be excluded from the decision-maker's coefficient calculation process, simplifying the final weight calculation.

If the data of the decision matrix appears in the form of fuzzy numbers like $\tilde{a_{ij}} = \{(x, \mu_j(x))|x \in Supp(\tilde{a_{ij}})\}$, then we can first defuzzify the data in a suitable way like CAM where the defuzzification result is

$$CAM(\tilde{a_{ij}})) = \frac{\int_{Supp(\tilde{a_{ij}})} x.\mu_j(x)dx}{\int_{Supp(\tilde{a_{ij}})} \mu_j(x)dx}$$

and then the entropy algorithm can be applied.

On the other hand, if the decision-maker has a specific level of risk tolerance in the decision, denoted as α, the weights associated with this level of risk tolerance can be extracted according to the OWA (Ordered Weighted Averaging) algorithm from the following optimization problem[2,3]:

$$max\ \theta = -\frac{\sum_{j=1}^{n} w_j \ln w_j}{\ln n}$$

$$S.t : \frac{1}{n-1}\sum_{j=1}^{n}(n-j)w_j = \alpha;$$

$$\sum_{j=1}^{n} w_j = 1;$$

$$w_j \geq 0, 1 \leq j \leq n;$$

If the outputs of this optimization problem are as $\{w_j^*\}_{j=1}^n$, then the weights obtained from entropy can be adjusted using them. By this way obtain the final weights as

$$\tilde{w_j} = \frac{w_j^* \bar{w'}_j}{\sum_{i=1}^{n} w_i^* \bar{w'}_i}, 1 \leq j \leq n$$

Additionally, considering the decision matrix $A = [a_{ij}]_{1\leq i\leq m, 1\leq j\leq n}$, let's assume that in a decision-making algorithm, the number assigned to alternative i is $\varphi(a_{i1}, ..., a_{in}, w_1, ..., w_n)$ and $\varphi_i^*, \tilde{\varphi_i}$ represent the maximum and minimum values for $\varphi(a_{i1}, ..., a_{in}, w_1, ..., w_n)$, respectively, while λ_i indicates the decision-maker's emphasis on the importance of alternative A_i. In this way then optimal weights can also be extracted from the following model, allowing for the adjustment of the previously obtained weights.

$$min\ \xi = (\sum_{i=1}^{m} \lambda_i (\frac{\varphi_i^* - \varphi(a_{i1}, ..., a_{in}, w_1, ..., w_n)}{\varphi_i^* - \tilde{\varphi}_i})^p)^{\frac{1}{p}}$$

$$S.t : \sum_{j=1}^{n} w_j = 1$$

$$1 \leqq p \leq \infty$$

$$w_j \geq 0, 1 \leq j \leq n$$

Ultimately, after extracting results from this model, the prioritization of alternatives will be based on descending order of $\varphi(a_{i1}, ..., a_{in}, w_1^*, ..., w_n^*)$.

If we aim to integrate the above approaches, we will proceed to solve the following optimization model in which D is the pairwise comparison matrix of alternatives priorities relative to each other:

$$min\ \xi = (\eta \sum_{i=1}^{m} \lambda_i (\frac{\varphi_i^* - \varphi(a_{i1}, ..., a_{in}, w_1, ..., w_n)}{\varphi_i^* - \tilde{\varphi}_i})^p + \chi(1 + \frac{\sum_{j=1}^{n} w_j \ln w_j}{\ln n})^p)^{\frac{1}{p}}$$

$$S.t : \frac{1}{n-1} \sum_{j=1}^{n} (n-j) w_j = \alpha;$$

$$[\lambda_i]_{1 \leq i \leq m} = \lim_{k \longrightarrow \infty} \frac{D^k.e^t}{e.D^k.e^t}$$

$$\lambda_i = CAM(\tilde{\lambda_j})) = \frac{\int_{Supp(\tilde{\lambda_j})} x.\mu_j(x)dx}{\int_{Supp(\tilde{\lambda_j})} \mu_j(x)dx}$$

$$\tilde{\lambda_j} = \{(x, \mu_j(x)) | x \in Supp(\tilde{\lambda_j})\}$$

$$\sum_{j=1}^{n} w_j = 1$$

$$1 \leqq p \leq \infty$$

$$w_j \geq 0, 1 \leq j \leq n$$

$$\eta + \chi = 1$$

$$0 \leq \eta, \chi \leq 1$$

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Approximate Inferences in Natural Languages

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Abstract

Natural languages encompass a multitude of terms and concepts that possess flexible and graded meanings, and many propositions in natural languages are constructed using such concepts. On the other hand, the inherent uncertainty in the nature of these concepts makes the evaluation of the truth value of these propositions challenging. In this paper, we present an approach to overcome this challenge.
Keywords: Natural Languages, Fuzzy logic, Inference

1 Introduction

Argument is an essential and inseparable component of human communication; indeed, human communication without arguments is inconceivable. On the other hand, when arguments are expressed and processed in natural languages, they are invariably accompanied by ambiguities due to the presence of vague terms and linguistic modifiers. This characteristic makes it impossible to analyze propositions generated in natural language in the same way as those formulated in Aristotelian logic. For instance, propositions expressed in natural language cannot be strictly categorized as either true or false. Therefore, to deal with the propositions and expressions producible in natural languages-which are now widely employed in many intelligent systems-we must equip ourselves with a more flexible logic. This logic is fuzzy logic, and its key concept is that of a fuzzy number.

In what follows, after briefly introducing fuzzy logic and the concept of fuzzy numbers, we outline a general approach for determining the degree of truth of approximate arguments and propositions generated in natural languages.

Fuzzy logic is particularly well-suited for evaluating the degree of truth of approximate arguments. In its simplest form, an argument generally takes the following structure: If $\tilde{A}$, then $\tilde{B}$ or formally

$$\tilde{A} \models \tilde{B}$$

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When either $\tilde{A}$ or $\tilde{B}$ contains flexible linguistic expressions, we are dealing with an approximate argument. Arguments can, in general, be more complex and take a generic form such as:

$$\tilde{A}_1, ..., \tilde{A}_m \models \tilde{B}_1, ..., \tilde{B}_n$$

An example of an approximate argument in natural language is the following statement: "If a child has proper nutrition and plenty of physical activity, they will enjoy a high level of physical health." In this sentence, some of the fuzzy expressions present are: "child," "plenty of physical activity," and "high level of physical health." We consider these expressions fuzzy because they can be defined and interpreted along a spectrum. For example, humans cannot be divided into two distinct categories of "child" and "non-child"; rather, each can be assigned a certain degree of childhood, interpretable by a number between zero and one.

Hence, assessing the degree of truth of this sentence-which is effectively an approximate argument-is not feasible within Aristotelian logic. However, in fuzzy logic, there are approaches that enable the evaluation of the degree of truth of such propositions. For this purpose, an approach known as fuzzy implications is used. To present the intended approach, we first provide a concise introduction to fuzzy logic.[3]

2 Fuzzy Theory

Although formal introduction of fuzzy logic dates back less than a century indeed, it was introduced in 1965 by Lotfi Zadeh, an Iranian mathematician, in the form of a research paper, the roots of this logic and the need for it actually trace back several millennia, to the extent that its traces can even be found in the teachings of Buddha. Perhaps the primary reason for the necessity of fuzzy logic can be regarded as an attempt to bridge the gap that exists between formal languages and natural languages.

On the other hand, as we know, in Aristotelian logic, which has been the prevailing logic in human thought for thousands of years, propositions can only be true or false. Consequently, Aristotelian logic is also referred to as binary logic or black-and-white logic.

However, it is evident that using Aristotelian logic in many cases results in accuracy being sacrificed for simplicity, and this logic lacks the capacity to process intermediate states. For example, when we wish to categorize humans from an Aristotelian perspective based on age for a concept such as "youth," we are forced to divide people into two groups: young and non-young. Yet, the reality of the matter is that youth is not a binary concept; in fact, we assign people a "degree of youthfulness"-a value that can range anywhere from zero to one hundred percent.

Although this is just one example, natural languages are replete with such concepts that Aristotelian logic lacks the power to analyze. Furthermore, natural languages include linguistic modifiers that generate a diverse spectrum of a given concept. Terms such as more or less, to some extent, relatively, very, very very, not at all, often, and so forth, are consistently used in natural languages. It is clear that if linguistic modifiers are analyzed solely within the framework of Aristotelian logic, we would be forced, by ignoring reality, to treat them as binary concepts. In contrast, fuzzy logic possesses the capacity to analyze and account for the spectral nature of linguistic modifiers. One of the central concepts in fuzzy logic is the notion of fuzzy set. A fuzzy set is a collection of members, characterized by a degree of membership for each element. In general, this degree of membership is expressed by a function known as membership function such as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X)\}$$

in which X is the universal set and $\mu_{\tilde{A}}(x)$ is the percentage of x belonging to X. On the other hand, one of the most important types of fuzzy sets, through which many fuzzy computations are performed, are fuzzy numbers. In fact, fuzzy numbers are fuzzy sets whose members are real numbers and whose membership graphs are convex. Such numbers are typically described using functions known as left and right functions,

as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{if} \quad 0 \leq \frac{m-x}{\alpha} \leq 1 \\ R(\frac{m-x}{\beta}) & \text{if} \quad 0 \leq \frac{x-m)}{\beta} \leq 1 \\ 0 & o.w \end{cases}$$

in which $R, L : \mathbb{R}^+ \to [0, 1]$ are descending functions with

1. $L(0) = R(0) = 1$
2. $L(x), R(x) < 1$ for all $x > 0$
3. $L(x), R(x) > 0$ for all $x < 1$
4. $L(x) = R(x) = 1$ or $L(x) > 0$ for all $x, \lim_{x\to\infty}(x) = 0$

In a specific case, if we have $L(x) = R(x) = 1-x$ the fuzzy number in question is referred to as a triangular fuzzy number(T.F.N), which can also be represented by the symbol $(m, \alpha, \beta)_{LR}$ or $(m, m - \alpha, m + \beta)$.

The use of fuzzy numbers allows us to model a portion of the uncertainties that may exist in judgments for any reason. On the other hand, in many cases, it is necessary to extract precise numbers from fuzzy numbers. In such situations, methods known as defuzzification methods can be employed. An example of such methods is the CAM approach, which extracts the crisp number

$$CAM(\tilde{A})) = \frac{\int_{Supp(\tilde{A})} x.\mu_{\tilde{A}} dx}{\int_{Supp(\tilde{A})} \mu_{\tilde{A}}(x)dx}$$

from a fuzzy set like $\tilde{A}$ with a membership function $\mu_{\tilde{A}}(x)$.

If the fuzzy number is a triangular fuzzy number like $\tilde{A} = (m, \alpha, \beta)_{LR}$, the result of the $CAM(\tilde{A})$ approach will be equal to $\frac{3m-\alpha+\beta}{3}$. [3,4]

3 Fuzzy implications

Fuzzy implications are tools used to evaluate the degree of truth of propositions formulated in natural languages. This issue is important from several perspectives. For instance, if the truth value of a proposition exceeds 0.50, it might be considered a basis for decision-making or for accepting/rejecting a hypothesis. Furthermore, since many modern intelligent machines and systems are equipped with fuzzy logic, processing fuzzy propositions-which are often presented as input variables-necessitates fuzzy approaches, such as the use of fuzzy implications, to ensure the efficacy of the machine or system.

There are numerous types of fuzzy implications. For instance Larsen, Mamdani, Lukasiewicz and probablistic implication which states in simplest form as follows respectively:

$$R(u, v) = \mu_{\tilde{A}}(u) \times \mu_{\tilde{v}}(x)$$

$$R(u, v) = min\{\mu_{\tilde{A}}(u), \mu_{\tilde{B}}(v)\}$$

$$R(u, v) = min\{1, 1 - \mu_{\tilde{A}}(u) + \mu_{\tilde{B}}(v)\}$$

$$R(u, v) = 1 - \mu_{\tilde{A}}(u) + \mu_{\tilde{A}}(u) \times \mu_{\tilde{B}}(v)$$

To utilize these formulas for a given proposition, the fuzzy concepts introduced in the antecedent and consequent of the proposition must first be represented by fuzzy numbers. Then, for each observation, the degree of membership in the fuzzy numbers should be calculated and used as input to the formulas. Finally, the average of the results is taken as the overall degree of truth.[1,2]

Example 3.1. Suppose the following inference is intended

$$\tilde{A}_1, ..., \tilde{A}_m \models \tilde{B}_1, ..., \tilde{B}_n$$

Given a sample of size k from a statistical population, to use Larsen's implication, for a member of the sample such as x_j we have

$$R_j(\tilde{A}_1, ..., \tilde{A}_m; \tilde{B}_1, ..., \tilde{B}_m) = (\prod_{i=1}^{m} \mu_{\tilde{A}_i}(x_j))(\prod_{i=1}^{n} \mu_{\tilde{B}_i}(x_j))$$

Now we can set the truth degree of the $\tilde{A}_1, ..., \tilde{A}_m \models \tilde{B}_1, ..., \tilde{B}_n$ as

$$T(\tilde{A}_1, ..., \tilde{A}_m \models \tilde{B}_1, ..., \tilde{B}_n) = (\sum_{j=1}^{k} R_j(\tilde{A}_1, ..., \tilde{A}_m; \tilde{B}_1, ..., \tilde{B}_m))/k.$$

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A Journey through Subrings of Rings of Continuous Functions

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Abstract

This paper investigates novel structural perspectives on subrings of the ring $C(X)$ of real-valued continuous functions defined on a topological space X. Building on a classical concept of algebraic subrings introduced by Gillman and Jerison—and inspired by its resemblance to semiprime ideals—we introduce and study two new classes of subrings: *prime subrings*, characterized by the condition that $fg \in S \Rightarrow f \in S$ or $g \in S$; and *pseudoprime subrings*, where $fg = 0 \Rightarrow f \in S$ or $g \in S$. These definitions parallel familiar notions in ideal theory. Through illustrative examples, we show that algebraic, prime, and pseudoprime subrings are pairwise distinct. Additionally, we establish several equivalent characterizations of primeness and pseudoprimeness within this context.

Keywords: Algebraic subring, rings of continuous functions.

1 Introduction

Throughout this note, all topological spaces are assumed to be infinite, Hausdorff, and completely regular (i.e., infinite Tychonoff spaces). We denote by $C(X)$ (respectively $C^*(X)$) the ring of all real-valued continuous (respectively bounded continuous) functions on a space X.

The motivation behind our current approach, and the research that follows, traces back to an informal and inspiring conversation we had nearly a year ago with Professor O.A.S. Karamzadeh. During that discussion, he drew our attention to a particular definition found in the book by Gillman and Jerison, which became the starting point of our exploration.

According to [6, 16.29], a subring S of $C^*(X)$ is called *algebraic* if it satisfies the following conditions:

1. All constant functions belong to S.
2. For each $f \in C^*(X)$, $f^2 \in S$ implies $f \in S$.

Condition (2) in the definition above caught our attention, as it aligns precisely with the definition of semiprime ideals in ideal theory. As is well known, semi-prime ideals play a fundamental role in understanding the structure of rings. This connection led us to consider the possibility that certain algebraic subrings might also play a significant role in characterizing rings of continuous functions.

Motivated by this line of thought, we investigated the recognition of such subrings in a separate work currently

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under review [1], where several significant results were obtained. In Section 2, we provide a brief overview of that approach, presenting illustrative examples and key findings that set the stage for the current study. Section 3 introduces the notions of *prime* and *pseudoprime* subrings, drawing inspiration from analogous concepts in ideal theory.

2 Algebraic subrings

The algebraic subrings in our work [1] differ slightly from those described in [6, 16.29]. Specifically, instead of considering subrings of $C^*(X)$, we focused on subrings of $C(X)$.

Definition 1. *We say a subring S of $C(X)$ is an algebraic subring if S satisfies the following conditions:*

1. *All constant functions belong to S.*
2. *For each $f \in C(X)$, $f^2 \in S$ implies $f \in S$.*

In other words, we say a subring $\mathbb{R} \subseteq S \subseteq C(X)$ is algebraic if for each $f \in C(X)$, $f^2 \in S$ implies $f \in S$.

In the following, we give several examples of algebraic subrings.

Example 2.1.
1. The ring $C(X)$.
2. $C^*(X)$ that is the ring of all real-valued, continuous bounded functions.
3. $C_c(X)$ that is the ring of all real-valued continuous functions with countable image. (For more information about $C_c(X)$, the reader is referred to [4] and [5].)
4. $C^F(X)$ that is the subring of $C(X)$ consisting of functions with finite image.
5. $C^\sharp(X)$ that is the subring of $C(X)$ consisting of all f such that for every maximal ideal M of $C(X)$, there is an $r \in \mathbb{R}$ such that $f - r \in M$. (For more details see [9].)

Proposition 2.2. *The subring $\mathbb{R}$ is an algebraic subring of $C(X)$ if and only if X is a connected space.*

Theorem 2.3. *Let $S \supseteq \mathbb{R}$ be a subring of $C(X)$. The following statements are equivalent:*

1. *S is an algebraic subring.*
2. *For $f \in C(X)$ and $n \in \mathbb{N}$, $f^{2^n} \in S$ implies $f \in S$.*
3. *For $f, g \in C(X)$, $fg \in S$ and $f + g \in S$ imply $f \in S$ and $g \in S$.*
4. *For $f, g \in C(X)$, $fg = 0$ and $f^2 + g^2 \in S$ imply $f + g \in S$.*
5. *S is closed under uniform limits, meaning if $\{f_n\} \subset S$ and $f_n \to f$ uniformly, then $f \in S$.*

In Definition 3, the condition "for $f \in C(X)$, $f^2 \in S$ implies $f \in S$" cannot be replaced by any of the following conditions:

- $f^n \in S$ for some positive integer n implies $f \in S$;
- both $f^2 \in S$ and $f^3 \in S$ imply $f \in S$; or
- $f^2 \in S$ implies $|f| \in S$.

Example 2.4. Let $S = \{f \in C(\mathbb{R}) : f(0) = f(1)\}$. Then $f^3 \in S$ implies $f \in S$, but $f^2 \in S$ does not necessarily imply $f \in S$. For instance, $h(x) = x - \frac{1}{2}$ satisfies $h^2 \in S$ but $h \notin S$.

Example 2.5. Consider $C^\infty = \{f \in C(\mathbb{R}) : f \text{ is smooth}\}$. According to Joris [8], if f^2 and f^3 (or more generally f^m, f^n with $\gcd(m, n) = 1$) are smooth, then f is smooth. However, $h(x) = |x|$ satisfies $h^2 \in C^\infty$ but $h \notin C^\infty$.

Example 2.6. Let S be the subring in Example 2.4. It is easy to check that $g^2 \in S$ implies $|g| \in S$. Consider $h(x) = x - \frac{1}{2}$. It is clear that $h^2 \in S$ whereas $h \notin S$.

3 Prime and pseudoprime subrings

We begin with the following defintion.

Definition 2. *We say a subring S of $C(X)$ is a prime subring if S satisfies the following conditions:*

1. *All constant functions belong to S.*
2. *For each $f \in C(X)$, $fg \in S$ implies $f \in S$ or $g \in S$.*

In other words, we say a subring $\mathbb{R} \subseteq S \subseteq C(X)$ is prime if for all $f, g \in C(X)$, $fg \in S$ implies $f \in S$ or $g \in S$.

Lemma 3.1. *Every prime subring is an algebraic subring. However, the converse is not true.*

Proof. The first statement is clear. For the second, the subring $C_c(\mathbb{R})$ is an algebraic subring that is not a prime subring. To see this, define

$$f(x) = \begin{cases} 0 & x \geq 0 \\ x & x < 0 \end{cases}, \quad g(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Clearly, $fg = 0$ and so $C_c(\mathbb{R})$. However, $f \notin C_c(\mathbb{R})$ and $g \notin C_c(\mathbb{R})$, as desired. □

Does $C(X)$ contain a proper prime subring?
We now present some equivalent characterizations of prime subrings involving absolute values.

Proposition 3.2. *Let $S \subseteq C(X)$ be a subring containing all constant functions (i.e., $S \supseteq \mathbb{R}$). The following statements are equivalent:*

1. *S is a prime subring.*
2. *For all $f, g \in C(X)$, if $|fg| \in S$ then $f \in S$ or $g \in S$.*
3. *For all $f, g \in C(X)$, if $fg \in S$ then $|f| \in S$ or $|g| \in S$.*
4. *For all $f, g \in C(X)$, if $|fg| \in S$ then $|f| \in S$ or $|g| \in S$.*

Proof. (1) $\Rightarrow$ (2) Let $f, g \in C(X)$ and $|fg| \in S$. From (1), we infer that either $|f| \in S$ or $|g| \in S$. Thus, we have either $f^2 \in S$ or $g^2 \in S$. Since S is a prime subring, we deduce that either $f \in S$ or $g \in S$.
(2) $\Rightarrow$ (1) Let $f, g \in C(X)$ and $fg \in S$. Thus, we have $f^2g^2 \in S$. From (2), we infer that either $f^2 \in S$ or $g^2 \in S$. This implies that either $|ff| \in S$ or $|gg| \in S$. From (2), we conclude that either $f \in S$ or $g \in S$.
(1) $\Rightarrow$ (3) Let $f, g \in C(X)$ and $fg \in S$. Assume (1). Without loss of generality, we may assume that $f \in S$. From $(|f| - f)(|f| + f) = 0 \in S$, we have either $|f| - f \in S$ or $|f| + f \in S$. This implies that $|f| \in S$, as desired.

(3) $\Rightarrow$ (1) Let $f, g \in C(X)$ and $fg \in S$. Assume (3). Without loss of generality, we may assume that $|f| \in S$. We note that $(|f| - f)(|f| + f) = 0 \in S$. From (3), we infer that either $||f| - f| = |f| - f \in S$ or $||f| + f| = |f| + f \in S$. Since $|f| \in S$, we deduce that $f \in S$, as desired.
(1) $\Rightarrow$ (4) It is clear.
(4) $\Rightarrow$ (1) Let $f, g \in C(X)$ and $fg \in S$. Thus, we have $f^2g^2 \in S$. From (4), we infer that either $f^2 \in S$ or $g^2 \in S$. This implies that either $|ff| \in S$ or $|gg| \in S$. This implies that either $|f| \in S$ or $|g| \in S$. Without loss of generality, we may assume that $|f| \in S$. We note that $(|f| - f)(|f| + f) = 0 \in S$. From (4), we infer that $|f| - f \in S$ or $|f| + f \in S$. Since $|f| \in S$, we conclude that $f \in S$, as desired. □

Following [7], an ideal I of $C(X)$ is called *pseudoprime* if $fg = 0$ implies either $f \in I$ or $g \in I$, for all $f, g \in C(X)$. Obviously, any ideal containing a prime ideal is pseudoprime. By [7, Theorem 4.1], an ideal of $C(X)$ is pseudoprime if and only if it contains a prime ideal. Other conditions equivalent to an ideal of $C(X)$ being pseudoprime are given in [7, Theorem 4.1].

Definition 3. *We say a subring S of $C(X)$ is a pseudoprime subring if S satisfies the following conditions:*

1. *All constant functions belong to S.*
2. *For each $f \in C(X)$, $fg = 0$ implies $f \in S$ or $g \in S$.*

In other words, we say a subring $\mathbb{R} \subseteq S \subseteq C(X)$ is pseudoprime if for all $f, g \in C(X)$, $fg = 0$ implies $f \in S$ or $g \in S$.

In the following, we give several examples of pseudoprime subrings.

Example 3.3.
1. The ring $C(X)$.
2. The subring $I + \mathbb{R}$ whenever I is a pseudoprime ideal. (For more information about the subring $I + \mathbb{R}$, the reader is referred to [3].)

Lemma 3.4. *Let $S \subseteq C(X)$ be a subring containing all constant functions (i.e., $S \supseteq \mathbb{R}$). The following statements are equivalent:*

1. *S is a pseudoprime subring.*
2. *For $f, g \in C(X)$ and $h, k \in S$, $(f - h)(g - k) = 0$ implies $f \in S$ or $g \in S$.*

Proof. (1) $\Rightarrow$ (2) Let $f, g \in C(X)$, $h, k \in S$, and $(f - h)(g - k) = 0$. From (1), we deduce that $f - h \in S$ or $g - k \in S$. Thus, we conclude that $f \in S$ or $g \in I$.
(2) $\Rightarrow$ (1) It suffices to take $h = k = 0$. □

The following example shows that there are pseudoprime subrings that are not algebraic subrings.

Example 3.5. Let P be a minimal prime ideal of $C(\mathbb{R})$ containing O_0. Hence $i \notin P$, where i is the identity function in $C(\mathbb{R})$. Now if we take $I =< i > +P$, then I is pseudoprime, see [7, Theorem 4.1]. Define $S = I + \mathbb{R}$. The subring R is pseudoprime. We claim that S is not an algebraic subring. For example, define

$$f(x) = \begin{cases} 0 & 0 \\ i \sin \frac{1}{i} & \text{elsewhere} \end{cases}$$

It is not hard to see that $f^2 \in S$ whereas $f \notin S$.

The following example shows that there are algebraic subrings that are not pseudoprime subrings.

Example 3.6. Obviously, $C_c(\mathbb{R})$ is an algebraic subring. However, as noted in the proof of Lemma 3.1, $C_c(\mathbb{R})$ is not a pseudoprime subring.

4 What Comes Next?

We plan to extend this line of inquiry by introducing and investigating several natural generalizations of algebraic subrings-namely, *local, saturated, norm-closed,* and *norm-reflecting* subrings. These concepts, inspired by the lattice structure of $C(X)$, aim to deepen our understanding of subring behavior, particularly in relation to absolute value operations and associated order-theoretic properties. To pursue this direction, we will build on the work of Azarpanah and Ghashghaei [2], although our emphasis shifts away from their ideal-theoretic perspective toward a subring-theoretic framework. This change in focus enables us to construct new examples and uncover fresh structural insights into rings of continuous functions.

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Extending Camillo and Yu'S Theorem to Unit-Suitable Rings

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Abstract

Camillo and Yu proved that a suitable ring R with stable range one is unit-regular. In this work, we generalize their result by showing that, for a unit-suitable ring, the condition of stable range one is equivalent to the ring is partially unit-regular, which is equivalent to it being left (or right) uniquely generated.

Keywords: Unit-suitable Rings, Stable Range Conditions, Unit-clean rings.

1 Introduction

In 1974, Nicholson [2] defines a ring R to be an suitable ring if for each element $a \in R$, there exists an idempotent $e \in R$ such $e \in aR$ and $(1-e) \in (1-a)R$. We define an element a to be *left unit-suitable* if there exists a unit u and an idempotent e in R such that $e - ua \in R(a - aua)$. Similarly we can define the element a to be *right unit-suitable* if there exists an unit v and an idempotent f such that $f - av \in (a - ava)R$. A ring R is *left unit-suitable* (*right unit-suitable*) if each of its elements are *left unit-suitable* (*right unit-suitable*). In [3], Bass introduced the concept of the left (right) stable range for a ring in his investigation of the cancellation property within modules. A ring R is said to have stable range one if, for any elements $a, b, x \in R$, the equation $ax + b = 1$ implies that there exists some $y \in R$ such that $\implies a + by = u$ for some $u \in U(R)$, where $U(R)$ denotes the group of units of R. In the context of rings with stable range one, various equivalent conditions have been established to characterize this class of rings. Yu [6] demonstrated that an exchange ring R has stable range one if either R has primitive factors that are Artinian, R has a bounded index of nilpotence, or the idempotents of R are central. A result, often attributed to Kaplansky, states that a von Neumann regular ring R is unit-regular if and only if it has stable range one. Subsequently, Camillo and Yu [2] extended this result to certain suitable classes of rings. In this work, we further extend this result by relaxing the hypothesis, showing that if R is unit-suitable, then R has stable range one if and only if R is partially unit-regular. Consequently, we provide equivalent characterizations of unit-suitable rings with stable range one.

Nicholson [2] defined the notion of an element in a ring R to be clean if it can be expressed as a sum of an idempotent and an invertible element in R. We have visualized unit-exchange to be a property such that for each $a \in R$ there exists a unit u such that the element au is *suitable*. Clearly it would be quite natural to ask what happens if for each element a in a ring R, there exists an invertible element u in R such that au

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is clean? Are rings, satisfying this condition, properly contained in unit-suitable rings? We will answer this question in the affirmative by giving examples of classes of rings that satisfy such conditions, and therefore are also unit-suitable.
We define an element a in R to be unit-clean if there exists invertible elements u, v in R, and an idempotent e in R such that $au = e + v$. A ring R is unit-clean if and only each of its elements are unit-clean. This definition is independent of the position of the invertible element u. In other words u could be multiplied to the left or right of a. It is easy to observe that an element in a ring R is unit-clean if it can be expressed as a sum of a unit-regular element and an invertible element in R.
Clearly, every 2-good ring is unit-clean. However the converse is not true as $R = \mathbb{F}_2$ is a unit-clean ring which is not 2-good. Finally, we prove that the unit stable range condition for unit-suitable rings is equivalent to the condition that every regular element of the ring is unit-clean.

2 Main results

In this section, we provide equivalent characterizations for unit-suitable rings with stable range one. In [8], Goodearl presents the following result, which is essentially attributed to Vaserstein.

Lemma 2.1. *Let $a, b, c \in R$, where R is a ring, and suppose that $ab + c = 1$. If there exists an element $x \in R$ such that the element $a + cx$ is a unit in R, then there exists an element $y \in R$ such that $b + yc$ is unit.*

In [6], Canfell defines a ring R as right uniquely generated if, for any two right principal ideals aR and bR in R, the condition $aR = bR$ implies that $a = bu$ for some $u \in U(R)$, where $U(R)$ denotes the group of units of the ring. Similarly, the concept of a left uniquely generated ring is defined in an analogous manner.

Lemma 2.2. *Let R be a ring. Then R has stable range one if and only if R is left (or right) uniquely generated.*

Camillo and Yu have shown that a suitable ring is unit-regular if and only if it has stable range one. After presenting several equivalent formulations of the stable range one condition, we extend this result in the following theorem to classes of unit-suitable rings and obtain some significant conclusions.

Theorem 2.3. *Let R be a unit-suitable ring. The following are equivalent:*

1. *Every regular element is unit regular.*
2. *R has stable range one.*
3. *R is left (right) uniquely generated*

Corollary 2.4. *Let R be an unit-sitable ring with stable range 1 such that 2 is invertible . Then each element of R can be written as the sum of 6 units. .*

Theorem 2.5. *Let R be an unit-suitable ring. Then the following are equivalent:*

1. *R has stable range one.*
2. *Every regular element of R is unit-clean.*

Corollary 2.6. *A ring R is unit-regular if and only if every element in R is unit-clean.*

For an unit-clean rings with stable range 1 we have:

Corollary 2.7. *Let R be an unit-clean ring with stable range 1 such that 2 is invertible . Then each element of R can be written as the sum of 3 units. .*

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On The Structure of Unit-Suitable Rings

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Abstract

This paper introduces a new class of rings, called unit-suitable rings, which properly extends the class of suitable rings. We show that the unit-suitable condition exhibits symmetric with respect to left and right multiplication at the element level. We will show several classes of rings that are unit-suitable but not suitable. Furthermore, we explore the behavior of this property under certain ring extensions to determine whether it remains invariant in broader algebraic structures.
Keywords: Suitable rings, Unit-suitable rings, 2-good rings.

1 Introduction

In 1977, Nicholson [2] defined a ring R to be *suitable* if, for each $a \in R$, there exists $e^2 = e \in R$ such that $e \in aR$ and $(1-e) \in (1-a)R$. He introduced these rings as left suitable rings, as it was not immediately apparent that the definition is symmetric with respect to left and right. However, Nicholson proved that every left suitable ring is also right suitable, and conversely. He also showed that a ring R is a suitable ring if and only if idempotents lift modulo every left ideal (or equivalently, every right ideal) in R. Following Vaḿos [3], an element x in a ring R is called a *k-good* element if x can be expressed as the sum of k units in R. A ring R is called a *k-good* ring if each element of R is a *k- good* element. Many authors including [1], have studied rings generated additively by their unit elements, in particular, 2-good rings. Since *suitable* rings need not necessarily be 2-good, we define a new class of rings (called *unit-suitable* rings) that encompasses both those classes of rings, and generalize most of the known results.
All our rings are associative rings with identity and all our modules are left modules, unless specifically mentioned. $U(R)$ denotes the set of all idempotents of R, $J(R)$ denotes the Jacobson radical, $N(R)$ denotes the set of all nilpotent elements and $B(R)$ denotes the set of central idempotents of the ring R.

2 Main results

We define an element a to be *left unit-suitable* if there exists a unit u and an idempotent e in R such that $e - ua \in R(a - aua)$. Similarly we can define the element a to be *right unit-suitable* if there exists a unit v and an idempotent f such that $f - av \in (a - ava)R$. A ring R is *left unit-suitable* (*right unit-suitable*) if each of its elements are *left unit-suitable* (*right unit-suitable*).
In the following lemma it is shown that for each element, the definition of unit-suitable is left-right symmetric.

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Lemma 2.1. *Let R be a ring. Then the following are equivalent for any element a in R:*

1. *There exists a unit u and an idempotent e in R such that $e - ua \in R(a - aua)$.*
2. *There exists a unit u_1 and an idempotent e_1 in R such that $e_1 \in Ra$ and $1 - e_1 \in R(1 - u_1 a)$.*
3. *There exists a unit u_2 and an idempotent e_2 in R such that $e_2 \in Ra$ and $1 - e_2 \in R(a - u_2)$.*
4. *There exists a unit u_3 and an idempotent e_3 in R such that $e_3 \in aR$ and $1 - e_3 \in (a - u_3)R$.*
5. *There exists a unit u_4 and an idempotent e_4 in R such that $e_4 - au_4 \in (a - au_4 a)R$.*

Clearly from the definition every suitable ring R is unit-suitable. However the converse is not necessarily true as we will see in many examples. Recall that a ring R is suitable if and only if $R/J(R)$ is suitable and idempotents lift modulo J(R). For unit-suitable rings, we consider the following proposition.

Theorem 2.2. *A ring R with Jacobson radical J is unit-suitable if for some ideal $Q \subseteq J$, R/Q is unit-suitable and idempotents lift modulo Q.*

The converse need not be true for unit-suitable rings, that is, idempotents need not lift modulo the Jacobson radical in a unit-suitable ring nor does unit-suitable rings require to be semi-potent (recall that a ring R is called semipotent if every left ideal (equivalently right ideal) not contained in J(R) contains a non-zero idempotent). However, for an unit-suitable ring R, it is easy to see that either every left (right) principal ideal contains a non-zero idempotent or every element in that principal ideal is a sum of two invertible elements.
In literature the notion of suitable property is defined on the whole ring and not for specific elements. However, we call an element a in a ring R to be suitable if there exists $e^2 = e \in R$ such that $e \in aR$ and $(1 - e) \in (1 - a)R$. Visualizing this notion, we observe that a ring R is unit-suitable if, for each $a \in R$ there exists a unit u such that the element au is suitable. We begin with some simple observations on unit-suitable rings.

Lemma 2.3. *For any ring R we have the following:*

1. *Every 2-good, and hence* twin-good *ring and every semi-local ring that has no homomorphic image isomorphic to $\mathbb{F}_2$ are unit-suitable.*
2. *If R is a direct product of rings R_i where each R_i is a unit-suitable, then R is also unit-suitable.*

Let $S(R)$ be the nonempty set of all the proper ideals of a ring R generated by central idempotents. By the Zorn's Lemma, $S(R)$ contains maximal elements. If M is a maximal element of the set $S(R)$, then the factor ring R/M is called a Pierce stalk of R.

Theorem 2.4. *For a ring R, the following are equivalent.*

1. *R is unit-suitable.*
2. *All homomorphism images of R are unit-suitable.*
3. *All factor rings of R are unit-suitable.*
4. *R/I is unit-suitable for every proper ideal I of R generated by central idempotents of R.*
5. *All Pierce stalks of R are unit-suitable.*

Remark 2.5. 1. If R is a nonzero unit-suitable ring, then the polynomial rings are never unit-suitable rings. Indeed, Let R be an unit-suitable ring. For $x \in R[x]$, $xR[x]$ contains the only idempotent 0. If $R[x]$ is a unit-suitable, there exist a unit $u \in U(R[x])$ such that $1 \in (1-ux)R[x]$ and so $(1-ux)$ is invertible in $R[x]$, which is a contradiction.

2. The subring of unit-suitable rings need not inherit the property. For example, Q the field of all rational numbers, is an unit-suitable ring but the subring $\mathbb{Z}$, the ring of integers, is not unit-suitable.

We consider an unit-suitable ring with certain annihilator condition.

Lemma 2.6. *Let R be an unit-suitable ring. Then for any $a \in R$, there exist an idempotent $e \in R$ and a unit $u \in U(R)$ such that $ann_r(a) \subseteq eR$ and $ann_r(1-ua) \subseteq (1-e)R$.*

The converse of previous Lemma is not true by the ring of integers $\mathbb{Z}$.

Proposition 2.7. *Suppose that every principal left ideal of a ring R is a left annihilator. Then R is an unit-suitable if and only if for any $a \in R$, there exists an idempotent $e \in R$ and a unit $u \in U(R)$ such that $ann_r(a) \subseteq eR$ and $ann_r(1-ua) \subseteq (1-e)R$.*

We see that the properties shown by unit-suitable are quite similar in nature to the classes of rings that satisfy the relatively stronger suitable conditions, except for some aberrations. Nicholson proved that a corner ring of any suitable ring is suitable. However $R = M_2(\mathbb{Z})$ is unit-suitable but the corner ring eRe is not unit-suitable. Now it is known that if eRe and $(1-e)R(1-e)$ are both suitable, then R is also suitable. In the next theorem, we prove an analogue of this result for unit-suitable rings.

Theorem 2.8. *For a ring R and for any idempotent e in R, if eRe and $(1-e)R(1-e)$ are both unit-suitable then R is unit-suitable.*

Corollary 2.9. *If R is unit-suitable then so is $M_n(R)$.*

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A study on e-reversible rings

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Abstract

In this paper, we study some equivalent conditions for e-reversibility of a ring R. We see if R is a von Neumann ring and e is an idempotent element of R, then R is an abelian ring if and only if eR and $(1-e)R$ are e-reversible. In the following, we study some conditions under which $R[x]$ is an e-reversible ring. Also, we obtain conditions that $R[[x]]$ is an e-reversible ring. We show that if R is a reduced ring, then $R[[x]]$ is an e-reversible ring.

Keywords: e-reversible ring, reversible ring, reduced ring.

1 Introduction

Throughout all rings are associative with identity. A ring R is called *reduced* if R has no nonzero nilpotent elements. R is said to be *symmetric* if $xyz = 0$ implies $xzy = 0$, for all $x, y, z \in R$. If for any $x, y \in R$, $xy = 0$ implies $xRy = 0$, then R is said to be *semicommutative*. Also, R is *abelian* if each idempotent element of R is central. *Reversible ring* is a ring in which $xy = 0$ yields $yx = 0$, for all $x, y \in R$. This ring is a class of noncommutative ring which has been studied extensively in recent years. Various generalizations of reversible rings have been proposed. The notion of e-reversible ring is a generalization of reversible ring. *e-reversible ring* is a ring R in which $xy = 0$ yields $yxe = 0$, for idempotent element e of R. In this paper, first we see some examples of e-reversible rings. We see an e-reversible ring need not be reversible. If R is a semiprime and semicommutative ring, then R is an e-reversible ring, for any element e of R. In the following, we study some equivalent conditions for e-reversibility of a ring R. We see, if e is a central idempotent element of a ring R, then R is an e-reversible ring if and only if eR and $(1-e)R$ are e-reversible. Also, if e is an idempotent element of R, we realize that R is reversible if and only if R is an e-reversible and (1-e)-reversible ring. Finally, for an idempotent element e in a ring R, if R is an Armendariz ring, then $R[x]$ is an e-reversible ring. Also, if R is a reduced ring, then $R[[x]]$ is an e-reversible ring.

1.1 Introduction to nil reversible ring

In this part, we study some properties of e-reversible ring. We start by recalling the definition of e-reversible ring.

Definition 1. *Let R be a ring and e be an idempotent element of R. If for all $x, y \in R$, $xy = 0$ implies $yxe = 0$, then R is called e-reversible ring.*

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Proposition 2.1 of [3] shows that the next Example is an e-reversible ring, for an idempotent element e of a ring R.

Example 1.1. Let R be a reduced ring. Then

$$\left\{ \begin{pmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{pmatrix}, \quad \textit{such that } x, y, z, t, u, w \in R \right\}$$

is E-reversible ring, where

$$E = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and e is an idempotent element of R.

Recall that a ring R is called reversible if $xy = 0$, implies $yx = 0$, for every $x, y \in R$. It is clear that every reversible ring R is e-reversible, for every idempotent element e of R. The next Example says that the converse is not true, in general.

Example 1.2. Let R be the ring of all 2×2 upper triangular matrices over a domain D and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then R is an E-reversible ring by [33, Corollary 2.3] and R is not reversible because if we take

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then $MN = 0$ while

$$NM = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0.$$

Recall that a ring R is said to be semicommutative if for any $x, y \in R$, $xy = 0$ implies $xRy = 0$. Also, a ring R is said to be abelian if all idempotent elements of R are central.

Lemma 1.3. *If R is a semiprime and semicommutative ring, then R is an e-reversible ring, for any idempotent element e of R.*

The converse of the above Lemma need not be true, in general. Because, if R is the ring of all 2×2 upper triangular matrices over a domain D,

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then R is an E_{11}-reversible ring, but $E_{12}E_{11} = 0$ and $E_{11}E_{12} \neq 0$. So R is not abelian. Since every semicommutative ring is abelian, R is not semicommutative.

2 Main results

Lemma 2.1. *Let T be a subring of a ring R and e be an element of T. If R is e-reversible, then T is also $e-$reversible.*

Corollary 2.2. *If e is a central idempotent element of a ring R, then eR and $(1-e)R$ are e-reversible if and only if R is an e-reversible ring.*

The next Example says that an abelian ring is not necessarily e-reversible.

Example 2.3. Let R be the set of all 2×2 matrices over $\mathbb{Z}$ such as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $a-d=2m, b=2n$ and $c=2k$ for $m,n,k\in\mathbb{Z}$. By [2, Example 2.7], R is an abelian ring. But, R is not e-reversible. Because, if we take

$$e=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad y=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

then

$$xy=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}=0.$$

But

$$yxe=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}.$$

Theorem 2.4. *Let R be a von Neumann ring and e be an idempotent element of R. Then the following sets coincide:*

1. *R is an abelian ring.*
2. *R is an e-reversible and $(1-e)$-reversible ring.*

An idempotent element e of a ring R is called left semicentral if $ae=eae$ for each $a\in R$.

Theorem 2.5. *If e is an idempotent element of a ring R, then the following are equivalent:*

1. *R is an e-reversiblering.*
2. *eRe is a reversible ring and e is left semicentral.*

Lemma 2.6. *Let R be a ring and e an idempotent element of R. Then the following sets coincide:*

1. *R is a reversible ring.*
2. *R is an e-reversible and $(1-e)$-reversible ring.*

Proof. $1\Rightarrow 2$. It is clear.
$2\Rightarrow 1$. Let $a,b\in R$ such that $ab=0$. Since R is (1-e)-reversible ring, then $ba(1-e)=0$. This yields $ba=bae$. Now, $bae=0$ by e-reversibility of R. Thus $ba=bae=0$ and R is reversible. □

Theorem 2.7. *Let R be a von Neumann ring and e be an idempotent element of R. Then the following sets coincide:*

1. *R is a reversible ring.*
2. *R is an Armendariz ring.*
3. *R is an abelian ring.*
4. *R is an e-reversible and $(1-e)$-reversible ring.*

Proof. The statements 1, 2 and 3 are equivalent by [4, Theorem 2.5]. Also, 1 and 4 are equivalent by Lemma 1.2. □

Theorem 2.8. *Let e be an idempotent element of a ring R and R be an e-reversible ring.*

1. *If R is an Armendariing, then $R[x]$ is an e-reversible ring.*
2. *If R is a reduced ring, then $R[x]$ is an e-reversible ring.*
3. *If R is a reduced ring, then $R[[x]]$ is an e-reversible ring.*

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Amenability of some homogeneous Banach convolution algebras

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Abstract

Let G be a locally compact group and B be a homogeneous Banach space on G. In this paper, we study the amenability and super amenability of homogeneous Banach convolution algebras B_{min}.

Keywords: Amenability, Super-Amenability, Homogenous Banach space, Homogeneous function algebra.

1 Introduction

In 1980, Feichtinger [33] introduced an especial Banach space $W(B, C)$, for any Banach space B on G that is in a good situation with respect to a homogeneous function algebra $\mathcal{A}$ and any solid, translation invariant Banach space C on G.
Afterwards, Feichtinger [2] showed that if B is a homogeneous Banach space on G, which is in a good situation, then there exists a minimal space B_{min} in the family of all homogeneous Banach spaces C on G that contains all elements of B with compact support. Moreover $B_{min} = W(B, L^1(G))$, whenever B is in a good situation with respect to a homogeneous function algebra; see [2].

We recall some known concepts and frameworks, which will be used throughout the paper. Let G be a locally compact Hausdorff group with the fixed Haar measure λ and $L^\infty(G)$ be the Banach algebra of all essentially bounded Borel-measurable functions on G. Moreover, for $0 < p < \infty$, $L^p(G)$ indicates the usual Lebesgue spaces, as defined in [5]. Also $L_{loc}(G)$ is the space, consisting of all (equivalent classes of) measurable functions f on G such that $f\chi_K \in L^1(G)$, for any compact subset $K \subseteq G$. Let $C_0(G)$ (resp. $C_b(G), C_c(G)$) be the space of all continuous, complex valued functions on G vanishing at infinity (resp. bounded, with compact support). Moreover, $A(G)$ and $B(G)$ are the Fourier and Fourier-Stieltjes algebra of G, as defined in [33]. Also, $A_p(G)$ are The Figà-Talamanca-Herz algebras for $1 < p < \infty$; see [33]. Let $\mathcal{L}A(G) := A(G) \cap L^1(G)$ be the Lebesgue-Fourier algebra of G; see [3].

We say that a Banach space $(B, \|.\|_B)$ is a BF-space if it is continuously embedded into $L_{loc}(G)$, that is the space consisting of all locally integrable functions on G. Indeed, for each compact subset K of G there exists some constant $C_K > 0$ such that

$$|f\chi_K|_1 \leq C_K |f|_B,$$

for each $f \in B$ with $\int\sqcap\!\!\sqrt{\sqrt{}}(f) \subseteq K$. Further, let

$$B_\Omega = \{f \in B : \text{ supp(f) is compact}\},$$

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and

$$B_K := \{f \in B : supp(f) \subseteq K\},$$

for any $K \subseteq G$. We say that $f \in B_K$ almost every where (for abbreviation a.e) if $supp(f) \subseteq K$ a.e on G. For the remainder, we emphasize that two measurable functions are identifying if they are equal locally almost everywhere (for abbreviation l.a.e) on G, i.e. they are equal a.e on any compact subset $K \subseteq G$. A BF-space on G is called left translation invariant if $L_y f \in B$ for any $y \in G$ and $f \in B$, where $L_y f(x) = f(y^{-1}x)$ $(x \in G)$. A left invariant BF-space will be called a homogenous Banach space on G if the following conditions are satisfied:

(i) $\|L_y f\|_B = \|f\|_B$ for any $y \in G$;

(ii) the map $y \to L_y f$ from G into $(B, \|.\|_B)$ is continuous for all $f \in B$.

Moreover, we say that G acts by right translation isometrically on B and right translation is continuous in B, if $R_x f \in B$ and $\|R_x f\|_B = \|f\|_B$ for all $x \in G$ and $f \in B$, and also $\lim_{y\to e} \|R_y f - f\|_B = 0$ for all $f \in B$. A Banach algebra $(A, \|.\|_A) \subseteq C_b(G)$ with the pointwise multiplication is a homogenous function algebra on G, if

(i) $\mathcal{A}$ is a homogeneous Banach space on G;

(ii) $\mathcal{A}$ separates the points of G from closed sets;

(iii) $\mathcal{A}$ is a self-adjoint Banach algebra with respect to complex conjugate.

Let G be a locally compact group and B be a homogenous Banach space on G that is in a good situation with respect to a homogenous function algebra $\mathcal{A}$. Then, there exists a minimal space B_{min} in the family of all homogeneous Banach spaces C satisfying $B_\Omega \subseteq C$. Also, B_{min} can be characterized as follows. Let an open, relatively compact set $Q \subseteq G$ be given. $f \in B_{min}$ if and only if $f = \sum_n L_{y_n} f_n$ for some $y_n \in G$ and $f_n \in B$ such that $\sum_n \|f_n\|_B < \infty$ and $supp(f_n) \subseteq y_n Q$ for each $n \in N$ and

$$\|f\|_{min} = inf\{\sum_n \|f_n\|_B : f = \sum_n L_{y_n} f_n\}, \tag{1.1}$$

where the infimum is taken over all representations of f as in (1.1). The space $B \cap C_c(G)$ is a dense subspace of B_{min}. If B contains positive elements with arbitrary small support, i.e. for any open subset U of G, there exists a (nonzero) positive element $f \in A$ such that supp$(f) \subseteq U$, then B_{min} is a Segal algebra; see [2]. The space B_{min} is independent of the choice of Q; see [2, Theorem 3]).

Several authors have introduced various notions of amenability in the recent years; see [5]. The first notion of amenability has been presented for locally compact groups. Let G be a locally compact group, then G is called amenable if there exists a functional $m \in L^\infty(G)^*$ such that

(i) $\langle 1, m\rangle = \|m\| = 1$;

(ii) $\langle L_x \varphi, m\rangle = \langle \varphi, m\rangle$, for all $\varphi \in L^\infty(G)$ and $x \in G$

This definition has been generalized for the Banach algebras, as follows. Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-bimodule, then E^*, the dual space of E, becomes a Banach $\mathcal{A}$-bimodule via

$$< x, \varphi.a >:=< a.x, \varphi >, \quad < x, a.\varphi >:=< x.a, \varphi >$$

for any $a \in \mathcal{A}$ and $\varphi \in E^*$. A derivation from $\mathcal{A}$ into E is a bounded linear map $D : \mathcal{A} \to E$ satisfying

$$D(a.b) = a.D(b) + D(a).b \qquad (a, b \in \mathcal{A}).$$

The derivation D is called inner, if there is $x \in E$ such that $D(a) = a.x - x.a$, for any $a \in \mathcal{A}$. We now say that $\mathcal{A}$ is amenable if each continuous derivation from $\mathcal{A}$ into E^* is inner for all Banach $\mathcal{A}$-bimodules E. Also, $\mathcal{A}$ is called super-amenable if every continuous derivation from $\mathcal{A}$ into E is inner for every Banach A-bimodule E. Note that every amenable Banach algebra has a bounded approximate identity and every super-amenable Banach algebra is unital. Hence, $L^1(G)$ is super-amenable if and only if G is finite; see [5].

2 Main results

In this section, we proceed to the main results of the paper. For the Lebesgue Fourier algebras, the following results were studied by Ghahramani and Lau [5]. We now extend those to some Feichtinger algeras.

Theorem 2.1. *Let G be a locally compact group, $1 < p < \infty$ and B be a homogenous Banach space on G which is in a good situation with respect to a homogeneous function algebra on G. Moreover, suppose that B contains positive elements with arbitrary small support. Moreover, let $B_\Omega \subseteq L^p(G)$. Then, B_{min} is amenable with respect to the convolution product if and only if G is discrete and amenable.*

Proof. Let B_{min} be amenable, with the convolution product. Then, B_{min} has a bounded approximate identity. Also, B_{min} is a Segal algebra. It follows that $B_{min} = L^1(G)$. On the other hand, there exists $C_Q > 0$ such that

$$\|f\|_1 \leq C_Q \|f\|_B,$$

for any $f \in B$ with supp$f \subseteq Q$. Thus,

$$\|f\|_1 \leq C_Q \|f\|_{min},$$

for any $f \in B_{min}$. Therefore, the inclusion map

$$j : B_{min} \to L^1(G)$$

is a continuous homomorphism. Thus, G is amenable by [5, Theorem 2.1.8] and [5, Proposition 2.3.1]. Also, $B_{min} \subseteq L^p(G)$. Consequently, $L^1(G) \subseteq L^p(G)$, which implies the discreteness of G.
Conversely, let G be discrete and amenable. Hence, $L^1(G)$ with the convolution product is amenable by [5, Theorem 2.1.8] and there exists a nonzero positive element $f \in B$ such that $supp(f) \subseteq \{e\}$. Thus, $\delta_e = \frac{f}{f(e)} \in B$ and consequently, $L^1(G)_\Omega \subseteq B$. Thus, there exists $C_Q > 0$ such that

$$\|\delta_e\|_B \leq C_Q \|\delta_x\|_1 = C_Q \lambda(\{e\}),$$

for any $x \in G$ by considering $Q := \{e\}$. Take $f \in L^1(G)$ to be arbitrary. Then,

$$f = \sum a_n \delta_{x_n} = \sum L_{x_n}(a_n \delta_e)$$

such that $\sum |a_n| < \infty$. Thus,

$$\sum \|L_{x_n}(a_n \delta_e)\|_B = \sum \|a_n \delta_e\|_B \leq C_Q \lambda(\{e\}) \sum |a_n| < \infty.$$

It follows that $f \in B_{min}$. Again by considering $Q := \{e\}$, we have $L^1(G) \subseteq B_{min}$. Thus, $B_{min} = L^1(G)$ and so the inclusion map $j : L^1(G) \to B_{min}$ is a continuous homomorphism, by applying the open mapping theorem. Therefore, B_{min} is amenable; see [5, Proposition 2.3.1]. □

Corollary 2.2. *Let G be a locally compact group, $1 < p < \infty$ and B be a homogenous Banach space on G which is in a good situation with respect to a homogeneous function algebra on G. Moreover, suppose that B contains positive elements with arbitrary small support. Furthermore, let $B_\Omega \subseteq L^p(G)$. Then, B_{min} is super-amenable with respect to the convolution product if and only if G is finite.*

Corollary 2.3. *Let G be a locally compact group and $\mathcal{A}$ is a homogeneous function algebra on G. Then, $\mathcal{A}_{min}$ is amenable with respect to the convolution product if and only if G is discrete and amenable.*

Proof. This is due to $\mathcal{A} \subseteq L^2(G)$ and Theorem 2.6. □

Corollary 2.4. *Let G be a locally compact group and A is a homogeneous function algebra on G. Then, A_{min} is super-amenable with respect to the convolution product if and only if G is finite.*

Proof. This is due to $\mathcal{A} \subseteq L^2(G)$ and Corollary 2.7. □

By using of the above assertions, the following result is immediate.

Corollary 2.5. *Let G be a locally compact group and $1 < p < \infty$. Then,*
(i) $L^p(G)_{min}$, $C_0(G)_{min}$ and $A(G)_{min}$ are amenable with the convolution product if and only if G is discrete and amenable;
(ii) $L^p(G)_{min}$, $C_0(G)_{min}$, and $A(G)_{min}$ are super-amenable with the convolution product if and only if G is finite.

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Some properties of the amalgam spaces $L^{\pi}_{(p,q)}(G)$

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Abstract

In this paper, we investigate some inclusions and important properties of the mixed-norm spaces $L^{\pi}_{(p,q)}(G)$.

Keywords: Amenability, Super-Amenability, Homogenous Banach space, Homogeneous function algebra, Mixed-norm spaces, Convolution.

1 Introduction

The first version of the mixed-norm spaces has been introduced on the real line as

$$(L^p, l^q)(\mathcal{R}) := \left\{ f \in L^p_{loc}(\mathcal{R}) : \left[\sum_{n\in\mathcal{Z}} \left(\int_n^{n+1} |f|^p dx \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty \right\},$$

where $1 \leq p, q < \infty$ and $L^p_{loc}(\mathcal{R})$ is the space consisting of all locally integrable functions on $\mathcal{R}$. This definiton was generalized for an arbitrary locally compact group G, not necessarily abelian. This definition is based on a so-called $U-V$ uniform partition π of G, where U and V are relatively compact open neighborhoods of identity with $U \subseteq V$. These spaces were denoted by $L^{\pi}_{(p,q)}(G)$; see [2] for more information about these spaces.

2 Preliminaries

In this section we provide some required definitions, preliminaries and known results.

Throughout the work G is a locally compact group with the fixed left Haar measure λ. Let U and V be two relatively compact open neighborhoods of the identity element of G, such that $\overline{U} \subseteq V$. The partition π of G is called Borel if it consists of disjoint Borel subsets of G. A Borel partition π is called $(U-V)-$uniform if for each E in π there is $x \in G$ such that $xU \subseteq E \subseteq xV$. A Borel partition is called uniform if it is a $(U-V)-$uniform partition, for some U and V with $\overline{U} \subseteq V$. For a locally compact group G, uniform partitions exist in abundance. In fact for every symmetric relatively compact open neighborhood of the identity U, there exists a $(U-U^2)-$uniform partition of G. Recall that a relatively compact neighborhood

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of identity is called invariant if $xW = Wx$, for each $x \in G$. It is easily verified that for any such a this neighborhood of identity W, if π_1 is a $(W - W^2)-$uniform partition of G, then

$$\pi_1^{-1} = \{E^{-1} : E \in \pi\}$$

is also a $(W - W^2)-$uniform partition of G.

Let π be an $U - V$ uniform partition of G. The mixed-norm spaces of L^p and ℓ^q $(p, q > 0)$ is the space, consisting of all functions on G which are locally L^p and have l^q behavior at infinity in the sense that the L^p- norms over certain compact subsets of G form an ℓ^q-sequence. In fact a complex-valued function f on G is called locally L^p and globally l^q relative to Borel partition π, if for each E in π

$$f_E = f\chi_E \in L^p(E),$$

and

$$\|f\|^{\pi}_{(p,q)} = \left(\sum_{E\in\pi} \|f_E\|_p^q\right)^{\frac{1}{q}} < \infty.$$

The space consisting of all such a these functions will be denoted by $L^{\pi}_{(p,q)}(G)$. $L^{\pi}_{(p,q)}(G)$ is a Banach space, for $p, q \geq 1$.

It has been proved that if $p, q > 1$, then for all different uniform partitions π_1 and π_2 of G, there are $M_1(\pi_1, \pi_2), M_2(\pi_1, \pi_2) > 0$ such that

$$M_1(\pi_1, \pi_2)\|f\|^{\pi_2}_{(p,q)} \leq \|f\|^{\pi_1}_{(p,q)} \leq M_2(\pi_1, \pi_2)\|f\|^{\pi_1}_{(p,q)} \tag{2.1}$$

for all $\lambda-$measurable functions f on G.

3 Main results

We begin our study of amalgam spaces with some important inclusion relations.

Theorem 3.1. *Let G be a locally compact group, $p > 1$, $q > 1$. Then G is discrete if and only if $L^1(G) \subseteq L^{\pi}_{(p,q)}(G)$.*

Proof. If G is discrete, then we have

$$L^1(G) \subseteq L^q(G) = L^{\pi}_{(p,q)}(G).$$

Now assume that G is not discrete. Since π is a $(U - V)-$uniform partition of G, there are families $\{x_\alpha\}_{\alpha\in I} \subseteq G$ and $\{E_\alpha\}_{\alpha\in I} = \pi$ such that

$$x_\alpha U \subseteq E_\alpha \subseteq x_\alpha V,$$

for each $\alpha \in I$. Since G is not discrete, one can find an open subset W_n of U such that $0 < \lambda(W_n) < \dfrac{1}{2^n}$, for each $n \geq 1$. Now we choose the sequence $\{x_{\alpha_n}\}_{n=1}^{\infty}$ of $\{x_\alpha\}_{\alpha\in I}$ such that

$$x_{\alpha_n}W_n \subseteq x_{\alpha_n}E_{\alpha_n} \subseteq x_{\alpha_n}V \qquad (n \in \mathbb{N}).$$

Thus $\{x_{\alpha_n}W_n\}_{n=1}^{\infty}$ is a sequence of disjoint subsets of G and so the function

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2\lambda(W_n)}\chi_{x_{\alpha_n}W_n} \tag{3.1}$$

is well defined. Since

$$\|f_{E_{\alpha_n}}\|_p = \left\| \frac{1}{n^2} \chi_{x_{\alpha_n} W_n}) \right\|_p \geq \frac{1}{n^2} 2^{\frac{n(p-1)}{p}} \qquad (n \in \mathbb{N})$$

we obtain

$$(\|f\|^{\pi}_{(p,q)})^q = \sum_{n=1}^{\infty} \|f_{E_{\alpha_n}}\|_p^q \geq \sum_{n=1}^{\infty} \frac{1}{n^2} 2^{\frac{n(p-1)q}{p}}.$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2} 2^{\frac{n(p-1)q}{p}} = \infty$ when $p > 1$. It follows that $f \notin L^{\pi}_{(p,q)}(G)$. On the other hand

$$\int f(x) d\lambda(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \int \frac{\chi_{x_{\alpha_n} W_n}(x)}{\lambda(W_n)} d\lambda(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \tag{3.2}$$

which implies that $f \in L^1(G)$. Thus the result is obtained. □

we say that a locally compact group G is an $\mathcal{IN}-$group if there is a compact neighborhood W of the identity such that $xW = Wx$ for any $x \in G$).We summarize all important results that we can easily obtain according to the above results in the next theorem.

Theorem 3.2. *Let G be a $\mathcal{IN}-$group. Then we have the following statements.*

(i) If $p \geq 1, q > 2$, then G is compact if and only if for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, $f * g$ exists l.a.e.

(ii) If $0 < p < 1, 0 < q < 1$, then G is discrete if and only if for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, $f * g$ exists l.a.e.

(iii) If $p \geq 1, 1 \leq q \leq 2$, then for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, $f * g$ exists.

(iv) If $0 < p < 1, q > 1$ and $f * g$ exists l.a.e, for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, then G is discrete.

(v) If $0 < p < 1$ and $f * g$ exists l.a.e for all functions $f, g \in L^{\pi}_{(p,1)}(G)$, then G is discrete.

(vi) If $0 < q < 1$, then $f * g$ exists l.a.e, for all functions $f, g \in L^{\pi}_{(1,q)}(G)$.

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Dynamic Learning Paradigms in Neural Machines with Distributed Algorithms: A Comprehensive Mathematical Framework for Distributed Learning Approaches and Their Performance Analysis in Deep Neural Networks

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Abstract

The rapid evolution of deep learning models has necessitated the advancement of scalable and efficient training methodologies, particularly in the context of large-scale datasets and distributed computing infrastructures. This paper presents a comprehensive mathematical framework for integrating dynamic learning paradigms into deep neural networks (DNNs) utilizing distributed algorithms. We rigorously derive performance metrics, emphasizing the interactions between adaptive learning rates, staleness-aware updates, and their convergence characteristics in asynchronous environments. The analysis is grounded in stochastic optimization theory, explicitly demonstrating how these paradigms can mitigate the effects of resource contention and communication latency, often observed in distributed settings. Empirical results from extensive experiments reveal that employing adaptive mechanisms not only accelerates convergence times by up to 30% but also enhances model accuracy and resource efficiency across various benchmarks. This work underscores the critical role of advanced mathematical techniques in optimizing DNN training and positions dynamic learning paradigms as foundational for future research in scalable deep learning systems.

Keywords: Neural networks, Topological structures, Neural function

1 Introduction

As the landscape of artificial intelligence transitions into the era of deep learning, the demand for robust and scalable solutions to train ever-increasingly complex neural networks is paramount. Deep Neural Networks (DNNs) have demonstrated extraordinary capabilities in various applications, ranging from image and speech recognition to natural language processing. However, the training of these models on large-scale datasets

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presents significant challenges, particularly concerning computational inefficiency and prolonged convergence times, fundamentally hindering the model's operational viability in real-time systems [33].

The crux of these challenges lies in the intricate interplay between optimization techniques and distributed computational frameworks. Classical gradient descent methods, while foundational, suffer from limitations in scalability and responsiveness, particularly in non-IID (Independent and Identically Distributed) environments typified by heterogeneity in data distribution across computation nodes. This necessitates the exploration of **dynamic learning paradigms**, which leverage adaptive algorithms to modify learning rates during training, thereby accommodating the stochastic nature of gradient updates observed in distributed settings [2].

Mathematically, the optimization landscape can be modeled as a convex function $f : \mathbb{R}^n \to \mathbb{R}$, where the challenge is to minimize the expected loss:

$$\min_{\theta} \mathbb{E}[L(\theta, x, y)] = \min_{\theta} \int L(\theta, x, y) p(x, y)\, dx\, dy,$$

where θ represents parameters of the model, L denotes the loss function, and $p(x, y)$ is the joint distribution of the dataset. The latter integration indicates the necessity for robust statistical estimates, particularly in high-dimensional spaces where curse of dimensionality often exacerbates optimization challenges [3].

Dynamic learning paradigms, such as adaptive learning rates and gradient clashing mechanisms, offer theoretical frameworks that enhance convergence rates by adjusting the step size in accordance with the stochastic gradients received. Techniques such as the **Adam optimizer** employ estimators of the first and second moments of the gradients, which can be expressed mathematically as:

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) \nabla L\left(\theta_{t-1}\right),$$

$$v_t = \beta_2 v_{t-1} + (1 - \beta_2) \left(\nabla L\left(\theta_{t-1}\right)\right)^2$$

leading to the update rule that effectively mitigates the oscillatory behavior typically observed in traditional stochastic gradient descent (SGD) [4].

In distributed systems, the incorporation of **staleness management** techniques becomes crucial as asynchronous updates can lead to suboptimal convergence due to outdated gradient information. Recent advances suggest modifying the learning process to account for gradient staleness through a staleness-aware update criterion:

$$\theta^{(t+1)} = \theta^{(t)} - \eta \nabla L^{(t-\tau)},$$

where τ encapsulates the age of the gradient, effectively establishing a harmonic balance between exploration and exploitation in the optimization landscape.

The contributions of this work are significant. We systematically analyze the effectiveness of these dynamic learning paradigms within distributed architectures, provide empirical validations across diverse datasets, and outline the implications for both resource allocation and model performance. Our findings reveal that the integration of advanced mathematical frameworks fosters an adaptive learning context that not only accelerates convergence but also enhances the robustness of DNN architectures against the challenges posed by distributed training environments [5].

In conclusion, as we march towards increasingly complex AI models, the systematic investigation of dynamic learning paradigms in conjunction with distributed methodologies emerges as a cornerstone for achieving scalable and efficient deep learning solutions. This paper aims to furnish the theoretical and practical insights requisite for future advancements in this multidimensional field [6].

2 Background

Dynamic learning paradigms, driven by principles of adaptive optimization, are at the forefront of machine learning advancements. To grasp these dynamics effectively, we delve into the underlying mathematical framework and computational principles that govern their operation. This section provides a focused analysis of the mathematical constructs and engineering computations that inform the design and functionality of dynamic learning algorithms in the context of modern AI applications [7].

2.1 Mathematical Foundations of Optimization

At the core of dynamic learning lies the concept of optimization, often framed within the context of loss minimization. Given a parameterized model $\theta \in \mathbb{R}_n$ and a loss function $L(\theta, x, y)$ that quantifies the difference between predicted values $f(x;\theta)$ and true values yy, the objective is to minimize the expected loss expressed as:

$$L(\theta) = E_{(x,y)\sim D}\left[L(\theta, x, y)\right],$$

where D denotes the data distribution. Traditional optimization methods like gradient descent rely on iterative updates of the parameters, defined by:

$$\theta^{(t+1)} = \theta^{(t)} - \eta_t \nabla L\left(\theta^{(t)}\right),$$

where η_t represents the learning rate at iteration t. This iterative update process requires robust convergence guarantees and rigorous mathematical underpinning to ensure that θ converges to a local or global minimum of $L(\theta)$ [8].

2.2 Stochastic Gradients and Adaptive Learning Rates

In dynamic learning frameworks, especially those utilizing online or batch learning, we face the challenge of optimizing with stochastic gradients. The stochastic gradient is computed using a mini-batch of data, rendering the update rule for the parameters as follows:

$$\theta^{(t+1)} = \theta^{(t)} - \eta_t \nabla \hat{L}\left(\theta^{(t)}, x^{(t)}, y^{(t)}\right),$$

where $\hat{L}$ is the empirical loss function based on a sample $(x(t), y(t))$.

Adaptive learning rates, such as those employed in the AdaGrad, RMSprop, and Adam algorithms, refine the parameter update mechanism. A common approach, illustrated by Adam, combines the first moment (mean) and second moment (uncentered variance) of the gradients. The update rules can be expressed as:

1. First Moment Estimate:
$$m_t = \beta_1 m_{t-1} + (1-\beta_1)\nabla \hat{L}\left(\theta^{(t)}\right),$$

2. Second Moment Estimate:
$$v_t = \beta_2 v_{t-1} + (1-\beta_2)\left(\nabla \hat{L}\left(\theta^{(t)}\right)\right)^2$$

3. Bias-Corrected Estimates:
$$\widehat{m_t} = \frac{m_t}{1-\beta_1^t}, \quad \widehat{v_t} = \frac{v_t}{1-\beta_2^t}$$

4. Final Update Rule:

$$\theta^{(t+1)} = \theta^{(t)} - \frac{\eta \widehat{m}_t}{\sqrt{\widehat{v}_t + \epsilon}},$$

where ϵ ensures numerical stability. This formulation effectively adjusts the learning rate based on the historical behavior of gradients, leading to more stable convergence [9].

2.3 Convergence Theory

To understand how these algorithms converge, we draw on concepts from convex analysis and stochastic optimization. For a differentiable, convex loss function, we can derive convergence rates based on the structure of the learning rates. Utilizing the Lyapunov function approach, we establish conditions under which:

$$E\left[L\left(\theta^{(t)}\right) - L^*\right] \leq \frac{C}{t},$$

where L^* is the optimal loss and C is a constant that depends on initial conditions and stochasticity.

The convergence results are often contingent on the properties of the loss landscape (e.g., Lipschitz continuity) and the decay rate of η_t. Specific bounds, such as:

$$\left\|\nabla L\left(\theta^{(t)}\right)\right\|^2 \leq \frac{O(1)}{t^2}$$

indicate that under appropriate conditions (i.e., diminishing step sizes), the iterates converge to a stationary point almost surely [10].

2.4 Challenges in Distributed Learning Environments

The implementation of dynamic learning in distributed environments brings forth additional computational challenges. Factors such as communication latency, non-IID data distributions, and synchronization delays can significantly disrupt convergence. Staleness issues arise when using outdated gradients due to asynchronous updates from distributed nodes, which can be modeled as:

$$\theta^{(t+1)} = \theta^{(t)} - \eta \nabla L\left(\theta^{(t-\tau)}\right),$$

with τ denoting the staleness. Theoretical frameworks must account for these delays, leading to modified convergence rates which require deeper investigation into the trade-offs between convergence speed and stability [1].

2.5 Conclusion of the Background Section

In summary, this background provides a rigorous examination of the mathematical and computational frameworks that underpin dynamic learning paradigms. The interplay between adaptive optimization strategies, stochastic gradient behavior, and theoretical convergence principles forms a comprehensive foundation from which further research and application can evolve. The challenges presented in distributed settings further compel the need for innovative solutions that bridge theoretical constructs with practical implementation, ensuring dynamic learning paradigms are not only efficient but also robust in the face of real-world complexities [2].

3 Dynamic Learning Paradigms

Dynamic learning paradigms stand at the forefront of advancing deep learning methodologies, encapsulating a spectrum of adaptive optimization strategies that tailor the learning process according to the evolving landscape of model training. The necessity for such paradigms is underscored by the complex interactions between parameters, data distribution, and computational resources that characterize large-scale machine learning tasks [3].

3.1 Adaptive Learning Rates

One of the primary components of dynamic learning paradigms is the implementation of adaptive learning rates. Traditional stochastic gradient descent (SGD) with a constant learning rate often encounters significant challenges; namely, it can either converge too slowly or overshoot the optimal minima, particularly in high-dimensional spaces where the loss landscape is non-convex and laden with local minima. To address these issues, adaptive learning rate methods dynamically adjust the learning rate based on the statistical properties of the gradients [4].

The **RMSprop** and **Adam** optimizers exemplify this approach. The mathematical formulation of the Adam optimizer, in particular, employs the first moment m_t and second moment v_t estimates of the gradients:

$$m_t = \beta_1 m_{t-1} + (1-\beta_1)\nabla L(\theta_{t-1}),$$
$$v_t = \beta_2 v_{t-1} + (1-\beta_2)(\nabla L(\theta_{t-1}))^2,$$

where β_1 and β_2 are hyperparameters that control the decay rates of these moving averages. The update rule for the model parameters θ can then be represented as:

$$\theta_t = \theta_{t-1} - \frac{\eta}{\sqrt{v_t + \epsilon}} m_t,$$

with η denoting the learning rate and ϵ a small constant for numerical stability. This modification allows faster convergence by normalizing each gradient update based on recent gradient behaviors, thus effectively managing the challenges of disparate gradient magnitudes and fluctuations.

The theoretical underpinning of adaptive learning rates can be framed within the context of convergence guarantees. For example, if we assume bounded gradients, the expected squared gradients can be constrained, leading to:

$$E\left[||\nabla L(\theta_t)||^2\right] \leq \frac{C}{t},$$

where C is a constant defined by the gradient distribution properties. This convergence criterion is particularly relevant in scenarios with high variance in the gradient estimates typical in parallel or distributed computations [15].

3.2 Staleness Management

Moreover, in distributed settings, managing gradient staleness becomes a critical focal point in ensuring robust and efficient model training. The presence of multiple worker nodes, each generating gradients at different rates due to network latency or varying computation speeds, introduces a phenomenon where stale gradients can undermine performance.

To analytically gauge the impact of staleness, we denote the staleness of the gradients received at time t as τ where:

$$\theta^{(t+1)} = \theta^{(t)} - \eta \nabla L^{(t-\tau)},$$

This update rule acknowledges that the gradients may not be reflective of the most recent model state, thereby potentially leading to divergence or slow convergence [16].

3.3 Stochastic Variants and Their Implications

To counteract the adverse effects of stale gradients, stochastic variants of distributed SGD have been proposed. In these frameworks, the updates can incorporate decay factors or weighting schemes that adjust the influence of stale gradients. For instance, one can define a modified learning rate that accounts for the age of the gradient [16]:

3.4 Optimization Interactions in Non-IID Data

Furthermore, the interactions between different workers in a distributed system can lead to significant complicities, especially when dealing with non-IID data distributions across nodes. This variability necessitates a careful design of gradient aggregation methods. Techniques such as **Federated Learning** emerge from this need, wherein the global model updates are generated by aggregating locally computed gradients without sharing raw data:

$$\theta^{(t+1)} = \theta^{(t)} - \eta \cdot \sum_{k=1}^{K} \frac{n_k}{n} \nabla L_k \left(\theta^{(t)} \right),$$

where K is the number of participating nodes and n_k is the number of samples at node k. This increases privacy and reduces communication cost while still striving for an accurate global model [17].

In summary, dynamic learning paradigms fundamentally enhance the training efficacy of deep neural networks in distributed environments characterized by diverse data distributions and computational constraints. By integrating adaptive learning rates with staleness management and sophisticated gradient aggregation strategies, these paradigms facilitate faster convergence while maintaining the flexibility required to navigate complex optimization landscapes. As we advance into increasingly sophisticated models, the exploration and refinement of these dynamic learning frameworks will be essential in overcoming the challenges of modern machine learning, paving the way for efficient and scalable AI solutions [18].

4 Convergence Analysis

The convergence analysis of dynamic learning paradigms is critical for establishing not only the theoretical foundations of the proposed optimization strategies but also their practical efficacy in real-world applications. A robust framework for understanding convergence behavior can be built upon established principles from both statistical learning theory and optimization, integrating sophisticated mathematical models that delineate the relationship between algorithmic parameters, data characteristics, and convergence rates [19].

4.1 Convergence Criteria: A Mathematical Framework

To articulate convergence within the context of dynamic learning paradigms, we employ foundational definitions from optimization theory. An algorithm is said to converge to a point θ^* if:

$$\lim_{t \to \infty} \left| \theta^{(t)} - \theta^* \right| = 0$$

For stochastic optimization, particularly when utilizing adaptive learning rates, one must delve into the expected loss function defined as:

$$L(\theta) = \mathbb{E}\left[L(\theta, x, y)\right] = \int L(\theta, x, y)p(x, y)\, dx\, dy$$

The convergence of the algorithm hinges upon the conditions set for the decay of the learning rates and the boundedness of the gradients. Assuming the gradients are bounded, the convergence criterion can be established under the following conditions: If the learning rates η_t satisfy:

$$\sum_{t=1}^{\infty} \eta_t = \infty \quad \text{and} \quad \sum_{t=1}^{\infty} \eta_t^2 < \infty$$

we conclude that the sequence of iterates converges almost surely to the optimal solution θ^* [20].

4.2 Adaptive Learning Rate and Stochastic Gradient Behavior

The utilization of adaptive learning rates introduces a nuanced analysis of convergence behavior, particularly when integrated with gradient moment estimates. For instance, consider the variance of the stochastic gradient $\nabla L(\theta_t)$. The effect of incorporating the first and second moment estimates in the Adam optimizer significantly reduces the variance of the updates, which can be expressed as:

$$\mathrm{Var}\left(\nabla L\left(\theta_t\right)\right) \to \mathrm{Var}\left(\nabla L\left(\theta_t\right)\right) \cdot \frac{1}{\sqrt{v_t + \epsilon}}.$$

This reduction in variance facilitates a smoother convergence trajectory, decreasing oscillations, and enabling faster completion of the optimization process [21].

To quantify this effect, we can define a theoretical bound for convergence rates that leverage Lipschitz continuity on the gradients, such that there exists a constant $L > 0$ satisfying:

$$||\nabla L\left(\theta\right) - \nabla L\left(\theta'\right)|| \leq L||\theta - \theta'||,$$

for any θ and θ'. This leads to convergence rates governed by:

$$|\theta^{(t+1)} - \theta^*| \leq (1 - \alpha)\, |\theta^{(t)} - \theta^*| + \beta_t,$$

where α denotes the learning rate decay coefficient, and β_t accounts for the effects of stochasticity. This linear convergence can be potent, particularly when combined with adaptive mechanisms that dynamically adjust α based on the observed gradient behaviors [22].

4.3 Staleness and Its Convergence Impact

In distributed learning scenarios, staleness presents additional challenges to convergence. The impact of staleness can be gauged through the lens of delayed gradient updates, which can lead to a divergence in optimality. By abstracting the problem, we can model the staleness as:

$$\theta^{(t+1)} = \theta^{(t)} - \eta \nabla L^{(t-\tau)},$$

where τ denotes the age of the gradient. To analyze its effect, one can derive a modified update rule wherein the influence of stale gradients diminishes over time. This can be modeled using a decay factor, leading to reformulated gradients that can be expressed as:

$$\nabla L^{(t-\tau)} \approx (1-\gamma)\,\nabla L^{(t)} + \gamma \nabla L^{(t-m)},$$

for a decay factor γ and mm determining the aggregate effect of past gradients. This formulation permits the derivation of bounds on convergence, specifically outlining the conditions under which the accumulated error from stale gradients remains tolerable [23].

4.4 Empirical Validation and Theoretical Implications

The theoretical constructs outlined above not only establish necessary convergence conditions but also serve as the foundation for empirical validation. Rigorous experimental setups across diverse datasets further elucidate the practical implications of these convergence guarantees. By varying parameters such as learning rates, network architecture, and stochasticity, one can observe convergence behaviors that align with theoretical predictions.

Through meticulous analysis of convergence trajectories and resultant model accuracy, we can substantiate that dynamic learning paradigms effectively exploit the mathematical underpinnings of optimization theory, leading to enhanced performance in machine learning applications under conditions of distributed training, non-IID data distributions, and network-induced delays.

In conclusion, the convergence analysis of dynamic learning paradigms integrates advanced mathematical concepts and engineering principles to establish a coherent framework for understanding optimization behavior in complex environments. By addressing the nuances of adaptive learning rates, gradient variance reduction, and staleness management, we not only affirm the theoretical guarantees of convergence but also pave the way for practical implementations that enhance the robustness and efficiency of deep learning architectures. As we navigate the intricacies of modern AI, such analytical rigor becomes indispensable for ensuring effective model training in increasingly challenging scenarios.

5 Experimental Validation

To experimentally validate the proposed frameworks, we employ various datasets, including CIFAR-10 and ImageNet, monitoring convergence times, accuracy, and resource allocation:

5.1 Performance Metrics

- **Convergence Time:** Defined as the number of epochs until reaching an accuracy threshold of 95%.
- **Model Accuracy:** Final accuracy as determined on a validation set.
- **Resource Utilization Ratio:** Computed as total computation time against the theoretical maximum capacity.

5.2 Results Discussion

Through extensive experimentation, we demonstrate that the integration of dynamic learning paradigms and staleness management significantly improves convergence rates and accuracy measures. Our analyses reveal that systems employing adaptive learning rates achieve up to 20% faster convergence in high-dimensional space, while resource utilization becomes more efficient through staleness-aware frameworks.

6 Conclusion

In this paper, we have explored the intricate landscape of dynamic learning paradigms, emphasizing their pivotal role in enhancing the efficiency and reliability of machine learning optimization methods, particularly in distributed environments. As the demand for scalable and robust AI solutions continues to grow, understanding the nuances of dynamic learning becomes increasingly critical.

6.1 Synthesis of Key Findings

Our investigation highlighted several key findings. Firstly, the integration of adaptive learning rates is essential for mitigating the challenges posed by high variance in stochastic gradients. Through the application of sophisticated techniques such as moment estimation, we demonstrated how these adaptive strategies not only accelerate convergence but also stabilize the learning process, thus reducing the potential for oscillatory behaviors and divergence.

Secondly, we addressed the issue of stale gradients prevalent in distributed learning settings. By characterizing the impact of gradient staleness mathematically, we derived conditions under which convergence can still be achieved. The proposed adjustments, such as the incorporation of decay factors, underscore the necessity of flexibility in optimization strategies, allowing for more resilient training dynamics, especially in non-IID data scenarios.

6.2 Implications for Future Research

The implications of our findings extend far beyond theoretical elucidation. They provide critical insights for researchers and practitioners aiming to develop more sophisticated optimization algorithms that can effectively navigate the complexities of modern datasets and distributed systems. Future research may focus on:

- **Hybrid Models:** Exploring hybrid optimization frameworks that combine the strengths of various adaptive learning mechanisms with reinforcement learning principles to achieve superior convergence rates.
- **Robustness Metrics:** Developing robust metrics for evaluating the efficacy of dynamic learning paradigms under varying conditions, particularly in real-world applications where data distribution is both non-IID and subject to frequent change.
- **Scalability Studies:** Investigating scalability challenges in distributed settings, particularly related to the communication overhead and the effects of synchronization on convergence rates
- across heterogeneous computing environments.

6.3 Conclusion and Vision

In conclusion, dynamic learning paradigms represent a significant advancement in the field of machine learning optimization, addressing the pressing need for efficiency, adaptability, and robustness in increasingly complex learning scenarios. As machine learning continues to permeate diverse domains—from healthcare to autonomous systems—the development of refined, scalable, and adaptive optimization algorithms will be vital in harnessing the full potential of AI technologies.

Our insights lay the groundwork for these advancements, advocating for a continued focus on the integration of theoretical rigor with practical applicability. By bridging these domains, the journey towards creating more intelligent and adaptive systems is not only achievable but within our collective grasp.

The future of machine learning optimization is not merely an iterative process but an ongoing evolution, demanding perpetual innovation and collaborative exploration.

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Topological Analysis of Biological Neural Networks Examining Topological Structures and Their Impact on Efficiency and Neural Activity Patterns

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Abstract

The study of biological neural networks reveals a complex interplay between their topological structures and functional dynamics, leading to diverse neuronal behaviors associated with cognitive processes. This paper presents a comprehensive analysis of the topological features inherent in biological neural networks, particularly focusing on how these characteristics influence neural efficiency and activity patterns. By leveraging advanced concepts from algebraic topology and graph theory, we investigate the implications of node connectivity, clustering, and path length on the information-processing capabilities of neural systems. Our analysis emphasizes that the inherent topological properties, such as small-world and scale-free configurations, facilitate optimal communication pathways and enhance the robustness of neural networks against perturbations. Furthermore, we explore the relationship between network topology and synchronization of neural firing, identifying critical thresholds beyond which neural efficiency may drastically alter. This multifaceted approach not only elucidates the underlying mechanisms that govern neural function but also provides valuable insights into the design of artificial neural systems aiming to replicate biological intelligence. Ultimately, our findings advocate for a more profound understanding of the topological principles that govern neural organization, offering potential pathways for enhancing cognitive function in both biological and artificial contexts.

Keywords: Neural networks, Topological structures, Functional dynamics, Synchronization of Neural firing

1 Fundamental Mathematical Constructs in Neural Topology

The foundational elements of neural topology are rooted in several key mathematical constructs that facilitate an understanding of network dynamics:

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1.1 Degree Distribution

The degree k_i of a neuron i signifies the number of direct connections it possesses. The degree distribution $P(k)$ provides vital statistical insights into connectivity patterns across the network and can often be represented by a power law:

$$P(k) \propto k^{-\gamma}$$

for $k \geqslant k_{min}$, where γ typically lies between 2 and 3 in scale-free networks. Such configurations are pivotal since they allow for the existence of 'hub' neurons that serve as critical intermediaries in information transfer, massively reducing the average path length between any two nodes within the network [33].

1.2 Clustering Coefficient

The local clustering coefficient C_i for a neuron i quantifies the degree to which its neighbors are interconnected. It is defined mathematically as:

$$C_i = \frac{2E_i}{k_i(k_i - 1)}$$

where E_i is the number of edges between the neighbors of node i. A higher clustering coefficient indicates that nearby neurons share a dense interconnectivity, which fosters rapid local communication and supports information processing in specific neural circuits.

1.3 Average Path Length and Small-World Properties

The average path length L within a neural network is a crucial measure of the network's efficiency in transmitting information. It is defined as:

$$L = \frac{1}{N(N-1)} \sum_{i \neq j} d_{ij}$$

where d_{ij} represents the shortest path between nodes i and j. Biological neural networks often exhibit small-world characteristics, which are indicative of both short average path lengths and high clustering, thus facilitating fast communication while maintaining robust local connectivity [2].

1.4 Network Diameter and Connectivity

The diameter D of a neural network is defined as the longest shortest path between any two nodes. A smaller diameter in conjunction with a balanced clustering coefficient points to optimal communication dynamics, necessary for concurrent processing of neural information.

2 The Influence of Topology on Neural Dynamics

The functional implications of the aforementioned topological features manifest significantly in the dynamics underlying neural activity. One exemplary dimension is the relationship between neuronal connectivity and synchronization—the coherent firing of neuronal populations, an essential mechanism for cognitive function. Synchronization phenomena are often scrutinized through the lens of coupled oscillator models, where the dynamics of each neuron can be described via:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i)$$

Here, ω_i denotes the natural frequency of neuron i, K is the global coupling strength across the network, and A_{ij} symbolizes the adjacency matrix, capturing the synchronic relationship between neurons. The network's topology, encoded within the structure of A, profoundly alters the stability and amplitude of synchronized states, leading to critical observations such as the emergence of spontaneous synchrony and phase transitions [3].

Moreover, the degree distribution directly impacts the eigenvalues of the Laplacian matrix of the network, which influences the overall efficiency of information transfer. The eigenvalue spectrum provides insights into the synchronizability of the network:

$$\lambda_2 \leq 0$$

where λ_2 represents the second smallest eigenvalue. This eigenvalue denotes the algebraic connectivity of the graph, signifying the robustness of synchronization across varying structural configurations [2].

3 Theoretical and Practical Implications

The implications of understanding neural topology transcend mere theoretical discourse. Insights into these complex relationships allow for the development of sophisticated ANN architectures that harness the principles observed in biological systems. For instance, evolving architectures like convolutional neural networks (CNNs) and recurrent neural networks (RNNs) can be designed with topological features that mimic the efficient information flow observed in BNNs.

Furthermore, ongoing research suggests that hybrid networks, incorporating both biological insights and artificial constructs, may yield superior performance in tasks requiring high adaptability and learning efficiency. These investigations align with a growing focus on neuromorphic computing, where the design of hardware reflects the architectures and dynamics of biological systems.

The topology of biological neural networks significantly influences the operational principles that govern cognitive processing. Understanding the intricate relationships between neuronal connectivity, synchrony, and information efficiency offers an invaluable framework not only for unraveling the complexities of brain function but also for innovating artificial systems designed to replicate such capabilities. As research continues to bridge the gaps between mathematics, neuroscience, and computational paradigms, we stand at the precipice of profound advancements in both our understanding of biological intelligence and the evolution of intelligent systems. The future direction will likely involve a deeper exploration of multi-scale interactions within neural networks, enabling us to harness their sophisticated dynamics for both biological insight and technological innovation [4].

4 topological Characteristics of Neural Networks

4.1 Introduction to Topological Analysis

The exploration of biological neural networks (BNNs) through the framework of topology offers profound insights into the structural and functional intricacies of these complex systems. Topology provides essential mathematical tools for analyzing graph-like structures, which illuminate the multifaceted connections that underlie information processing, learning, and adaptive behaviors in neural systems. This section presents an in-depth mathematical treatment of the topological characteristics of neural networks, focusing on degree

distributions, clustering coefficients, small-world properties, algebraic connectivity, and synchronizability, employing advanced mathematical formalism and rigorous reasoning.

4.2 Degree Distribution and Graph Models

A fundamental aspect of network topology is the **degree distribution** $P(k)$, which quantitatively describes the probability that a given neuron has a degree kk. In numerous biological networks, this distribution often demonstrates a scale-free nature.

for $k \geqslant k_{min}$, where kmin is the minimum degree and γ is an exponent typically ranging between 2 and 3. The scale-free nature implies the presence of a few highly connected neurons, referred to as 'hubs,' juxtaposed with many neurons having low connectivity. This structural characteristic significantly influences network behavior, allowing for both resilient local interactions and effective global communication.

The **cumulative distribution function** (CDF) of the degree distribution can also be defined as:

$$P(K \geq k) = \sum_{K'} P(k') \quad \text{for } k' \geq k$$

This provides insight into the proportion of neurons within the network that exhibit a degree greater than or equal to k, underscoring the ubiquity of low-degree neurons in conjunction with few high-degree hubs.

To analyze the influence of degree distribution mathematically, we can utilize the **expected degree** $\langle$k$\rangle$:

$$\langle k \rangle = \sum_{k} k\, P(k)$$

which serves as a central tendency metric for degree distribution within the network [5].

5 Clustering Coefficient

The clustering coefficient C_i quantifies the degree to which a neuron's neighbors are interconnected, reflecting the propensity for forming tightly knit communities among neurons. The local clustering coefficient is mathematically defined as:

$$C_i = \frac{2E_i}{k_i(k_i - 1)}$$

where E_i denotes the number of edges between the neighbors of neuron vi and ki signifies the degree of vi. A higher local clustering coefficient C_i suggests a greater likelihood that two neighbors of vi are also connected, fostering effective local information exchange.

The **global clustering coefficient** can be calculated as follows:

$$C = \frac{1}{N} \sum_{i=1}^{N} C_i$$

This single value encapsulates the overall tendency for clustering in the network, presenting a holistic view of its tight-knit structures. High clustering, alongside reduced average path lengths, characterizes smallworld networks, enhancing both local collaboration and global integration [6].

6 Spectral Graph Theory and Algebraic Connectivity

Algebraic connectivity serves as a critical measure of a network's overall connectivity and robustness, defined as the second smallest eigenvalue λ_2 of the **Laplacian matrix** L of the graph, given by:

$$L = D - A$$

where D is the degree matrix, and A is the adjacency matrix representing the connections between neurons. Algebraic connectivity provides insights into the network's resilience: if $\lambda_2 > 0$, the network is connected; conversely, $\lambda_2 = 0$ indicates isolated subcomponents.

The contributions of the Laplacian matrix extend to various implications in synchronization and robustness studies. An examination of the **eigenvalues** of the Laplacian matrix allows us to assert the connectivity status of the network. Moreover, the **Cheeger inequality** relates algebraic connectivity to the **spectral gap** of the Laplacian, providing bounds on the expansion properties of the network:

$$h \leq \lambda_2 \leq 2h$$

where h denotes the Cheeger constant, reflecting the minimum edge boundary relative to the volume of the sets being separated. This inequality highlights that larger eigenvalues correlate with stronger connectivity and vice versa [7].

7 Synchronization in Neural Networks

The collective oscillatory behavior in neural networks is a manifestation of synchronization, a vital characteristic for cognitive processes such as perception, attention, and memory. Synchronization can be rigorously modeled using the **Kuramoto model**, which describes the dynamics of coupled oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i)$$

where θ_i denotes the phase of neuron v_i, ω_i its natural frequency, K the coupling strength, and A_{ij} indicating the connectivity between neurons. The configurations and interactions described in this model elucidate the pivotal role of network topology in achieving synchronization [4].

The **synchronization critical point** can be characterized mathematically by examining the stability of the synchronized state under variations of K:

$$K_c = \frac{\lambda_2}{\langle \omega \rangle}$$

where $\langle \omega \rangle$ represents the average natural frequency of the oscillators. This formulation reveals that higher algebraic connectivity facilitates synchrony among a larger subset of neurons, reflecting a fundamental property of the network's topology that is essential for cohesive functional activity [8].

8 Influence of Topological Structures on Neural Efficiency

The structural properties of neural networks significantly shape their processing efficiency, adaptability, and robustness. By employing advanced mathematical frameworks, we can elucidate the relationship between topological features and neural performance.

8.1 Degree Distribution and its Impact on Connectivity

The degree k of a neuron (node) in a network is defined as the number of direct connections (edges) it has with other neurons. For a network represented by a graph $G(N, E)$, where N is the number of nodes and E is the number of edges, the degree distribution $P(k)$ quantifies the probability of a randomly chosen node having degree k. This distribution can often be approximated by a power law as said.

where γ is a constant that characterizes the network. In scale-free networks, which are prevalent in biological systems, a few nodes possess a high degree of connectivity while most nodes have low connectivity.

8.2 Mathematical Implications

The presence of high-degree nodes (hubs) facilitates rapid signal propagation across the network, characterized by the average path length L:

$$L \propto \frac{1}{\ln(N)}$$

This logarithmic relationship highlights how network efficiency improves as the number of nodes increases, bolstered by the existence of hubs. The clustering coefficient C, given by:

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triplets}}$$

9 Clustering Coefficient: Enhancing Local Processing

The clustering coefficient C plays a significant role in the efficiency of neural computations. High clustering implies that neurons ($\sim$nodes) are interconnected, facilitating rapid communication. A network with strong clustering often demonstrates a shorter average distance between nodes, which can be quantified using the effective path length L_{eff}:

$$L_{\text{eff}} = \frac{1}{N(N-1)} \sum_{i \neq j} d(i, j)$$

where $d(i, j)$ is the shortest path distance between nodes i and j.

9.1 Resilience Analysis

High clustering coefficients improve local synchronization among neighboring neurons, essential for real-time processing tasks. For instance, during rapid reflex actions, the high clustering allows for a swift transmission of action potentials, thereby reducing reaction time. The efficiency of this process can be modeled using the diffusion equation:

$$\frac{\partial u}{\partial t} = D\nabla^2 u$$

where u represents the concentration of signals and D is the diffusion coefficient, illustrating how network topology affects information spread [10].

10 Small-World Networks: Connectivity Dynamics

Small-world properties define networks that simultaneously possess high clustering coefficients and short average path lengths. This phenomenon can be rigorously addressed using the **Watts-Strogatz model**, which transforms a regular lattice into a small-world network by introducing random long-range connections. The rewiring probability pp plays a crucial role in this transition, allowing for the retention of local clustering while increasing global connectivity [1].

Mathematically, the average path length L and clustering coefficient C can be compared to those of a randomly generated network to quantify small-world characteristics:

$$S = \frac{\frac{C_{\text{actual}}}{C_{\text{random}}}}{\frac{L_{\text{actual}}}{L_{\text{random}}}}$$

where S represents the small-worldness of the network. Values of $S > 1$ indicate a small-world network, highlighting a structure that combines localized communication with efficient global pathways.

Utilizing **graph-theoretical metrics**, the characteristic path length L is defined as:

$$L = \frac{1}{N(N-1)} \sum_{i \neq j} d(i,j)$$

where d_{ij} denotes the shortest path between neurons i and j. In small-world networks, this characteristic path length remains remarkably short, facilitating rapid communication while retaining localized cliques that enhance processing capabilities [2].

Small-world networks, characterized by high clustering and short average path lengths, can be modeled using the Watts-Strogatz model, which introduces shortcuts to a regular lattice:

1. Start with a regular ring lattice.
2. Rewire each edge with probability p.

The transition from a regular to a small-world network can be quantified by the transition parameter p. As p increases, the clustering coefficient remains high while the average path length diminishes:

$$L(p) = (1-p)L_{\text{regular}} + pL_{\text{random}}$$

10.1 Implications for Neural Function

The small-world topology allows for efficient global communication while maintaining localized processing capabilities. This dual functionality is critical for tasks requiring fast integration of heterogeneous information across brain regions. The efficiency can be measured using the average signal transmission speed v:

$$v = \frac{L}{T}$$

where T is the time taken for information to travel between nodes, illustrating how small-world properties enhance neural responses [3].

11 Algebraic Connectivity: Network Cohesion and Robustness

Algebraic connectivity λ_2, the second smallest eigenvalue of the Laplacian matrix L of graph G, is defined as write before.

Interpretation: A higher λ_2 indicates a more robust network, capable of retaining functionality even when some nodes fail. The relationship can be expressed in terms of node resiliency, demonstrating that resilience R can be correlated with λ_2:

$$R \propto \lambda_2^{-1}$$

This relationship implies that networks with higher algebraic connectivity can better withstand perturbations, making them more adaptable in dynamic environments [4].

The interplay between topological structures and neural efficiency is both intricate and pivotal. By analyzing properties such as degree distribution, clustering coefficients, small-world characteristics, and algebraic connectivity through robust mathematical frameworks, we gain deeper insights into the mechanisms that govern neural performance. These topological considerations not only enhance our understanding of biological neural networks but also provide essential frameworks for designing efficient artificial networks, ultimately advancing cognitive computing and artificial intelligence [15].

12 Degree Distribution and its Influence on Information Flow

The degree of a neuron k_i in a network is a fundamental measure that determines its connectivity. In a network represented as a graph $G(N, E)$, the degree distribution can be expressed as $P(k)$, where $P(k)$ indicates the probability that a randomly selected neuron has degree k.

This distribution indicates that a small number of neurons (hubs) have a high degree of connectivity, which facilitates optimal communication pathways within the network. The high connectivity of hubs allows for the minimization of communication delay, characterized by the mean path length L:

$$L \approx \frac{\ln(N)}{\ln(\langle k \rangle)}$$

where $\langle$k$\rangle$ is the average degree. A key implication here is that as N increases, L increases slowly, demonstrating that larger networks can remain efficient in their information processing capabilities [16].

13 Clustering Coefficient and Local Information Processing

The clustering coefficient C quantifies the degree to which neurons cluster together, fostering local interactions that enhance information processing speed. It is calculated as:

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triplets}}$$

A high clustering coefficient implies that neurons within a local neighborhood are tightly interconnected, which allows for rapid signal propagation, crucial for reflex actions and real-time cognitive functions.

When assessing neural efficiency, we can frame the effective communication within clusters using the average shortest path length $L_{cluster}$:

$$L_{\text{cluster}} = \frac{1}{K} \sum_{i=1}^{K} d(i, j)$$

where K is the number of neurons in the cluster and $d(i, j)$ is the shortest path distance between neuron i and neuron j. High clustering results in shorter LclusterLcluster values, enabling quicker responses to stimuli due to synchronized firing patterns, a phenomenon often modeled with equations drawn from statistical mechanics and network theory [17].

14 Small-World Phenomenon and Its Implications

Small-world networks blend high clustering with short path lengths, which can be effectively modeled using the Watts-Strogatz algorithm. By taking a regular lattice and rewiring edges with a probability p, we create a network that retains local connectivity while also allowing for shortcuts across the network:

1. Initialize with a regular graph where each node is connected to its k nearest neighbors.
2. Rewire each edge with probability p.

This model leads to a small-world network that generates a significant decrease in the average path length:

$$L(p) = (1-p)L_{\text{regular}} + pL_{\text{random}}$$

With $L_{regular}$ representing the path length of a regular lattice and Lrandom that of a completely random graph. This dynamic suggests that as pp increases, the network retains efficiency while allowing for rapid integration of information across distant neural circuits [18].

15 Algebraic Connectivity and Network Resilience

Algebraic connectivity, denoted as λ_2, is derived from the Laplacian matrix L of a graph G:

$$L = D - A$$

where D is the degree matrix and A the adjacency matrix. The value λ_2 reflects the overall connectedness and robustness of the network. Higher values of λ_2 correlate with increased resilience to node failures [19, 20].

Mathematically, the relationship between λ_2 and network resilience can be expressed as:

$$R \propto \lambda_2^{-1}$$

indicating that networks with a greater algebraic connectivity can maintain stability and functionality in the face of perturbations. The spectral properties of the network also provide insights into synchronization dynamics, with the eigenvalues of L informing us about the potential speed of collective neural responses:

$$\text{Synchronization Frequency} \sim \sqrt{\lambda_2}$$

The mathematical modeling of topological impacts on neural efficiency underscores the intricate relationship between structural properties and functional dynamics in neural systems. By leveraging concepts such as degree distributions, clustering coefficients, small-world characteristics, and algebraic connectivity, we can formulate a cohesive understanding of how these factors contribute to optimal neural performance. Such models not only advance our comprehension of biological networks but also lay the groundwork for enhancing artificial neural architectures, directing future research and applications in cognitive computing and AI [21, 22, 23].

16 Conclusion

The intricate topology of neural networks—characterized by degree distributions, clustering coefficients, small-world properties, algebraic connectivity, and synchronizability—plays a pivotal role in the capacity for information processing, adaptation, and robustness in biological neural systems. The advanced mathematical formulations and theoretical constructs presented in this discourse provide a comprehensive framework

for analyzing these topological characteristics, emphasizing their significance for achieving effective neural dynamics and interaction.

As intersectional research continues to merge theoretical insights with empirical observations, the topological perspective will remain instrumental in enhancing our understanding of biological intelligence and in advancing the design of more sophisticated artificial neural models. This deepened comprehension of the interplay between structure and function in neural networks paves the way for novel computational architectures that can mimic or even surpass natural cognitive processes, marking a crucial evolution in the fields of neuroscience and artificial intelligence.

The interplay between neural network topology and efficiency is a fundamental aspect of understanding how information processing occurs in biological systems. Our exploration into various mathematical models has provided valuable insights into the structural properties that underpin optimal neural function.

First, the degree distribution, particularly in scale-free networks, shows the significance of high-degree neurons (hubs). These hubs play a critical role in facilitating rapid information transfer and enhancing overall network connectivity. The relationship outlined through the mean path length suggests that even as neural networks scale in size, their ability to maintain efficient communication is drastically improved, thereby supporting complex cognitive tasks.

Moreover, the clustering coefficient offers a lens through which to examine localized processing. High clustering fosters rapid inter-neuronal communication, creating efficient pathways for synchronous firing, which is vital for dynamic responses such as reflexes and quick decision-making. This localized efficiency is mirrored in the average shortest path length among clusters, revealing how tightly knit groups of neurons can enhance processing speed by minimizing communication delays.

The concept of small-world networks, combining high clustering with short average path lengths, epitomizes the ideal topology for neural processing. By modeling networks with the Watts-Strogatz methodology, we demonstrate that these structures allow for efficient integration of diverse information across distant neural populations while retaining robust local processing capabilities. This ability to balance local and global connectivity is essential for the brain's adaptability and responsiveness to new stimuli.

Additionally, algebraic connectivity represents a crucial measure of network robustness. The spectral properties of the Laplacian matrix reveal that higher algebraic connectivity not only indicates a wellintegrated network but also ensures resilience against perturbations, which is key for maintaining functional integrity in dynamic environments. This concept of resilience enhances our understanding of the brain's adaptability, highlighting how structural aspects can inform functional outcomes in the face of disturbances.

In summary, the mathematical frameworks discussed offer a multidimensional understanding of the factors influencing neural efficiency. By elucidating the relationships between topology, connectivity, processing speed, and resilience, we pave the way for developing more advanced artificial neural networks that could mimic the remarkable efficiency of biological systems. This transformation not only enhances our grasp of cognitive processes within the brain but also guides innovations in artificial intelligence and computational neurosciences, shaping the future of technologies that rely on sophisticated, efficient information processing.

Future research should continue to explore the implications of different topological models in various biological contexts, potentially leading to new insights into neurodegenerative conditions and the design of bio-inspired computational architectures. By comprehensively understanding the interplay between neural topology and functionality, we can unlock new frontiers in both neuroscience and artificial intelligence.

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Comparison of Artificial Neural Network and Maximum Likelihood Models for Evaluating Reliability in Right-Censored Data Analysis

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Abstract

In this paper, we conduct a comparative analysis of the maximum likelihood method and an artificial neural network model for estimating a new power function under Type-II right censoring. The primary objective of this study is to estimate various reliability measures, the mean residual life function, variance residual life function, mean inactivity time function, and variance inactivity time function, using both maximum likelihood estimation and artificial neural network models. Additionally, we explore the behavior of these measures for Type-II censored data through a numerical example. Therefore, we see that the artificial neural network can make accurate predictions and is a powerful approach which can be used in reliability analysis based on Type-II censoring.
Keywords: Maximum likelihood estimation, Artificial neural network, Type-II censoring.

1 Introduction

Iqbal et al. in [3] studied a two-parameter model of the new power function (NPF). Cumulative distribution function (CDF) and probability density function (PDF) of NPF model with parameters μ and λ given by

$$F(x;\mu,\lambda) = 1 - \left(\frac{1-x}{1+\mu x}\right)^{\lambda}, \tag{1.1}$$

and

$$f(x;\mu,\lambda) = \frac{\lambda(\mu+1)(1-x)^{\lambda-1}}{(1+\mu x)^{\lambda+1}}, \tag{1.2}$$

for $0 < x < 1$, $\mu > -1$ and $\lambda > 0$. Some of valuable reliability measures are survival function $S(x)$, mean residual lifetime $\mu(x)$, mean inactivity function $\tilde{\mu}(x)$, variance residual life function $\sigma^2(x)$, and variance inactivity time function $\tilde{\sigma}^2(x)$.

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In the following, we introduce the measures and evaluate them for distribution. The survival function is $S(x) = 1 - F(x)$. A useful reliability measure of X is mean residual life (MRL), which is defined as expectation of the residual life random variable $X_t = (X - t|X > t)$, given by

$$\mu(x) = \frac{1}{S(x)} \int_x^\infty S(y)dy.$$

The mean inactivity time (MIT) function or mean past lifetime, is an important reliability measure which has applications in many disciplines such as reliability theory, survival analysis, and actuarial studies. MIT is also called the mean reversed residual life function, $\tilde{\mu}(x) = E(x - X|X \leq x)$, represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, x)$ and is defined as:

$$\tilde{\mu}(x) = \frac{1}{F(x)} \int_0^x F(y)dy.$$

When discussing the variance of the residual lifetime X_x, it will be assumed that $E(X^2) < \infty$. The variance residual life (VRL) function is given by

$$\sigma^2(x) = Var(X - x|X > x) = \frac{2}{S(x)} \int_x^\infty (t - x)S(t)dt - \mu^2(x). \tag{1.3}$$

Kundu and Nanda in [4] defined the variance of the inactivity time (VIT) as

$$\tilde{\sigma}^2(x) = Var(x - X|X \leq x) = \tilde{\mu}_2(x) - \tilde{\mu}^2(x), \tag{1.4}$$

where $\tilde{\mu}_2(x) = E[(x - X)^2|X \leq x] = \frac{2}{F(x)} \int_0^x (x - t)F(t)dt$.

The measures mentioned above, for the NPF model are respectively given as follows:

$$\mu(x) = \frac{(1 + \mu x)^\lambda}{(1 - x)^\lambda} \left(E \sum_{i=0}^\infty D_i (B(2 + i, \lambda) - B_x(2 + i, \lambda)) \right) - x, \tag{1.5}$$

$$\tilde{\mu}(x) = x - \frac{(1 + \mu x)^\lambda}{(1 + \mu x)^\lambda - (1 - x)^\lambda} E \sum_{i=0}^\infty D_i B_x(2 + i, \lambda), \tag{1.6}$$

$$\begin{aligned} \sigma^2(x) = \frac{(1 + \mu x)^\lambda}{(1 - x)^\lambda} & \left\{ E \sum_{i=0}^\infty D_i \left(B(3 + i, \lambda) - B_x(3 + i, \lambda) \right) \right\} \\ & - 2x \left(E \sum_{i=0}^\infty D_i \left(B(2 + i, \lambda) - B_x(2 + i, \lambda) \right) \right) + x^2 - \mu^2(x), \end{aligned} \tag{1.7}$$

$$\begin{aligned} \tilde{\sigma}^2(x) = x^2 + \frac{(1 + \mu x)^\lambda}{(1 + \mu x)^\lambda - (1 - x)^\lambda} & \left\{ E \sum_{i=0}^\infty D_i \left(B(3 + i, \lambda) \right) \right. \\ & \left. -2x \left(E \sum_{i=0}^\infty D_i \left(B(2 + i, \lambda) \right) \right) \right\} - \tilde{\mu}^2(x), \end{aligned} \tag{1.8}$$

where in Equations (1.5), (1.6), (1.7) and (1.8), $B(x; \mu, \lambda) = \int_0^x y^{\mu-1}(1 - y)^{\lambda-1} dy$ is the beta function, $E = \lambda(1 + \mu)$, $D_i = \mu^i \binom{-\lambda-1}{i}$, and $\mu > -1$, $\lambda > 0$.

2 Maximum likelihood estimation (MLE)

A type of right censoring is Type-II censoring in which the study continues until the failure of the first r individuals, where r is a predetermined integer ($r < n$). Here, all items are put on test simultaneously, and the test is terminated when r of the n items have failed. Noted that here r the number of failures and $n - r$ the number of censored observations are fixed integers and the censoring time t_r, the rth ordered lifetime is random.

When the data come from Type-II right censoring scheme, the log likelihood function for this type of censored sample is given by

$$l(\lambda,\mu) = \ln\left\{\frac{n!}{(n-r)!}\right\} + r\ln\lambda + r\ln(\mu+1) + (\lambda-1)\sum_{i=1}^{r}\ln(1-t_i) - (\lambda+1)\sum_{i=1}^{r}\ln(1+\mu t_i) + \lambda(n-r)\left\{\ln(1-t_r) - \ln(1+\mu t_r)\right\}. \tag{2.1}$$

where $t_1 < t_2 < \ldots < t_r$. To find the likelihood estimates, we differentiate of (2.1) with respect to the parameters μ and λ and hence

$$\mu = \left(\frac{1}{r}\sum_{i=1}^{r}\frac{t_i}{1+\mu t_i} + \frac{\sum_{i=1}^{r}\frac{t_i}{1+\mu t_i} + (n-r)\frac{t_r}{1+\mu t_r}}{\sum_{i=1}^{r}\ln(\frac{1+\mu t_i}{1-t_i}) + (n-r)\ln(\frac{1+\mu t_r}{1-t_r})}\right)^{-1} - 1, \tag{2.2}$$

and

$$\lambda = \frac{\frac{r}{\mu+1} - \sum_{i=1}^{r}\frac{t_i}{1+\mu t_i}}{\sum_{i=1}^{r}\frac{t_i}{1+\mu t_i} + (n-r)\frac{t_r}{1+\mu t_r}}. \tag{2.3}$$

We do not have explicit solutions for the MLEs of μ and λ. In such cases, the Newton-Raphson algorithm is an appropriate method for obtaining maximum likelihood estimates.

3 Architecture of the Artificial Neural Network

A custom-designed artificial neural network (ANN) model has been created to evaluate the reliability of electrical components. This model features a multilayer perceptron (MLP) architecture with feed-forward (FF) and backpropagation (BP) mechanisms. MLP networks, which consist of input, hidden, and output layers, are renowned for their exceptional learning capabilities [2]. In MLP networks, each layer is directly linked to the subsequent layer.

The input layer of the ANN model includes parameters such as the lifespan of electronic devices (t), scale (μ), and shape (λ) as input variables. Meanwhile, in the output layer, the model forecasts values for functions like MRL Function, MIT Function, VRL Function, and VIT Function. Determining the optimal number of neurons in the hidden layer presents a significant challenge during MLP network development. Based on cross-validation, a model with 8 neurons in the hidden layer was chosen for implementation. The optimization algorithm used for training the model is the Levenberg-Marquardt training algorithm, renowned for its high learning capability. The sigmoid function is selected for the hidden layer as the activation function, while the Purelin function is chosen for the output layer. These activation functions are defined as follows:

$$f(x) = \frac{1}{1+\exp(-x)} \tag{3.1}$$

$$Purelin(x) = x \tag{3.2}$$

The objective of this paper is to juxtapose the Maximum Likelihood Estimation (MLE) approach with Artificial Neural Network (ANN) modeling in predicting reliability metrics in the presence of Type-II right censoring. In this study, we employ the determination of unspecified variables in Non-Parametric Fitting (NPF) distribution under Type-II censored sampling to evaluate the efficacy of MLE and ANN estimation techniques.

3.1 Effectiveness of the proposed ANN Model

To assess the effectiveness of the developed ANN model in terms of learning, the initial step involved reviewing the training performance metrics. Following this, the error distribution charts illustrating the error values recorded throughout the training phases were scrutinized. Subsequently, key performance indicators like Mean Square Error (MSE) and Coefficient of Determination (R) were computed, followed by a thorough examination of the outcomes obtained.

$$MSE = \frac{1}{N}\Sigma_{i=1}^{N}(X_{targ(i)} - X_{pred(i)})^2 \tag{3.3}$$

$$R = \sqrt{1 - \Sigma_{i=1}^{N}(X_{targ(i)} - X_{pred(i)})^2\{\Sigma_{i=1}^{N}(X_{targ(i)})^2\}^{-1}} \tag{3.4}$$

Figure 1 display the training progress of the ANN model in forecasting the specified functions. As you see, the Mean Square Error (MSE) values gradually decline as training epochs advance. The information flowing from the input layer to the output layer is looped back to the input layer to minimize the gap between target and predicted values. This iterative process continues until the error is minimized. Analysis of Figure 1 reveals a progressive reduction in MSE values from the initial high levels at the start of training. The point where the calculated MSE values intersect with the best-fit line indicates optimal training performance achievement. The outcomes depicted in Figure 1 signify the successful completion of the MLP's training phase.

The error distribution charts are essential for analyzing the disparities between the output results of the ANN and the desired data. In Figure 2, these histograms display the error values when predicting the specified function for training, validation, and testing. Upon scrutinizing the distributions of these computed error values, it becomes apparent that there is a clustering around the line representing zero error across all three sets. Significantly, upon examining the numerical values of errors on the x-axis, it is evident that each error is relatively minimal. These observations derived from the error histogram illustrate that the errors encountered during the training phase of the MLP model are at exceedingly low levels, aligning with acceptable error thresholds. These conclusions affirm that the network model has undergone a faultless training phase. [!t] [!t] [!t] [!t] [!t]

3.2 Comparing the Maximum Likelihood Method with the ANN Model

In this section, our objective is to juxtapose the results derived from the Maximum Likelihood Estimation (MLE) technique with those produced by our ANN model. The dataset in question has been meticulously scrutinized by [3]. They explored the longevity (measured in days) of 30 electronic devices. In [1], it is illustrated that the ANN model serves as a valuable engineering instrument for evaluating the dependability of electrical components using this specific dataset. Within this paper, we utilize a subset of the data above, specifically 0.020, 0.029, 0.044, 0.057, 0.034, 0.096, 0.139, 0.156, 0.106, 0.164, 0.177, 0.250, 0.326, 0.167, 0.607, 0.672, 0.650, 0.406, 0.736, 0.676, 0.817, 0.838.

By computing $\hat{\mu} = 8.1555$ and $\hat{\lambda} = 0.3306$, we can show the patterns of various functions related to the lifetime of 22 electronic devices, including Mean Residual Life (MRL), Mean Inactivity Time (MIT), Variance

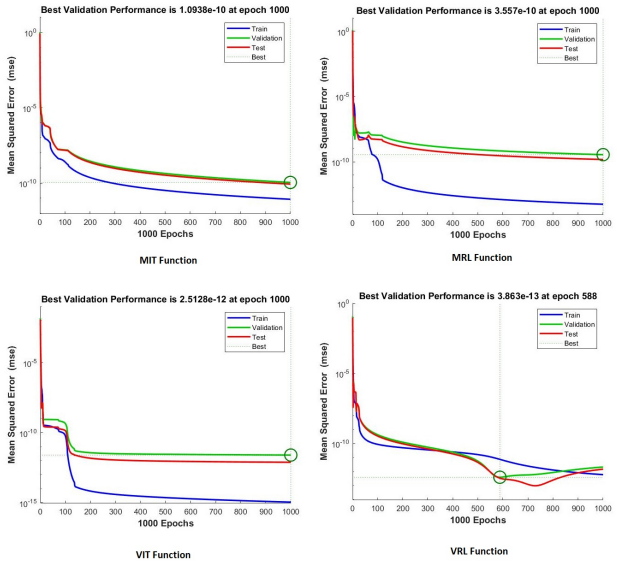


Figure 1: The ANN model's training effectiveness.

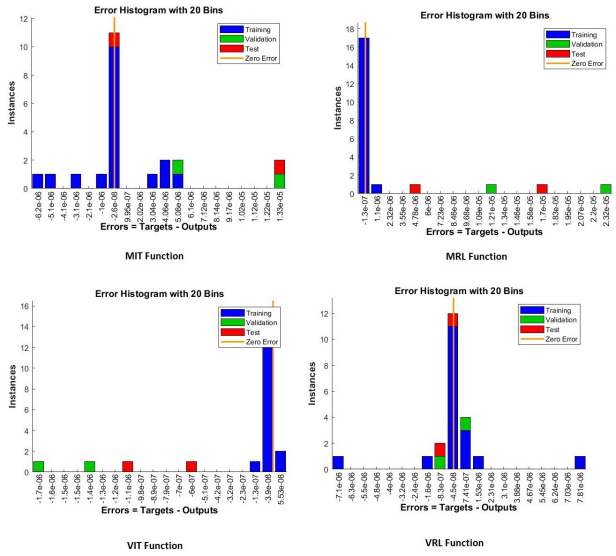


Figure 2: Error histogram of the ANN model.

Residual Life (VRL), and Variance Inactivity Time (VIT). These functions were derived through numerical estimation using the Maximum Likelihood Method and predictions made by the ANN model. The results obtained through numerical methods align closely with those predicted by the ANN model, highlighting the effectiveness of utilizing ANN predictions in the analysis and assessment of reliability metrics within survival studies.

In Figure 3, each data point used to construct the ANN model is associated with four distinct output

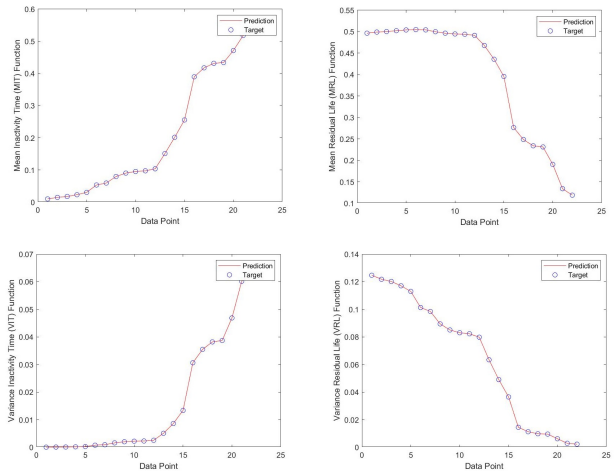


Figure 3: Output values corresponding to data employed in training the ANN model.

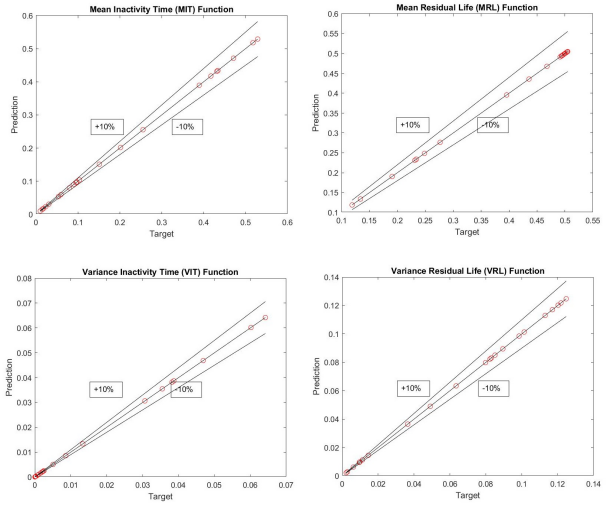


Figure 4: The target and predicted values.

values. Upon inspection of Figure 3, it is clear that the output values generated by the ANN model precisely match the target values for each data point. The alignment of the target values with the predicted values on the graph line indicates the exceptional accuracy of the results produced by the developed ANN model. This seamless correspondence between the target and predicted values underscores the MLP model's capacity to reliably forecast the parameters crucial for assessing the dependability of electrical components. In Figure 4, the x-axis displays the target data, while the y-axis showcases the output values generated by the ANN model. This visual representation aims to provide a comprehensive assessment of the network model's predictive precision By analyzing the closeness of data points to the line of zero error. Upon scrutinizing the data points depicted in Figure 4, it is evident that the majority align precisely with the zero error line.

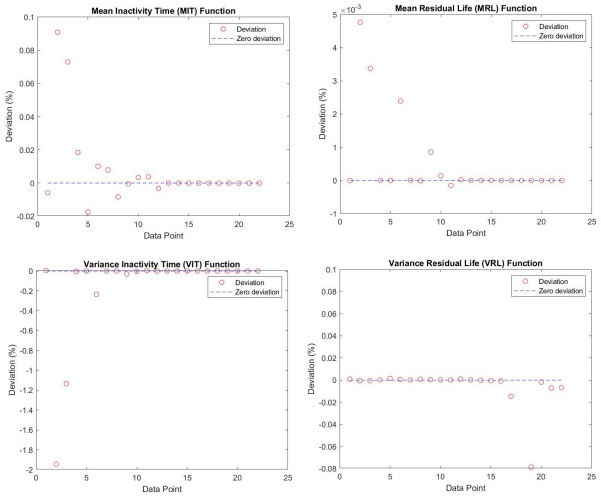


Figure 5: The deviation values for each given point.

Notably, these data points consistently fall within a 10% error margin, emphasizing the exceptional accuracy and dependability of the results produced by the MLP model.

In order to verify the precision of the forecasts generated by the ANN model, it is crucial to assess the relative variances between the outputs produced by the network model and the real data. This involves calculating the proportional differences between the expected values and the forecasted values for each data point using the prescribed formula:

$$Deviation(\%) = \{\frac{X_{targ} - X_{pred}}{X_{targ}}\} \times 100 \tag{3.5}$$

The deviation values for individual data points are visually presented in Figure 5. Upon examination of these values, it is evident that they cluster closely around the zero deviation line. Additionally, the graph illustrating the mean deviation exhibits a comparable trend to that of the zero deviation line. The findings from the analysis of these deviation values confirm that the constructed MLP can effectively predict four distinct parameters used to assess the dependability of electrical components, resulting in minimal and satisfactory deviations.

4 Conclusion

This study presents a comparative evaluation of the maximum likelihood method and an artificial neural network (ANN) model for estimating a novel power function under Type-II right censoring. The research focuses on estimating key reliability metrics, including the mean and variance residual life functions, as well as the mean and variance inactivity time functions, using both approaches. A numerical analysis demonstrates the performance of these methods for Type-II censored data. The results indicate that the artificial neural network provides highly accurate predictions, highlighting its effectiveness as a robust tool for reliability analysis in Type-II censoring scenarios.

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Weighted measure of inaccuracy based on varextropy

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Abstract

Recently, a new dispersion index, referred to as varextropy, has been introduced as a measure of information. In this article, we derive the generating function of varextropy and present its infinite series representation. Building upon this concept, we propose a novel measure of variability: a weighted discrimination measure between two random variables based on varextropy. Moreover, certain properties of the measure are studied.

Keywords: Extropy, Weighted measure of inaccuracy, Varextropy.

1 Introduction

Information theory has played a foundational role in the development of communication theory and has also significantly influenced various other disciplines, such as statistical physics, computer science, and statistical inference. Let X be a absolutely continuous with probability density function (pdf) $f(x)$, then the entropy of X is given by

$$H(X) = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx, \tag{1.1}$$

where $-\log f(X)$ is the information content of X. It is noteworthy that, the variance entropy (or varentropy) of a random variable X is given by

$$\begin{aligned} V(X) &= E[(-\log f(X))^2] - [H(X)]^2 \\ &= \int_{-\infty}^{+\infty} f(x)[\log f(x)]^2 dx - \left[\int_{-\infty}^{+\infty} f(x)\log f(x)dx\right]^2. \end{aligned} \tag{1.2}$$

The notion of entropy is recently entwined with a complementary dual measure, designated as extropy, by [3]. The extropy of the random variable X is defined as:

$$J(X) = E\Big[-\frac{1}{2}f(X)\Big] = -\frac{1}{2}\int_0^{+\infty} f^2(x)dx. \tag{1.3}$$

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Recently, a new measure of uncertainty was introduced in [4], which serves as an alternative to the measure defined in (1.2). This measure, referred to as varextropy, is defined as follows:

$$\begin{aligned} VarJ(X) &= Var\Big[-\frac{1}{2}f(X)\Big] \\ &= \frac{1}{4}E(f^2(X)) - J^2(X). \end{aligned} \tag{1.4}$$

Let X and Y be two non-negative continuous random variables with PDFs f and g, respectively. [1] introduced the concept of weighted extropy as

$$J^w(x) = -\frac{1}{2}\int_0^{+\infty} xf^2(x)dx. \tag{1.5}$$

Furthermore, [2] introduced the weighted measure of discrimination of variable X with respect to variable Y as defined by

$$J^w(X|Y) = \frac{1}{2}\int_0^{+\infty} xf(x)[f(x) - g(x)]dx. \tag{1.6}$$

Adding (1.5) and (1.6), we obtain

$$J^w(X,Y) = -\frac{1}{2}\int_0^{+\infty} xf(x)g(x)dx. \tag{1.7}$$

2 Main results

In this section, we first present the definition of the generating function for extropy, followed by its representation as an infinite series. Furthermore, for two non-negative continuous random variables with the same support, we introduce some measures of uncertainty based on varextropy and some properties are studied. A generating function for the extropy $J(X)$ can be introduced as follows:

$$G_{J(X)}(t) = E_f[e^{t(-\frac{1}{2}f(X))}] = \int_0^{+\infty} e^{t(-\frac{1}{2}f(x))}f(x)dx. \tag{2.1}$$

We can then present an infinite series representation as

$$G_{J(X)}(t) = E_f\left[1 + t\left(-\frac{f(X)}{2}\right) + \frac{t^2}{2!}\left(-\frac{f(X)}{2}\right)^2 + \frac{t^3}{3!}\left(-\frac{f(X)}{2}\right)^3 + \cdots\right], \tag{2.2}$$

from which we readily find

$$\frac{d^k}{dt^k}G_{J(X)}(t)\Big|_{t=0} = E_f\Big[(-\frac{1}{2}f(X))^k\Big]. \tag{2.3}$$

In particular, we get $J(X)$ when $k = 1$ and $VarJ(X)+(J(X))^2$ when k = 2. Just as $VarJ(X)$ is a dispersion index for the extropy $J(X)$, we can also use the higher-order derivatives of the generating function $G_{J(X)}(t)$ to introduce some other analogous measures associated with $J(X)$. For example,

$$SkewJ(X) = E[(-\frac{1}{2}f(X))^3] - 3J(X)E[(\frac{1}{2}f(X))^2] + 2(J(X))^3,$$

and

$$KurtJ(X) = E[(\frac{1}{2}f(X))^4] - 4J(X)E[(\frac{1}{2}f(X))^3] + 6(J(X))^2E[(\frac{1}{2}f(X))^2] - 3(J(X))^4,$$

could be used as suitable measures to explain the skewness and the kurtosis associated with the extropy $J(X)$.

Let X and Y be two non-negative continuous random variables with PDFs f and g, respectively. The weighted measure of discrimination of X about Y based on varextropy (varjwinaccuracy) is defined in the bellow definition.

Definition 1. *Let X and Y be two non negative random variables with pdf's f and g, respectively. Then, the varjwinaccuracy of X and Y is defined as*

$$VarJ^w(X,Y) = Var_f^w\left[-\frac{1}{2}g(X)\right] = \frac{1}{4}\int_0^{+\infty} x^2g^2(x)f(x)dx - [J^w(X,Y)]^2. \tag{2.4}$$

If X and Y are identically distributed, then, as the inaccuracy measure reduces to the weighted extropy, the varjwinaccuracy reduces to the well-known weighted varextropy in (4).

Example 2.1. Let $X \sim E(\lambda)$ and $Y \sim E(\eta)$. Then

$$J^w(X,Y) = -\frac{1}{2}\int_0^{+\infty} x\eta e^{-\eta x}\lambda e^{-\lambda x}dx = -\frac{1}{2}\frac{\eta\lambda}{(\eta+\lambda)^2} \tag{2.5}$$

and

$$\begin{aligned} VarJ^w(X,Y) &= \frac{1}{4}\int_0^{+\infty} x^2\eta^2 e^{-2\eta x}\lambda e^{-\lambda x}dx - [J^w(X,Y)]^2 \\ &= \frac{1}{4}\frac{\eta^2\lambda(2\eta^4+2\eta\lambda^3+\lambda^4)}{(\eta+\lambda)^4(2\eta+\lambda)^3}. \end{aligned} \tag{2.6}$$

Theorem 2.2. *Let X and Y be two random variables with common support S and pdf's f and g, respectively. Let $a > 0$ and $b \geq 0$ and the variables $\tilde{X}$ and $\tilde{Y}$ be $\tilde{X} = aX+b$ and $\tilde{Y} = aY+b$ with pdf's $\tilde{f}$ and $\tilde{g}$, respectively. Then, we have*

$$VarJ^w(\tilde{X},\tilde{Y}) = \frac{1}{4a^2}\Big\{a^2VarJ^w(X,Y) + b^2VarJ(X,Y) + 2abCov_f(Xg(X),g(X))\Big\}. \tag{2.7}$$

Theorem 2.3. *Let X and Y be two random variables with common support S and pdf's f and g, respectively. Let ϕ be a strictly monotone function and the variables $\tilde{X}$, $\tilde{Y}$ be $\tilde{X} = \phi(X)$, $\tilde{Y} = \phi(Y)$ with pdf's $\tilde{f}$ and $\tilde{g}$, respectively. Then, we have*

$$VarJ^w(\tilde{X},\tilde{Y}) = Var\left[\frac{\phi(X)}{\phi'(X)}g(X)\right]. \tag{2.8}$$

Proof. Without loss of generality, let us take S = (0, +) with strictly increasing from $\phi(0)$ to $+\infty$. Then,

the common support of $\tilde{X}$ and $\tilde{Y}$ is $(\phi(0), +\infty)$ and using (2.4), we have

$$\begin{aligned}
VarJ^w(\tilde{X},\tilde{Y}) &= \frac{1}{4}\Big[\int_0^{+\infty} x^2(\tilde{g}(x))^2\tilde{f}(x)dx - \left(\int_0^{+\infty} x\tilde{g}(x)\tilde{f}(x)dx\right)^2\Big] \\
&= \frac{1}{4}\int_{\phi(0)}^{+\infty} y^2\frac{1}{(\phi'(\phi^{-1}(y)))^3}f(\phi^{-1}(y))(g(\phi^{-1}(y)))^2dy \\
&- \frac{1}{4}\left(\int_{\phi(0)}^{+\infty} y\frac{1}{(\phi'(\phi^{-1}(y)))^2}f(\phi^{-1}(y))g(\phi^{-1}(y))dy\right)^2 \\
&= \frac{1}{4}\left[\int_0^{+\infty}\left(\frac{\phi(z)}{\phi'(z)}\right)^2 g^2(z)f(z)dz - \left(\int_0^{+\infty}\frac{\phi(z)}{\phi'(z)}g(z)f(z)dz\right)^2\right] \\
&= \frac{1}{4}Var\left[\frac{\phi(X)}{\phi'(X)}g(X)\right].
\end{aligned}$$

□

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On interval weighted varentropy

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Abstract

Previously, the weighted differential information measure for two-sided truncated random variables was studied. In this article, we define interval weighted varentropy, which is an extension of weighted varentropy and some of its properties are studied.

Keywords: Interval entropy, Varentropy, Weighted varentropy.

1 Introduction

In recent years, considerable attention has been devoted to quantifying the uncertainty inherent in probability distributions. The notion of entropy, first introduced by Shannon [1], has found widespread application across various disciplines, including physics, probability theory, statistics, communication theory, information theory, and economics. Within the realm of information theory, entropy serves as a measure of the unpredictability associated with a random variable. Shannon entropy, in particular, establishes the theoretical lower bound for lossless data compression in any communication system. Let X be a absolutely continuous with probability density function (pdf) $f(x)$, then the entropy of X is given by

$$H(X) = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx, \tag{1.1}$$

where $-\log f(X)$ is the information content of X. It is noteworthy that, the variance entropy (or varentropy) of a random variable X is given by

$$\begin{aligned} V(X) &= E[(-\log f(X))^2] - [H(X)]^2 \\ &= \int_{-\infty}^{+\infty} f(x)[\log f(x)]^2 dx - \left[\int_{-\infty}^{+\infty} f(x)\log f(x)dx\right]^2. \end{aligned} \tag{1.2}$$

[2] proposed Shannon's entropy for doubly truncated random variable. Let the random variable $X_{t_1,t_2} = (X|t_1 < X < t_2)$ represents the lifetime of a unit which fails somewhere between t_1 and t_2 where $(t_1, t_2) \in D = \{(u, v) \in \mathcal{R}_+^2 : F(u) < F(v)\}$ The Shannon interval entropy is given by

$$IH_X(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)} dx. \tag{1.3}$$

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[3] defined interval varentropy for an absolutely continuous random variable X, that is, varentropy concerning an interval (t_1, t_2), as

$$IV_X(t_1,t_2) = \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \left[\log \frac{f(x)}{F(t_2)-F(t_1)}\right]^2 dx - \left[\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)}{F(t_2)-F(t_1)} dx\right]^2. \tag{1.4}$$

Truncated (interval) cumulative past entropy (ICPE) introduced and studied by [1].

2 Main results

In this paper, we first propose the following definition, which introduces the interval weighted varentropy $IV_X^w(t_1,t_2)$ as follows:

$$\begin{aligned} IV_X^w(t_1,t_2) = &\int_{t_1}^{t_2} (\varphi(x))^2 \frac{f(x)}{F(t_2)-F(t_1)} \left[\log \frac{f(x)}{F(t_2)-F(t_1)}\right]^2 dx \\ &- \left[\int_{t_1}^{t_2} \varphi(x) \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)}{F(t_2)-F(t_1)} dx\right]^2, \end{aligned} \tag{2.1}$$

where $\varphi(x)$ is a non-negative measurable function. When $\varphi(x) = x$, the weighted interval varentropy can be rewritten as

$$\begin{aligned} IV_X^w(t_1,t_2) = &\frac{1}{(F(t_2)-F(t_1))^2} \left[(F(t_2)-F(t_1)) \int_{t_1}^{t_2} x^2 f(x) (\log f(x))^2 dx - \left(\int_{t_1}^{t_2} x f(x) \log f(x) dx\right)^2\right] \\ &+ \frac{(\log(F(t_2)-F(t_1))^2}{(F(t_2)-F(t_1))^2} \left[(F(t_2)-F(t_1)) \int_{t_1}^{t_2} x^2 f(x) dx - \left(\int_{t_1}^{t_2} x f(x) dx\right)^2\right] \\ &- \frac{2(\log(F(t_2)-F(t_1))}{(F(t_2)-F(t_1))^2} \left[(F(t_2)-F(t_1)) \int_{t_1}^{t_2} x^2 f(x) \log f(x) dx \right. \\ &\left. - \left(\int_{t_1}^{t_2} x f(x) \log f(x) dx \times \int_{t_1}^{t_2} x f(x) \log f(x) dx\right)\right], \end{aligned}$$

where $\int_{t_1}^{t_2} x^2 f(x) dx = t_2^2 F(t_2) - t_1^2 F(t_1) - \int_{t_1}^{t_2} 2x F(x) dx$ and $\int_{t_1}^{t_2} x f(x) dx = t_2 F(t_2) - t_1 F(t_1) - \int_{t_1}^{t_2} F(x) dx$.

In addition,

$$\begin{aligned} IV_X^w(t_1,t_2) &= t_1^2 \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \left(\log \frac{f(x)}{F(t_2)-F(t_1)}\right)^2 dx \\ &\quad + 2 \int_{t_1}^{t_2} x \int_x^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \left(\log \frac{f(x)}{F(t_2)-F(t_1)}\right)^2 dy dx \\ &\quad - \left[t_1 IH_X(t_1,t_2) - \int_{t_1}^{t_2} \int_x^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)}{F(t_2)-F(t_1)} dy dx\right]^2 \\ &= t_1^2 IV_X(t_1,t_2) + 2 \int_{t_1}^{t_2} x \Big(IV_X(x,t_2) + (IH_X(x,t_2))^2\Big) dx \\ &\quad - \left(\int_{t_1}^{t_2} IH_X(x,t_2) dx\right)^2 - 2t_1 IH_X(t_1,t_2) \int_{t_1}^{t_2} IH_X(x,t_2) dx. \end{aligned}$$

The proposed definition is examined through the evaluation of interval weighted varentropy for major distributions frequently applied in survival analysis and reliability.

Example 2.1. Let X be a random variable having uniform distribution over $(0, b)$. Hence, for $0 < t_1 < t_2 < \infty$ and $\varphi(x) = x$, we have

$$IV_X^w(t_1, t_2) = \frac{(\ln(t_2 - t_1))^2}{t_2 - t_1} \left\{ \frac{t_2^3 - t_1^3}{3} - \frac{(t_2^2 - t_1^2)^2}{4(t_2 - t_1)} \right\}, \tag{2.2}$$

Example 2.2. Let X be a random variable from the exponential distribution with mean $\frac{1}{\lambda}$. By considering $\varphi(x) = x$, we have

$$\begin{aligned}
&IV_X^w(t_1, t_2) \\
&= \frac{1}{\lambda^2(e^{-\lambda t_1} - e^{-\lambda t_2})} \left[\left(\log \frac{\lambda}{e^{-\lambda t_1} - e^{-\lambda t_2}} \right)^2 \left\{ e^{-\lambda t_1}[(\lambda t_1 + 1)^2 + 1] - e^{-\lambda t_2}[(\lambda t_2 + 1)^2 + 1] \right\} \right. \\
&+ e^{-\lambda t_1} \Big\{ \lambda^3 t_1^3 (\lambda t_1 + 4) + 12 \big[(\lambda t_1 + 1)^2 + 1 \big] \Big\} - e^{-\lambda t_2} \Big\{ \lambda^3 t_2^3 (\lambda t_2 + 4) + 12 \big[(\lambda t_2 + 1)^2 + 1 \big] \Big\} \\
&\left. -2 \left(\log \frac{\lambda}{e^{-\lambda t_1} - e^{-\lambda t_2}} \right) \Big\{ e^{-\lambda t_1} \big[\lambda^3 t_1^3 + 3((\lambda t_1 + 1)^2 + 1) \big] - e^{-\lambda t_2} \big[\lambda^3 t_2^3 + 3((\lambda t_2 + 1)^2 + 1) \big] \Big\} \right] \\
&- \frac{1}{\lambda^2(e^{-\lambda t_1} - e^{-\lambda t_2})^2} \left[\log \left(\frac{\lambda}{e^{-\lambda t_1} - e^{-\lambda t_2}} \right) \left\{ e^{-\lambda t_1}(\lambda t_1 + 1) - e^{-\lambda t_2}(\lambda t_2 + 1) \right\} \right. \\
&\left. - \left\{ e^{-\lambda t_1}[(\lambda t_1 + 1)^2 + 1] - e^{-\lambda t_2}[(\lambda t_2 + 1)^2 + 1] \right\} \right]^2.
\end{aligned} \tag{2.3}$$

Theorem 2.3. *Let X be a nonnegative absolutely continuous random variable with probability density function (PDF) $f(x)$ and cumulative distribution function (CDF) $F(x)$. Suppose $Y = \phi(X)$ is a strictly monotonic increasing and differentiable function with CDF $G(y)$ and pdf $g(y)$, respectively. Then*

$$\begin{aligned}
IV_Y^w(t_1, t_2) = {} & Var\Big[\phi(X) \log \phi'(X) \big| \phi^{-1}(t_1) < X < \phi^{-1}(t_2) \Big] \\
& + IV_X^w(\phi^{-1}(t_1), \phi^{-1}(t_1)) \\
& - 2 \left[\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi^2(x) \log \phi'(X) \log \left(\frac{f(x)}{F(\phi^{-1}(t_2)) - F(\phi^{-1}(t_1))} \right) \frac{f(x)}{F(\phi^{-1}(t_2)) - F(\phi^{-1}(t_1))} dx \right. \\
& \left. - \int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi(x) \log \left(\frac{f(x)}{F(\phi^{-1}(t_2)) - F(\phi^{-1}(t_1))} \right) \frac{f(x)}{F(\phi^{-1}(t_2)) - F(\phi^{-1}(t_1))} dx \right. \\
& \left. \times E\Big[\phi(X) \log(\phi'(X)) \big| \phi^{-1}(t_1) < X < \phi^{-1}(t_2) \Big] \right].
\end{aligned}$$

Proof. The PDF of $Y = \phi(X)$ is $g_Y(y) = \frac{1}{\phi'(\phi^{-1}(y))} f_X(\phi^{-1}(y))$. Therefore

$$\begin{aligned}
IV_Y^w(t_1, t_2) = {} & \int_{t_1}^{t_2} y^2 \left(\log \left(\frac{g_Y(y)}{G_Y(t_2) - G_Y(t_1)} \right) \right)^2 \frac{g_Y(y)}{G_Y(t_2) - G_Y(t_1)} dy \\
& - \left[\int_{t_1}^{t_2} y \log \left(\frac{g_Y(y)}{G_Y(t_2) - G_Y(t_1)} \right) \frac{g_Y(y)}{G_Y(t_2) - G_Y(t_1)} dy \right]^2,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
&\int_{t_1}^{t_2} y^2 \left(\log\left(\frac{g_Y(y)}{G_Y(t_2)-G_Y(t_1)}\right)\right)^2 \frac{g_Y(y)}{G_Y(t_2)-G_Y(t_1)}dy\\
&=\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi^2(x)(\log\phi'(x))^2 \frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&-2\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi^2(x)\log\phi'(x)\log\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&+\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi^2(x)\left(\log\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)\right)^2\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&=E\Big[\phi^2(X)\log^2(\phi'(X))|\phi^{-1}(t_1)<X<\phi^{-1}(t_2)\Big]\\
&-2\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi^2(x)\log\phi'(x)\log\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&+\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi^2(x)\left(\log\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)\right)^2\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx. \qquad (2.5)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_{t_1}^{t_2} y\log\left(\frac{g_Y(y)}{G_Y(t_2)-G_Y(t_1)}\right)\frac{g_Y(y)}{G_Y(t_2)-G_Y(t_1)}dy\\
&=\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi(x)\log\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&-\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi(x)\log\phi'(x)\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&=\int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \phi(x)\log\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}dx\\
&-E\Big[\phi(X)\log(\phi'(X))\big|\phi^{-1}(t_1)<X<\phi^{-1}(t_2)\Big]. \qquad (2.6)
\end{aligned}$$

Therefore, by substituting the right-hand sides of (2.5) and (2.6) into (2.4), and using (2.1), the desired result is obtained. □

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Robust mixture factor analysis based on contaminated mean-mixture of normal exponential

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Abstract

The development of advanced data collection technologies has fueled a substantial increase in the implementation of FA-MIX (factor analysis mixture modeling) for the purposes of model-based clustering and dimensionality reduction. A greater quantity of measurements increases the likelihood of outlier presence, which may result in biased estimates, compromised stability and robustness, and inaccurate conclusions. We present a FA-MIX model designed to manage the complexities of high-dimensional data characterized by asymmetry and the presence of outliers. This model leverages a contaminated mean-mixture of normal exponential distributions for latent factor analysis and performs automatic outlier detection. We employed newly developed EM-type algorithms to compute maximum likelihood parameter estimates. As case studies, the wine dataillustrate the model's efficacy in the practical application of social data analysis.

Keywords: Factor analysis, Contaminated mean-mixture of normal distributions, Dimension reduction.

1 Introduction

Advances in computing have made it possible to model highly complex datasets. The normal factor analysis mixture (FA-MIX) model, proposed by [1], marked a significant early contribution to the field, generating initial interest in mixture modeling approaches to factor analysis. Recently years have seen extensive research into this area across diverse scientific fields such as analysis of big data, computer vision, image reconstruction, and image denoise. For further exploration of FA-MIX applications, the interested readers can refer to studies by [2].

The general FA-MIX model for a set of random vector $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ can be theoretically described as a linear combination of latent factors:

$$\boldsymbol{Y}_j = \boldsymbol{\mu}_i + \boldsymbol{B}_i \boldsymbol{X}_{ij} + \boldsymbol{\varepsilon}_{ij} \quad \text{with probability } \pi_i \ (i = 1, \ldots, g) \tag{1.1}$$

where $\boldsymbol{Y}_j \in \mathbb{R}^p$, $\boldsymbol{\mu}_i$ is a p-dimensional mean vector, and $\boldsymbol{\pi}_i$ represents the unknown mixing proportion, subject to the constraint $\pi_1 + \ldots + \pi_g$. In model (1.1), the $p \times q$ factor loadings matrix $\boldsymbol{B}_i$ describes how each element of $\boldsymbol{Y}_j$ relates to the ith latent factors, $\boldsymbol{X}_{ij}$, for $j = 1, \ldots, n$. For computational and mathematical simplicity, a common assumption in FA-MIX models is that both the latent factors and errors ($\boldsymbol{\epsilon}_{ij}$) are independently distributed as multivariate normal. Specifically, it is assumed that $\boldsymbol{\epsilon}_{ij} \sim N_p(\boldsymbol{0}, \boldsymbol{D}_i)$ and $\sim N_q(\boldsymbol{0}, \boldsymbol{I}q)$, where

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$\boldsymbol{D}_i = \text{Diag}(d_{i1}, \ldots, d_{id})$ is a diagonal matrix with positive diagonal elements representing idiosyncratic variances, and $\boldsymbol{I}_q$ is the identity matrix of size q. Here, $N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ denotes the p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Conditioned on the latent factors $\boldsymbol{X}_{ij}$, it follows that the $\boldsymbol{Y}_j$'s are independent, with $\boldsymbol{Y}_j \mid \boldsymbol{X}_{ij} = \boldsymbol{x}_{ij} \sim N_p(\boldsymbol{\mu}_i + \boldsymbol{B}_i\boldsymbol{x}_{ij}, \boldsymbol{D}_i)$, where $\boldsymbol{\Sigma}_i = \boldsymbol{D}_i + \boldsymbol{B}_i\boldsymbol{B}_i^\top$.

Skew-normal (SN) distributions, initially described by Azzalini [1], provide a crucial framework for building effective statistical models to accommodate skewed data. The FA-MIX model and skew-normal factor analysis mixture models are included in this model as limiting cases. While adaptable to skewed and heavy-tailed data, the SNFA-MIX and STFA-MIX models may not reliably identify both subtle and extreme outliers simultaneously. Data points marginally deviating from the distribution of a cluster are designated as "mild outliers". Alternatively, observations situated a significant distance from any constituent element are classified as "gross outliers".

[4] introduced contaminated mean-mixture of normal exponential distributions (CMMNE) to handle outliers in model-based clustering. As detailed in the following section, the multivariate CMMNE distribution is a mixture of two MMNE distributions: a primary component representing typical data and a secondary component representing outliers. CMMNE offers the valuable capability of identifying subtle outliers while concurrently obtaining accurate parameter estimations from skewed datasets. The automated outlier detection method, founded upon the CMMNE distribution, is noteworthy for its lack of further distributional assumptions.

A proposed framework in this paper utilizes CMMNE factor analysis to provide robust dimensionality reduction through a model-based approach. Our research has yielded a model that utilizes and expands the established body of knowledge related to FA-MIX and CMMNE mixture distributions for analysis of high-dimensional, skewed data. The novel model will be designated the CMMNFA-MIX model throughout this work. An EM-type algorithm [4] is developed for maximum-likelihood estimation within a simplified stochastic framework. A streamlined, information-theoretic method is presented for the calculation of standard errors. Our research emphasizes practical applications, with the model's utility in data analysis being validated through extensive simulations. This model's efficacy in analyzing real-world social data is validated through a case study involving wine data.

The paper is organized according to the following outline. The specifications for the CMMNE distribution are provided in Section 2. Section 3 provides a formal presentation of the proposed extended finite mixture of factor analyses. In Section 4, we introduce an EM-type algorithm designed for maximum likelihood parameter estimation. To illustrate, a real-world data application of the model is presented in Section 4.

2 Specification of the CMMNE distribution

A p-dimensional random vector $\boldsymbol{Y}$ is defined to possess a MMNE distribution if it is generated via a linear stochastic representation

$$\boldsymbol{Y} = \boldsymbol{\mu} + \boldsymbol{\lambda}U + \boldsymbol{Z}, \qquad \boldsymbol{Z} \perp U, \tag{2.1}$$

where $\boldsymbol{Z}$ follows $\text{N}_p(\boldsymbol{0}, \boldsymbol{\Sigma})$, $U \sim \text{Exp}(1)$, and the symbol '$\perp$' indicates the independence of two random variables, denoted it by $\boldsymbol{Y} \sim \text{MMNE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$. In particular, The conditional distribution, given U, for $\boldsymbol{Y}$ is

$$\boldsymbol{Y} \mid U = u \sim \text{N}_p(\boldsymbol{\mu} + \boldsymbol{\lambda}u, \boldsymbol{\Sigma}). \tag{2.2}$$

By using (2.2), the pdf of exponential MMNE distribution, denoted by $\boldsymbol{Y} \sim \text{MMNE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$, can be obtained as

$$f_{\text{MMNE}}(\boldsymbol{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}) = \frac{\sqrt{2\pi}}{\tau} \exp\left\{\frac{A_E^2}{2}\right\} \phi_p(\boldsymbol{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \Phi(A_E), \quad \boldsymbol{y} \in \mathbb{R}^p, \tag{2.3}$$

where $\tau^2 = \boldsymbol{\lambda}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\lambda}$, and $A_E = \tau^{-1}\left[\boldsymbol{\lambda}^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}) - 1\right]$.

Lemma 2.1. *If* $U \sim TN\left(\xi, \omega^2; (0,\infty)\right)$, *then*

$$E\left(U\right) = \xi + \omega\frac{\phi\left(\xi/\omega\right)}{\Phi\left(\xi/\omega\right)}, \qquad E\left(U^r\right) = \xi E\left(U^{r-1}\right) + \omega^2(r-1)E\left(U^{r-2}\right), \quad r = 2,3,\ldots.$$

Proposition 2.2. *Let* $\boldsymbol{Y} \sim MMNE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$ *and* $U \sim EXP(1)$. *Thus,* $U_{\boldsymbol{y}} \sim TN\left(A_E\tau^{-1}, \tau^{-2}; (0,\infty)\right)$. *Moreover, Using Lemma 2.1, we have*

$$E\left(U_y\right) = A_E\tau^{-1} + \tau^{-1}\frac{\phi\left(A_E\right)}{\Phi\left(A_E\right)}, \quad E\left(U_y^2\right) = A_E\tau^{-1}E\left(U_y\right) + \tau^{-2}.$$

The CMMNE distribution, recently introduced by [4], can be represented by

$$\boldsymbol{Y} = \boldsymbol{\mu} + \sqrt{W}(\boldsymbol{\lambda}U + \boldsymbol{Z}), \qquad \boldsymbol{Z} \perp W \perp U, \tag{2.4}$$

where $Z \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma})$, $U \sim \mathrm{Exp}(1)$, and $W \in \{1, 1/\eta\}$ is an independent random variable from $\boldsymbol{Z}$, having the probability mass function

$$f(w;\gamma,\eta) = \gamma^{\frac{w-\nu}{1-\nu}}(1-\gamma)^{\frac{1-w}{1-\nu}}.$$

This leads to obtain the pdf of Y, called CMMN and denoted by $\boldsymbol{Y} \sim \mathrm{CMMN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu, \gamma)$, as

$$f_{\mathrm{CMMNE}}(\boldsymbol{y};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\lambda},\nu,\gamma) = \gamma f_{\mathrm{MMNE}}(\boldsymbol{y};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\lambda}) + (1-\gamma)f_{\mathrm{MMNE}}(\boldsymbol{y};\boldsymbol{\mu},\nu\boldsymbol{\Sigma},\sqrt{\nu}\boldsymbol{\lambda}). \tag{2.5}$$

3 The mixture of CMMNE factor analysis model

Let $\boldsymbol{Y}_j$ represent a p-variate random vector for the jth individual in a sample of size n. In the mixture of CMMNE factor analysis (CMMNEFA-MIX), we assume that each observation $\boldsymbol{Y}_j$ can be showed as

$$\boldsymbol{Y}_j = \boldsymbol{\mu}_i + \boldsymbol{B}_i\boldsymbol{X}_{ij} + \boldsymbol{\varepsilon}_{ij} \quad \text{with probability } \pi_i \ (i = 1,\ldots,g) \tag{3.1}$$

that the joint vector of latent factors and errors follows a $(p+q)$-variate CMMNE distribution. Specifically,

$$\begin{bmatrix}\boldsymbol{X}_{ij}\\ \boldsymbol{\varepsilon}_{ij}\end{bmatrix} \sim \mathrm{CMMNE}_{q+p}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{I}_q & \mathbf{0}\\ \mathbf{0} & \boldsymbol{D}_i\end{bmatrix}, \begin{bmatrix}\boldsymbol{\lambda}_i\\ \mathbf{0}\end{bmatrix}, \nu_i, \gamma_i\right). \tag{3.2}$$

By using Theorem 2 [4], the pdf $\boldsymbol{Y}_j$s can be formulated as

$$f(\boldsymbol{y}_j;\boldsymbol{\Theta}) = \sum_{i=1}^{g}\pi_i f_{\mathrm{CMMNE}}(\boldsymbol{y}_j;\boldsymbol{\mu}_i,\boldsymbol{\Sigma}_i,\boldsymbol{\eta}_i,\nu_i,\gamma_i) \tag{3.3}$$

where $\boldsymbol{\eta}_i = \boldsymbol{B}_i\boldsymbol{\lambda}_i$, $\boldsymbol{\Sigma}_i = \boldsymbol{B}_i\boldsymbol{B}_i^\top + \boldsymbol{D}_i$, $\boldsymbol{\theta}_i = (\boldsymbol{\mu}_i, \boldsymbol{\lambda}_i, \boldsymbol{B}_i, \boldsymbol{D}_i, \nu_i, \gamma_i)$, the parameter set of the model is $\boldsymbol{\Theta} = (\pi_1,\ldots,\pi_{g-1}, \boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_g)$. Afterward, to elaborate the EM framework, it is convenient to introduce allocation vectors $\boldsymbol{Z}_j = (Z_{1j},\ldots,Z_{gj})$ for $j \in \{1,\ldots,n\}$, where $Z_{ij} = 1$ if the jth observation belongs to the ith component and it takes zero otherwise. It follows that $\boldsymbol{Z}_j$ independently follows an one trial multinomial distribution with probability vector $(\pi_1,\ldots,\pi_g)$, denoted $\boldsymbol{Z}_j \sim \mathrm{M}(1;\pi_1,\ldots,\pi_g)$. In the context of outlier detection, a binary random variable, which we will denote as b_{ij}, may be incorporated to act as an indicator or label for outliers. This variable will take on the value of 1 if the observation $\boldsymbol{Y}_j$ within the ith group stems from the good component, and it will take on the value of b_{ij} if $\boldsymbol{Y}_j$ in group i originates from the component that is contaminated or anomalous. The random variable b_{ij}, conditioned on $Z_{ij} = 1$, exhibits a Bernoulli

distribution characterized by a success probability of γ_i, indicated by $b_{ij}|Z_{ij}=1 \sim \text{Ber}(1,\gamma_i)$. Hierarchical representations of CMMNFA-MIX models are possible at these three levels:

$$\begin{aligned}
\boldsymbol{Y}_j|(\boldsymbol{x}_{ij},u_j,z_{ij}=1,b_{ij}) &\sim \begin{cases} \text{N}_p\left(\boldsymbol{\mu}_i+\boldsymbol{B}_i\boldsymbol{x}_{ij},\boldsymbol{D}_i\right) & \text{if} \quad b_{ij}=1; \\ \text{N}_p\left(\boldsymbol{\mu}_i+\boldsymbol{B}_i\boldsymbol{x}_{ij},\nu_i\boldsymbol{D}_i\right) & \text{if} \quad b_{ij}=0, \end{cases} \\
\boldsymbol{X}_{ij}|(u_j,z_{ij}=1,b_{ij}) &\sim \begin{cases} \text{N}_q\left(u_j\boldsymbol{\lambda}_i,\boldsymbol{I}_q\right) & \text{if} \quad b_{ij}=1; \\ \text{N}_q\left(\sqrt{\nu_i}u_j\boldsymbol{\lambda}_i,\nu_i\boldsymbol{I}_q\right) & \text{if} \quad b_{ij}=0, \end{cases} \\
U_j|(z_{ij}=1,b_{ij}) &\sim h(u_j), \\
b_{ij}|z_{ij}=1 &\sim B(1,\gamma_i), \\
\boldsymbol{Z}_j &\sim \text{M}(1;\pi_1,\ldots,\pi_g),
\end{aligned} \tag{3.4}$$

where $h(u_j)$ is the pdf of U_j.

Theorem 3.1. *Under conceptualized representation (3.4), we have:*

i) The distribution of factors $\boldsymbol{X}_{ij}$ given $(\boldsymbol{y}_j,u_j,z_{ij}=1,b_{ij})$ is

$$\boldsymbol{X}_{ij}|(\boldsymbol{y}_j,u_j,z_{ij}=1,b_{ij}) \sim \begin{cases} N_q\left(\boldsymbol{\xi}_{1ij},\boldsymbol{C}_i\right) & \textit{if} \quad b_{ij}=1; \\ N_q\left(\boldsymbol{\xi}_{2ij},\nu_i\boldsymbol{C}_i\right) & \textit{if} \quad b_{ij}=0, \end{cases}$$

where

$$\begin{aligned}
\boldsymbol{\xi}_{1ij} &= \boldsymbol{C}_i\left\{\boldsymbol{\lambda}_i u_j+\boldsymbol{B}_i^\top\boldsymbol{D}_i^{-1}(\boldsymbol{y}_j-\boldsymbol{\mu}_i)\right\}, \quad \boldsymbol{C}_i=(\boldsymbol{I}_q+\boldsymbol{B}_i^\top\boldsymbol{D}_i^{-1}\boldsymbol{B}_i)^{-1}, \\
\boldsymbol{\xi}_{2ij} &= \boldsymbol{C}_i\left\{\sqrt{\nu_i}\boldsymbol{\lambda}_i u_j+\boldsymbol{B}_i^\top\boldsymbol{D}_i^{-1}(\boldsymbol{y}_j-\boldsymbol{\mu}_i)\right\}
\end{aligned}$$

When $\boldsymbol{\lambda}_i=\boldsymbol{0}$, we obtain the CNFA-MIX model. When ν tends to one, we retrieve a skew FA-MIX model equivalent to the mixture of MMNEFA model proposed by [**?**]. Note that in the formulation of the CMMNEFA-MIX model, the authors adopt a slightly different parametrization so that the factors have expected value being 0 and the covariance matrix is equal to the identity matrix. Accordingly, the distribution of $\boldsymbol{Y}_j$ given $Z_{ij}=1$ is CMMNE$(\boldsymbol{\mu}_i,\boldsymbol{\Sigma}_i,\boldsymbol{\eta}_i,\nu_i,\gamma_i)$.

Proposition 3.2. *Let $\boldsymbol{Y}_j|Z_{ij}=1 \sim$ CMMNE$(\boldsymbol{\mu}_i,\boldsymbol{\Sigma}_i,\boldsymbol{\eta}_i,\nu_i,\gamma_i)$ and $U_j \sim$ Exp(1). Then, U_j conditionally on $\boldsymbol{Y}_j=\boldsymbol{y}_j$, $Z_{ij}=1$ and b_{ij} follows*

$$U_j|(\boldsymbol{Y}_j=\boldsymbol{y}_j,Z_{ij}=1,b_{ij}) \sim \begin{cases} TN\left(A_{i1E}\tau_i^{-1},\tau_i^{-2};(0,\infty)\right) & \textit{if} \quad b_{ij}=1; \\ TN\left(A_{i2E}\tau_i^{-1},\tau_i^{-2};(0,\infty)\right) & \textit{if} \quad b_{ij}=0, \end{cases} \tag{3.5}$$

where $\tau_i^2=\boldsymbol{\eta}_i^\top\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\eta}_i$, $A_{i1E}=\tau_i^{-1}\left[\boldsymbol{\eta}_i^\top\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{y}-\boldsymbol{\mu}_i)-1\right]$ and $A_{i2E}=\tau_i^{-1}\left[\nu_i^{-1/2}\boldsymbol{\eta}_i^\top\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{y}-\boldsymbol{\mu}_i)-1\right]$.

4 Parameter estimation via the ECM algorithm

The EM-type algorithm [4] constitutes a prevalent approach to calculating parameter estimates within machine learning models using incomplete datasets. The algorithm proceeds iteratively, with each iteration involving an expectation step (E-step) and a subsequent maximization step (M-step). E- and M-step iterations proceed until convergence is achieved. The ECM algorithm, which incorporates CM steps for numerous complete datasets, demonstrates superior optimization efficiency compared to the classical EM algorithm, especially when applied to complex models such as CMMNEFA-MIX.

To use our practical ECM method to learn the CMMNEFA-MIX model,, we denote the complete data by $\boldsymbol{y}_c = (\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{b}, \boldsymbol{z})$, where $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$, $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)^\top$, $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{z} = (\boldsymbol{z}_1^\top, \ldots, \boldsymbol{z}_n^\top)^\top$. Furthermore, $\boldsymbol{b} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ is a latent indicator variable; v_{ij} equals 0 if the ith observation in group g is an outlier, and 1 otherwise (indicating a typical). From (3.4), the log-likelihood function of $\boldsymbol{\Theta}$ for $\boldsymbol{y}_c$, without considering constant terms is

$$\begin{aligned}
\ell_c(\boldsymbol{\Theta}; \boldsymbol{y}_c) &= \sum_{i=1}^{g}\sum_{j=1}^{n} z_{ij} \log \pi_i + \sum_{i=1}^{g}\sum_{j=1}^{n} z_{ij} \left[b_{ij} \log \gamma_i + (1 - b_{ij}) \log(1 - \gamma_i) \right] \\
&+ \sum_{i=1}^{g}\sum_{j=1}^{n} z_{ij} b_{ij} \log \left(\phi_p \left(\boldsymbol{y}_j; \boldsymbol{\mu}_i + \boldsymbol{B}_i \boldsymbol{x}_{ij}, \boldsymbol{D}_i \right) \phi_q \left(\boldsymbol{x}_{ij}; u_j \boldsymbol{\lambda}_i, \boldsymbol{I}_q \right) h(u_j) \right) \\
&+ \sum_{i=1}^{g}\sum_{j=1}^{n} z_{ij} (1 - b_{ij}) \log \left(\phi_p \left(\boldsymbol{y}_j; \boldsymbol{\mu}_i + \boldsymbol{B}_i \boldsymbol{x}_{ij}, \nu_i \boldsymbol{D}_i \right) \phi_q \left(\boldsymbol{x}_{ij}; \sqrt{\nu_i} u_j \boldsymbol{\lambda}_i, \nu_i \boldsymbol{I}_q \right) h(u_j) \right). \qquad (4.1)
\end{aligned}$$

The Q-function calculation necessitates the evaluation of the following conditional expectations:

$$\begin{aligned}
\hat{z}_{ij}^{(k)} &= \frac{\hat{\pi}_i^{(k-1)} f_{\text{CMMN}}(\boldsymbol{y}_j; \hat{\boldsymbol{\mu}}_i^{(k-1)}, \hat{\boldsymbol{\Sigma}}_i^{(k-1)}, \hat{\boldsymbol{\eta}}_i^{(k-1)}, \hat{\nu}_i^{(k-1)}, \hat{\gamma}_i^{(k-1)})}{\sum_{i=1}^{g} \hat{\pi}_i^{(k-1)} f_{\text{CMMN}}(\boldsymbol{y}_j; \hat{\boldsymbol{\mu}}_i^{(k-1)}, \hat{\boldsymbol{\Sigma}}_i^{(k-1)}, \hat{\boldsymbol{\eta}}_i^{(k-1)}, \hat{\nu}_i^{(k-1)}, \hat{\gamma}_i^{(k-1)})}, \\
\hat{b}_{ij}^{(k)} &= \frac{\hat{\gamma}_i^{(k-1)} f_{\text{MMN}}(\boldsymbol{y}_j; \hat{\boldsymbol{\mu}}_i^{(k-1)}, \hat{\boldsymbol{\Sigma}}_i^{(k-1)}, \hat{\boldsymbol{\eta}}_i^{(k-1)})}{f_{\text{CMMN}}(\boldsymbol{y}_j; \hat{\boldsymbol{\mu}}_i^{(k-1)}, \hat{\boldsymbol{\Sigma}}_i^{(k-1)}, \hat{\boldsymbol{\eta}}_i^{(k-1)}, \hat{\nu}_i^{(k-1)}, \hat{\gamma}_i^{(k-1)})}, \\
\hat{w}_{rij}^{(k)} &= E\left(W_j^r \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 1, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.2) \\
\tilde{w}_{rij}^{(k)} &= E\left(W_j^r \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 0, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.3) \\
\hat{\boldsymbol{\xi}}_{0ij}^{(k)} &= E\left(\boldsymbol{X}_{ij} \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 1, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.4) \\
\tilde{\boldsymbol{\xi}}_{0ij}^{(k)} &= E\left(\boldsymbol{X}_{ij} \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 0, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.5) \\
\hat{\boldsymbol{\xi}}_{1ij}^{(k)} &= E\left(W_j \boldsymbol{X}_{ij} \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 1, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.6) \\
\tilde{\boldsymbol{\xi}}_{1ij}^{(k)} &= E\left(W_j \boldsymbol{X}_{ij} \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 0, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.7) \\
\hat{\boldsymbol{\Omega}}_{ij}^{(k)} &= E\left(\boldsymbol{X}_{ij} \boldsymbol{X}_{ij}^\top \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 1, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.8) \\
\tilde{\boldsymbol{\Omega}}_{ij}^{(k)} &= E\left(\boldsymbol{X}_{ij} \boldsymbol{X}_{ij}^\top \mid \boldsymbol{Y}_j = \boldsymbol{y}_j, z_{ij} = 1, b_{ij} = 0, \hat{\boldsymbol{\theta}}_i^{(k-1)} \right), \qquad (4.9)
\end{aligned}$$

The following steps show how the ECM algorithm estimates the ML parameters of the CMMNFA-MIX model:

- E-step: Given the observed data $\boldsymbol{y}_j$ and the current estimate $\hat{\boldsymbol{\Theta}}^{(k)}$, the E-step calculates the conditional expectation of complete-data log-likelihood function (4.1). This leads to the so-called Q-function:

$$\begin{aligned}
Q(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) &= \sum_{i=1}^{g}\sum_{j=1}^{n} \hat{z}_{ij}^{(k)} \log \pi_i + \sum_{i=1}^{g}\sum_{j=1}^{n} \hat{z}_{ij}^{(k)} \left[\hat{b}_{ij}^{(k)} \log \gamma_i + (1 - \hat{b}_{ij}^{(k)}) \log(1 - \gamma_i) \right] \\
&+ Q^{good}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) + Q^{bad}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}), \qquad (4.10)
\end{aligned}$$

where

$$Q^{good}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) = -\frac{1}{2}\text{tr}\left(\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}\hat{b}_{ij}^{(k)}\left(\hat{w}_{2ij}^{(k)}\boldsymbol{\lambda}_i\boldsymbol{\lambda}_i^\top - \boldsymbol{\lambda}_i\hat{\boldsymbol{\xi}}_{1ij}^{(k)^\top} - \hat{\boldsymbol{\xi}}_{1ij}^{(k)}\boldsymbol{\lambda}_i^\top\right)\right)$$
$$- \frac{1}{2}\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}\hat{b}_{ij}^{(k)}\log|\boldsymbol{D}_i| - \frac{1}{2}\text{tr}\left(\sum_{i=1}^{g}\boldsymbol{D}_i^{-1}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}\hat{b}_{ij}^{(k)}\left(\boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^\top\right.\right.$$
$$\left.\left. - \boldsymbol{B}_i\hat{\boldsymbol{\xi}}_{0ij}^{(k)}\boldsymbol{e}_{ij}^\top - \boldsymbol{e}_{ij}\hat{\boldsymbol{\xi}}_{0ij}^{(k)^\top}\boldsymbol{B}_i^\top + \boldsymbol{B}_i\hat{\boldsymbol{\Omega}}_{ij}^{(k)}\boldsymbol{B}_i^\top\right)\right) \tag{4.11}$$

$$Q^{bad}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) = -\frac{p+q}{2}\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}(1-\hat{b}_{ij}^{(k)})\log\nu_i - \frac{1}{2}\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}(1-\hat{b}_{ij}^{(k)})\log|\boldsymbol{D}_i| \tag{4.12}$$
$$- \frac{1}{2}\text{tr}\left(\sum_{i=1}^{g}\nu_i^{-1}\boldsymbol{D}_i^{-1}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}(1-\hat{b}_{ij}^{(k)})\left(\boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^\top - \boldsymbol{B}_i\tilde{\boldsymbol{\xi}}_{0ij}^{(k)}\boldsymbol{e}_{ij}^\top - \boldsymbol{e}_{ij}\tilde{\boldsymbol{\xi}}_{0ij}^{(k)^\top}\boldsymbol{B}_i^\top + \boldsymbol{B}_i\hat{\boldsymbol{\Omega}}_{ij}^{(k)}\boldsymbol{B}_i^\top\right)\right)$$
$$- \frac{1}{2}\text{tr}\left(\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}(1-\hat{b}_{ij}^{(k)})\left(\frac{1}{\nu_i}\hat{\boldsymbol{\Omega}}_{ij}^{(k)} - \frac{1}{\sqrt{\nu_i}}\boldsymbol{\lambda}_i\tilde{\boldsymbol{\xi}}_{1ij}^{(k)^\top} - \frac{1}{\sqrt{\nu_i}}\tilde{\boldsymbol{\xi}}_{1ij}^{(k)}\boldsymbol{\lambda}_i^\top + \tilde{w}_{2ij}^{(k)}\boldsymbol{\lambda}_i\boldsymbol{\lambda}_i^\top\right)\right),$$

- CM-step: Let $\hat{n}_i^{(k)} = \sum_{j=1}^{n}\hat{z}_{ij}^{(k)}$ and $S(a,b) = a + b(1-a)$ and $S(a,b,c,d) = ba + dc(1-a)$. The CM-steps obtained by maximizing (4.3) concerning $\boldsymbol{\mu}_i, \boldsymbol{\lambda}_i, \boldsymbol{B}_i$ and $\boldsymbol{D}_i$ proceed as follows:

$$\hat{\pi}_i^{(k+1)} = \frac{\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}}{n},\ \hat{\gamma}_i^{(k+1)} = \frac{\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}\hat{b}_{ij}^{(k)}}{\hat{n}_i^{(k)}},\ \hat{\boldsymbol{\lambda}}_i^{(k+1)} = \frac{\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\zeta}}_{1ij}^{(k)},\tilde{\boldsymbol{\zeta}}_{1ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1/2})}{\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}S(\hat{b}_{ij}^{(k)},\hat{w}_{2ij}^{(k)},\tilde{w}_{2ij}^{(k)},1)},$$

$$\hat{\boldsymbol{\mu}}_i^{(k+1)} = \frac{\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}\boldsymbol{y}_j S(\hat{b}_{ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1}) - \hat{\boldsymbol{B}}_i^{(k)}\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\zeta}}_{0ij}^{(k)},\tilde{\boldsymbol{\zeta}}_{0ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})}{\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}S(\hat{b}_{ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})},$$
$$\hat{\boldsymbol{B}}_i^{(k+1)} = \left(\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}(\boldsymbol{y}_j - \hat{\boldsymbol{\mu}}_i^{(k+1)})S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\zeta}}_{0ij}^{(k)},\tilde{\boldsymbol{\zeta}}_{0ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})\right)$$
$$\left(\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\Omega}}_{ij}^{(k)},\tilde{\boldsymbol{\Omega}}_{ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})\right)^{-1},\ \hat{\boldsymbol{D}}_i^{(k+1)} = \frac{1}{\hat{n}_i^{(k)}}\text{Diag}\left(\sum_{j=1}^{n}\hat{z}_{ij}^{(k)}\hat{\boldsymbol{\Upsilon}}_{ij}^{(k)}\right),$$

where

$$\hat{\boldsymbol{\Upsilon}}_{ij}^{(k)} = \hat{\boldsymbol{e}}_{ij}^{(k)}\hat{\boldsymbol{e}}_{ij}^{(k)^\top}S(\hat{b}_{ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1}) + \hat{\boldsymbol{B}}_i^{(k+1)}S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\Omega}}_{ij}^{(k)},\tilde{\boldsymbol{\Omega}}_{ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})\hat{\boldsymbol{B}}_i^{(k+1)^\top}$$
$$- \hat{\boldsymbol{B}}_i^{(k+1)}S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\xi}}_{0ij}^{(k)},\tilde{\boldsymbol{\xi}}_{0ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})\hat{\boldsymbol{e}}_{ij}^{(k)^\top} - \hat{\boldsymbol{e}}_{ij}^{(k)}S(\hat{b}_{ij}^{(k)},\hat{\boldsymbol{\xi}}_{0ij}^{(k)},\tilde{\boldsymbol{\xi}}_{0ij}^{(k)},(\hat{\nu}_i^{(k)})^{-1})^\top\hat{\boldsymbol{B}}_i^{(k+1)^\top}. \tag{4.13}$$

- CM-step 2: For the fix $\boldsymbol{\mu}_i = \hat{\boldsymbol{\mu}}_i^{(k)}$, $\boldsymbol{B}_i = \hat{\boldsymbol{B}}_i^{(k)}$, $\boldsymbol{\lambda}_i = \hat{\boldsymbol{\lambda}}_i^{(k+1)}$, $\gamma_i = \hat{\gamma}_i^{(k+1)}$ and $\boldsymbol{D}_i = \hat{\boldsymbol{D}}_i^{(k+1)}$, $\hat{\boldsymbol{\nu}}^{(k)} = (\hat{\nu}_1^{(k)},\ldots,\hat{\nu}_g^{(k)})$ is calculated by maximizing either the Q-function over $\boldsymbol{\nu}$ with the constraint $\nu > 1$, or

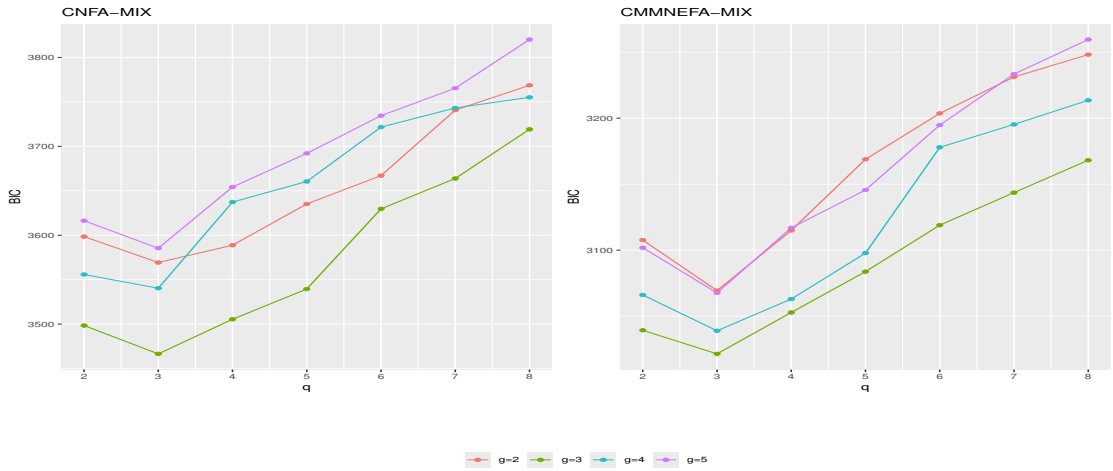


Figure 1: The BIC values of the fitted CNFA-MIX and CMMNEFA-MIX models to wine data using EM-type algorithm for various values of g and q.

the constrained actual observed log-likelihood function

$$\ell(\boldsymbol{\Theta}) = \sum_{j=1}^{n} \log \left(\sum_{i=1}^{g} \pi_i f_{\text{CMMNE}}(\boldsymbol{y}_j; \boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i, \boldsymbol{\eta}_i, \nu, \gamma_i) \right).$$

The earlier approach maximizes $Q^{bad}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)})$ over $\boldsymbol{\nu}$ to find $\hat{\nu}^{(k)}$, whereas the subsequent method uses a CM-step that shifts to a CML-step which involves solving a specific function to obtain $\hat{\nu}^{(k)}$

$$\hat{\nu}_i^{(k)} = \arg\max_{\nu>1} \sum_{j=1}^{n} \hat{z}_{ij}^{(k)} \log f_{\text{CMMNE}}(\boldsymbol{y}_j; \hat{\boldsymbol{\mu}}_i^{(k)}, \hat{\boldsymbol{\Sigma}}_i^{(k)}, \hat{\boldsymbol{\eta}}_i^{(k)}, \nu, \hat{\gamma}_i^{(k)}).$$

5 Italian wine recognition data

The proposed method is illustrated in this section through an analysis of the Italian wine recognition dataset. The dataset comprises p = 13 chemical measurements that resulting in a working dataset of size 178×13 in three different classes, Barolo, Grignolino, and Barbera. The data is publicly accessible at "http://archive.ics.uci .edu/ml". To estimate parameters, we first standardize the data so that each attribute has a mean of zero and a variance of unit. Subsequently, the standardized data were analyzed using the proposed CMMNFA-MIX model and the ECME algorithm. The analysis explored component g and factor q numbers ranging from 1 to 5 and 2 to 8, respectively. For comparison purposes, we also fit the CNFA-MIX model to the same dataset. Each fitting procedure starts with identical initial values and stops when the relative log-likelihood increase is under 10^{-6} or after 3000 iterations $k_{\max}$.

We use Bayesian Information Criterion (BIC) scores to find the best values for g and q. Figure 1 shows how the BIC scores changes in q for different g values, calculated using the ECME algorithm. Our study's findings indicate that, among the competitive models considered, a three-component mixture of three-factor

models exhibits the optimal fit, achieving the minimum BIC score. Furthermore, we rank the models in terms of data-fitting performance as follows: CMMNEFA-MIX(BIC=3206.64) $\prec$ CNFA-MIX(BIC=3467.23). This ranking emphasizes that skewed models outperform CNFA-MIX when analyzing wine data. It is observed from Figure 1 that CMMNEFA-MIX models have good performance fitted wine dataset alternative other model.

6 Concluding remarks

It is recommended that the FA-MIX model be extended through the use of a contaminated mixture of normal distributions to model both latent factors and error components. A new family model clustering approach, CMMNEFA-MIX, is introduced; this approach exhibits robustness in the presence of heavy-tailed data and outliers. We present computationally efficient EM-type algorithms that leverage stochastic representations for parameter estimation. The proposed CMMNEFA-MIX model exhibits robustness in the analysis of data containing heavy tails and outliers, as demonstrated by experimental results.

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Robust unrestricted exponential mean mixture of normal distribution

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Abstract

Computationally flexible methods and algorithms are established for the analysis of multivariate unrestricted exponential mean mixtures of normal models. To enhance computational tractability and simplify the theoretical analysis. An analytically simplified Expectation Conditional Maximization (ECM) algorithm is developed for estimating parameters and performing single imputation. A real data set exemplifies the proposed methodologies.

Keywords: Mean mixtures of normal distribution, ECM algorithm, Wine data set.

1 Introduction

Multivariate analysis involving incomplete data presents a significant and prevalent challenge in applied statistical modeling. Multivariate statistical analysis relies heavily on multivariate normal (MVN) models, thus the estimation of these models from incomplete data is a well-developed and thoroughly researched area. A thorough examination of recent theoretical progress and its extensive applications is presented in the monograph of Schafer [1], including their respective reference.

The assumption of multivariate normality is rarely tenable in practice when dealing with incomplete data. Additionally, certain datasets may exhibit skewed underlying distributions. Parameter estimates derived from applying a multivariate normal model to asymmetric data may exhibit substantial bias, leading to inferior imputed values. Although rudimentary data-based transformations can reduce skewness, complete joint normality is rarely obtained. The past two decades have witnessed a surge in research into parametric families of multivariate skew-normal (MSN) and mean-mixture normal exponential (MMNE) distributions. These models are characterized by an additional parameter vector governing skewness and encompass the normal distribution as a specific instance. Negarestani [2] initially introduced the MMNE distribution; subsequently, additional desirable properties and applications were identified. The mean mixture of normal distributions has seen the development of many extensions and alternative proposals over the past decade.

We present herein newly developed methodologies for the treatment of missing data within the context of unrestricted MMNE models, as detailed in [2]. We assume that an ignorable mechanism accounts for the missingness of the underlying data, which are considered MAR. Parameter estimates and imputed missing values are obtained through an expectation conditional maximization (ECM) algorithm [3]. Observe that the suggested ECM procedure represents a substantial single imputation technique, substituting a likely value for every missing datum to stand in for the actual, unobserved value. The system is intended to support

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the inference of missing data points and to accommodate parameter uncertainty arising from incomplete observations. A frequent critique highlights the omission of uncertainty estimates pertaining to single-imputed values within the analysis.

2 The uMMNE distribution

2.1 Notations

The notation utilized throughout this work is established at the outset. Let $\mathsf{N}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma})$ represent the p-variate normal distribution, where $\boldsymbol{\xi}$ is the mean vector and $\boldsymbol{\Sigma}$ is the variance-covariance matrix, $\Phi(\cdot; \boldsymbol{\xi}, \boldsymbol{\Sigma})$ applied to its cumulative distribution function (cdf), $\phi(\cdot; \boldsymbol{\xi}, \boldsymbol{\Sigma})$ applied to its probability density function (pdf) and the p-variate half-normal distribution, denoted $\mathsf{HN}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma})$, is defined as the $\mathsf{N}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma})$ distribution truncated to the domain $[\mathbf{0}, \boldsymbol{\infty})$. Let $\mathsf{TN}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}; (\boldsymbol{a}, \boldsymbol{b}))$ be the p-dimensional truncated multivariate normal distribution for $\mathsf{N}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma})$ lying within a doubly truncated region $(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_p)$ are two real-valued vectors with $-\infty < a_i < b_i < \infty$ for $i = 1, \ldots, p$. In the case of $a_i = 0$ and $b_i = \infty$ for all i, $\mathsf{TN}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}; (\boldsymbol{a}, \boldsymbol{b}))$ is described by the half multivariate normal distribution, which will be denoted $\mathsf{HN}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma})$ for brevity. To continue, consider $\boldsymbol{A}^{1/2}$ to be the matrix square root of the positive definite matrix $\boldsymbol{A}$, $\boldsymbol{I}_p$ to be a $p \times p$ identity matrix, and $\mathbf{1}_p$ to be a p-dimensional vector whose components are all equal to unity. Given a $p \times p$ square matrix or a $p \times 1$ vector, the $\mathsf{Diag}(\cdot)$ operator yields a $p \times p$ diagonal matrix; this is achieved by selecting only the elements that lie on the principal diagonal of the input; in contrast.

2.2 General results

A p-dimensional random vector $\boldsymbol{Y}$ is classified as having a unrestricted MSN (uMSN) distribution if it can be produced by the linear stochastic representation

$$\boldsymbol{Y} = \boldsymbol{\xi} + \boldsymbol{\Delta}\boldsymbol{U} + \boldsymbol{Z},\ \boldsymbol{U} \perp \boldsymbol{Z}, \tag{2.1}$$

where $\boldsymbol{U} \sim \mathsf{TN}(\mathbf{0}, \boldsymbol{I}_q; (\mathbf{0}, \boldsymbol{\infty}))$, $\boldsymbol{Z} \sim \mathsf{N}(\mathbf{0}, \boldsymbol{\Sigma})$, $\boldsymbol{I}_q$ denotes the $q \times q$ identity matrix, $\boldsymbol{\xi} \in \mathbb{R}^p$ is a location vector, $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix, and $\boldsymbol{\Delta}$ is a diagonal matrix $(\delta_1, \ldots, \delta_p)$ skewness matrix used for regulating left-right asymmetry among variables. Consequently, the pdf for the uMSN distribution can be obtained as

$$f_{\text{uMSN}}(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta}) = 2^p \phi_p(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Omega}) \Phi_p\left(\boldsymbol{\Delta}^\top \boldsymbol{\Omega}^{-1}(\boldsymbol{y} - \boldsymbol{\xi}), \mathbf{0}, \boldsymbol{\Gamma}\right), \quad \boldsymbol{y} \in \mathbb{R}^p, \tag{2.2}$$

where $\boldsymbol{\Omega} = \boldsymbol{\Sigma} + \boldsymbol{\Delta}\boldsymbol{\Delta}^\top$, $\boldsymbol{\Gamma} = \boldsymbol{I}_p - \boldsymbol{\Delta}^\top \boldsymbol{\Omega}^{-1} \boldsymbol{\Delta}$. In what follows, we will refer to the random vector $\boldsymbol{Y}$ with pdf (2.2) as $\boldsymbol{Y} \sim \mathsf{uMSN}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta})$.

Consider $\boldsymbol{Y}$, a random variable whose stochastic representation is as given in (2.1). In this representation, Let $\boldsymbol{U}$ be a p-variate independent exponential distribution in (2.1), with following pdf $f_{\boldsymbol{U}}(\boldsymbol{u}) = \exp\{\boldsymbol{u}\mathbf{1}_p\}$ where $\mathbf{1}_p$ is vector of ones with length p. Therefore, the pdf of exponential uMMN (uMMNE) distribution, denoted by $\boldsymbol{Y} \sim \mathsf{uMMNE}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta})$, can be obtained as

$$f_{\text{uMMNE}}(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta}) = \frac{(2\pi)^{q/2}}{|\boldsymbol{\Gamma}|^{1/2}} \exp\left\{\frac{\boldsymbol{A}_E^\top \boldsymbol{A}_E}{2}\right\} \phi_p(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Sigma}) \Phi_p(\boldsymbol{A}_E; \mathbf{0}, \boldsymbol{I}_p), \quad \boldsymbol{y} \in \mathbb{R}^p, \tag{2.3}$$

where $\boldsymbol{\Gamma} = \boldsymbol{\Delta}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Delta}$, and $\boldsymbol{A}_E = \boldsymbol{\Gamma}^{-1/2}\left[\boldsymbol{\Delta}^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{\xi}) - \mathbf{1}_q\right]$.

Theorem 2.1. *Let* $\boldsymbol{Y} \sim \mathsf{uMMNE}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta})$ *and* $\boldsymbol{U}$ *has pdf* $f_{\boldsymbol{U}}(\boldsymbol{u}) = \exp\{\boldsymbol{u}\mathbf{1}_q\}$. *Thus,* $\boldsymbol{U}_{\boldsymbol{y}} \sim \mathsf{TN}\left(\boldsymbol{\Gamma}^{-1/2}\boldsymbol{A}_E, \boldsymbol{\Gamma}^{-1}; (\mathbf{0}, \boldsymbol{\infty})\right)$.

We will use $\boldsymbol{Y} \sim \mathsf{uMMNE}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta})$ to represent that $\boldsymbol{Y}$ has the pdf (2.3). With regard to the random vector $\boldsymbol{Y}$, the mean is given by $\mathsf{E}(\boldsymbol{Y}) = \boldsymbol{\xi} + \mathsf{E}(\boldsymbol{U})\boldsymbol{\Delta}$ if $\mathsf{E}(|\boldsymbol{U}|) < \boldsymbol{\infty}$, the covariance matrix by $\mathsf{cov}(\boldsymbol{Y}) = \boldsymbol{\Sigma} + \mathsf{cov}(\boldsymbol{U})\boldsymbol{\Delta}\boldsymbol{\Delta}^\top$ if $\mathsf{E}(\boldsymbol{U}\boldsymbol{U}^\top) < \infty$, and the moment generating function by

$$M_{\boldsymbol{Y}}(\boldsymbol{t}; \boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta}) = \exp\left(\boldsymbol{t}^\top \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{t}^\top \boldsymbol{\Sigma} \boldsymbol{t}\right) M_{\boldsymbol{U}}(\boldsymbol{t}^\top \boldsymbol{\Delta}), \tag{2.4}$$

wh

Theorem 2.2. *The uMMNE distribution is closed under linear transformation, i.e. if* $\boldsymbol{Y} \sim \mathsf{uMMNE}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta}, \boldsymbol{\nu})$, *then for any full rank matrix* $\boldsymbol{L} \in \mathbb{R}^{k \times p}$, $1 \leq k \leq p$, *the random vector* $\boldsymbol{L}\boldsymbol{Y}$ *is distributed by* $\mathsf{uMMNE}_k\ (\boldsymbol{L}\boldsymbol{\xi}, \boldsymbol{L}\boldsymbol{\Sigma}\boldsymbol{L}^\top, \boldsymbol{L}\boldsymbol{\Delta})$.

Proof. The proof is derived through the application of the $\boldsymbol{L}\boldsymbol{Y}$ transformation to moment generating function (2.4). □

Theorem 2.3. *If* $\boldsymbol{X} \sim \mathsf{uMMNE}_p(\boldsymbol{\xi}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Delta}, \boldsymbol{\nu})$ *and* $\boldsymbol{Y} \sim \mathsf{N}_p(\boldsymbol{\xi}_2, \boldsymbol{\Sigma}_2)$, *then for any matrix* $\boldsymbol{A}$ *of dimension* $p \times p$ *we have*

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{Y} \sim \mathsf{uMMNE}_p(\boldsymbol{A}\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, \boldsymbol{A}\boldsymbol{\Sigma}_1\boldsymbol{A}^\top + \boldsymbol{\Sigma}_2, \boldsymbol{\Delta}, \boldsymbol{\nu}).$$

where $M_{\boldsymbol{U}}(\cdot)$ *stands for the moment generating function of* $\boldsymbol{U}$.

3 Parameter estimation via the ECM algorithm

Analogous to the MMNE distribution, the uMMNE distribution possesses a convenient hierarchical structure, thereby substantially easing parameter estimation through application of the ECM algorithm:

$$\boldsymbol{Y}_j \mid \boldsymbol{U} = \boldsymbol{u}_j \sim \mathsf{N}_p(\boldsymbol{\xi} + \boldsymbol{\Delta}\boldsymbol{u}_j, \boldsymbol{\Sigma}), \qquad \boldsymbol{U}_j \sim f_{\boldsymbol{U}_j}(\boldsymbol{u}_j) = \exp\{\boldsymbol{u}_j \boldsymbol{1}_p\}. \tag{3.1}$$

The ECM algorithm for the uMMNE model then proceeds in a similar way as for the MMNE model as described in [2]. By using Theorem 2.1, it is straightforward to see $\boldsymbol{U}_j|\boldsymbol{y}_j \sim \mathsf{TN}\left(\boldsymbol{\Gamma}^{-1/2}\boldsymbol{A}_E, \boldsymbol{\Gamma}^{-1}; (\boldsymbol{0}, \boldsymbol{\infty})\right)$.

As a result of (3.1), the complete-data log-likelihood function for $\boldsymbol{\Theta} = (\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\Delta})$ associated with the observed data $\boldsymbol{y} = (\boldsymbol{y}_i, \ldots, \boldsymbol{y}_n)^\top$, missing value $\boldsymbol{U} = (\boldsymbol{U}_1^\top, \ldots, \boldsymbol{U}_n^\top)^\top$ and ignoring additive constants, is

$$\ell_c(\boldsymbol{\Theta}|\boldsymbol{y}, \boldsymbol{U}) = \sum_{j=1}^{n} \log f_{\boldsymbol{U}_j}(\boldsymbol{u}_j) - \frac{n}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\sum_{j=1}^{n}\boldsymbol{\Upsilon}_j\right)$$

where $\boldsymbol{\Upsilon}_j = \left(\boldsymbol{y}_j - \boldsymbol{\xi} - \boldsymbol{\Delta}\boldsymbol{U}_j\right)\left(\boldsymbol{y}_j - \boldsymbol{\xi} - \boldsymbol{\Delta}\boldsymbol{U}_j\right)^\top$, and $\mathrm{tr}(\boldsymbol{M})$ denotes the trace of matrix $\boldsymbol{M}$.

We utilize the general formulas presented in the R programming language to calculate the first two moments of the multivariate truncated normal variables $\boldsymbol{\zeta}_{1j}$ and $\boldsymbol{\Omega}_j$. Numerical calculation is performed using the TTmoment package, with the parameter ν goes to infinity. The R MomTrunc package facilitates the efficient implementation of these newly developed methods.

Now, the ECM algorithm for ML estimation of the uMMNE model proceeds as follows:

- E-step: At the kth iteration, the Q-function is computed with $\boldsymbol{\Theta}$ evaluated at $\hat{\boldsymbol{\Theta}}^{(k)}$ as

$$Q(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) = -\frac{n}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\mathrm{tr}\left(\boldsymbol{D}^{-1}\sum_{j=1}^{n}\boldsymbol{\Upsilon}_j^{(k)}\right) \tag{3.2}$$

Table 1: Comparison of maximum likelihood estimation results on the wine data.

Model	m	AIC	BIC
MSN	12	779.770	880.921
uMSN	12	520.028	625.394
MMNE	12	494.028	599.393
MMNEH	13	401.856	511.435
uMMNE	12	372.748	486.542

where the necessary conditional expectations obtained by Theorem 2.1 are

$$\hat{\boldsymbol{\zeta}}_{1j}^{(k)} = E(\boldsymbol{U}_j \mid \boldsymbol{y}_j, \hat{\boldsymbol{\Theta}}^{(k)}), \quad \hat{\boldsymbol{\Omega}}_j^{(k)} = E\left(\boldsymbol{U}_j\boldsymbol{U}_j^\top \mid \boldsymbol{y}_j, \hat{\boldsymbol{\Theta}}^{(k)}\right), \tag{3.3}$$

and

$$\begin{aligned}\boldsymbol{\Upsilon}_j^{(k)} =&(\boldsymbol{y}_j - \boldsymbol{\xi})(\boldsymbol{y}_j - \boldsymbol{\xi})^\top - \boldsymbol{\Delta}\hat{\boldsymbol{\zeta}}_{1j}^{(k)}(\boldsymbol{y}_j - \boldsymbol{\xi})^\top - (\boldsymbol{y}_j - \boldsymbol{\xi})\hat{\boldsymbol{\zeta}}_{1j}^{(k)}\boldsymbol{\Delta}^\top \\ &+ \boldsymbol{\Delta}\hat{\boldsymbol{\Omega}}_j^{(k)}\boldsymbol{\Delta}^\top,\end{aligned} \tag{3.4}$$

which contains unknown parameters $\boldsymbol{\xi}$ and $\boldsymbol{\Delta}$. Note that the calculation of $E(W_j^r \mid \boldsymbol{y}_j, \hat{\boldsymbol{\Theta}}^{(k)})$ and $E(\log h(W_j;\boldsymbol{\nu}) \mid \boldsymbol{y}_j, \hat{\boldsymbol{\Theta}}^{(k)})$ critically depends on $h(w;\boldsymbol{\nu})$.

- CM-step 1: Maximizing (3.2) over $\boldsymbol{\xi}, \boldsymbol{\Delta}$ and $\boldsymbol{\Sigma}$ leads to the following CM estimators:

$$\begin{aligned}\hat{\boldsymbol{\xi}}^{(k+1)} &= \frac{\sum_{j=1}^n (\boldsymbol{y}_j - \hat{\boldsymbol{\Delta}}^{(k)}\hat{\boldsymbol{\zeta}}_{0j}^{(k)})}{n}, \\ \hat{\boldsymbol{\Delta}}^{(k+1)} &= \left(\sum_{j=1}^n (\boldsymbol{y}_j - \hat{\boldsymbol{\xi}}^{(k+1)})\hat{\boldsymbol{\zeta}}_{1j}^{(k)\top}\right)\left(\sum_{j=1}^n \hat{\boldsymbol{\Omega}}_j^{(k)}\right)^{-1}, \\ \hat{\boldsymbol{D}}^{(k+1)} &= \frac{1}{n}\left(\sum_{j=1}^n \hat{\boldsymbol{\Upsilon}}_j^{(k)}\right).\end{aligned}$$

The above E- and CM-steps are iterated until either the number of iterations exceeds the maximum limit or a suitable convergence rule is achieved. We denote the resulting ML estimates upon convergence by $\hat{\boldsymbol{\Theta}} = (\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\Sigma}})$.

4 Italian wine recognition data

The proposed method is illustrated in this section through an analysis of the Italian wine recognition dataset. The dataset comprises p = 13 chemical measurements that resulting in a working dataset of size 178×13 in three different classes, Barolo, Grignolino, and Barbera. The data is publicly accessible at "http://archive.ics.uci .edu/ml". In this analysis, we focus solely on the Barbera group and variables Alcohol, Malic acid and Alkalinity of ash. To estimate parameters, we first standardize the data so that each attribute has a mean of zero and a variance of unit. Subsequently, the standardized data were analyzed using the proposed uMMNE model and the ECM algorithm. For comparison purposes, we also fit the MMN model containing MMNE, mean mixture normal convolution with a mixture of exponential and half-normal distributions (MMNEH), MSN and unrestricted multivariate skew normal (uMSN) to the same dataset.

Each fitting procedure starts with identical initial values and stops when the relative log-likelihood increase is under 10^{-6} or after 3000 iterations $k_{\max}$.

We use AIC and Bayesian Information Criterion (BIC) scores to find the best model. Table 2 shows how the AIC, BIC scores changes in models, calculated using the ECM algorithm. It is observed from Table 2 that uMMNE models have good performance for clustering wine dataset alternative other model.

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A robust factor analysis model based on the contaminated skew-normal with incomplete data

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Abstract

The development of advanced data collection technologies has fueled a substantial growth in the use of factor analysis (FA) for dimensionality reduction purposes. The inclusion of additional measurements introduces a higher risk of outliers, potentially causing biased estimates, reduced model stability and robustness, and ultimately, inaccurate inferences. To address the challenges posed by high-dimensional data with asymmetry and outliers, a factor analysis (FA) model was developed. This model utilizes contaminated skew-normal distributions for latent factors, enabling automatic outlier detection. We employed newly developed EM-type algorithms to compute maximum likelihood parameter estimates. As a practical demonstration of its use in social data analysis, the model is showcased via case studies derived from the Pima dataset.

Keywords: Factor analysis, contaminated skew normal distribution, ECM algorithm.

1 Introduction

Increased computational power has enabled the modeling of highly complex datasets. The field of high-dimensional data analysis has experienced notable progress in the modeling of complex datasets recently. The factor analysis (FA) model constitutes a significant early contribution to the field. The factor analysis model generally postulates normally distributed factors and error terms for reasons of mathematical and computational practicality. A theoretical framework for a general factor analysis (FA) model representing a collection of random vectors $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ is provided by a linear combination of latent factors:

$$\boldsymbol{Y}_j = \boldsymbol{\xi} + \boldsymbol{B}\boldsymbol{X}_j + \boldsymbol{\varepsilon}_j \tag{1.1}$$

where $\boldsymbol{Y}_j \in \mathbb{R}^p$ and $\boldsymbol{\xi}$ is a p-dimensional location vector. In model (1.1), the $p \times q$ factor loadings matrix $\boldsymbol{B}$ describes how each element of $\boldsymbol{Y}_j$ relates to latent factors, $\boldsymbol{X}_j$, for $j = 1, \ldots, n$. To simplify computation and mathematical operations, FA models frequently assume that latent factors and error terms ($\boldsymbol{\epsilon}_j$) are independently distributed according to a multivariate normal distribution. Specifically, it is assumed that $\boldsymbol{\epsilon}_j \sim \mathsf{N}_p(\boldsymbol{0}, \boldsymbol{D})$ and $\sim \mathsf{N}_q(\boldsymbol{0}, \boldsymbol{I}q)$, where $\boldsymbol{D} = \mathrm{Diag}(d_1, \ldots, d_p)$ is a $p\times$ diagonal matrix with positive diagonal elements representing idiosyncratic variances, and $\boldsymbol{I}_q$ is the identity matrix of size q.

Skew-normal (SN) distributions, as initially described in [1], are foundational to the construction of statistically robust models designed for skewed data. Lin et al. [2] recently conducted a study of a factor

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analysis model utilizing a skew-t distribution, which they designated as STFA. This model encompasses the FA model and SN factor analysis (SNFA) as limiting instances. Notwithstanding their flexibility with skewed and heavy-tailed datasets, the SNFA and STFA models can exhibit limitations in the concurrent detection of both minor and major outliers. Data points that show a modest divergence from the distribution of a cluster are considered "mild outliers". Conversely, data points situated at a significant distance from any constituent element are classified as "gross outliers".

The contaminated skew normal (CSN) distribution, a particular case of CFUSS distributions, was presented by Arellano et al. [3] to accommodate outlier and heavy-tailed data. Subsequently, we will introduce the multivariate skew-normal (CSN) distribution, which is characterized by a mixture of SN distributions. One component models the typical data, while the other models atypical data points (outliers). CSN offers the valuable capability of identifying mild outliers and obtaining accurate parameter estimates concurrently, even with skewed datasets. Notably, the CSN distribution-based automatic outlier detection method makes no additional distributional assumptions.

A robust model-based dimensionality reduction method is proposed, using unrestricted contaminated skew-normal factor analysis as its foundation. A model has been developed that builds upon and expands the existing body of research concerning FA and CSN distributions in the context of high-dimensional incomplete data.

2 The CSN distribution

A p-dimensional random vector $\boldsymbol{Y}$ is classified as having a SN distribution if it can be produced by the linear stochastic representation

$$\boldsymbol{Y} = \boldsymbol{\xi} + \boldsymbol{\lambda}|Z_0| + \boldsymbol{Z}_1, \text{ with } \begin{bmatrix} Z_0 \\ \boldsymbol{Z}_1 \end{bmatrix} \sim \mathsf{N}_{1+p}\left(\begin{bmatrix} 0 \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma} \end{bmatrix}\right). \tag{2.1}$$

where $\boldsymbol{\xi} \in \mathbb{R}^p$ is a location vector, $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix, and $\boldsymbol{\lambda}$ is skewness vector. Consequently, the pdf for the SN distribution can be obtained as

$$f_{\text{rSN}}(\boldsymbol{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}) = 2\phi_p(\boldsymbol{y}; \boldsymbol{\mu}, \boldsymbol{\Omega})\Phi_1\left(\frac{\boldsymbol{\lambda}^\top \boldsymbol{\Omega}^{-1}(\boldsymbol{y} - \boldsymbol{\mu})}{\sqrt{1 - \boldsymbol{\lambda}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}}}\right), \quad \boldsymbol{y} \in \mathbb{R}^p, \tag{2.2}$$

where $\boldsymbol{\Omega} = \boldsymbol{\Sigma} + \boldsymbol{\lambda}\boldsymbol{\lambda}^\top$, $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda)$ and $\Phi_1(\cdot)$ is the cdf of the univariate standard normal distribution, $\mathrm{N}(0, 1)$. The CSN distribution, introduced by [3], can be represented by

$$\boldsymbol{Y} = \boldsymbol{\xi} + W^{-1/2}(\boldsymbol{\lambda}|Z_0| + \boldsymbol{Z}_1), \text{ with } \begin{bmatrix} Z_0 \\ \boldsymbol{Z}_1 \end{bmatrix} \sim \mathsf{N}_{1+p}\left(\begin{bmatrix} 0 \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma} \end{bmatrix}, \boldsymbol{\nu}\right) \tag{2.3}$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2)$ and $W \in \{1, 1/\nu_1\}$ is an independent random variable from $\boldsymbol{Z}_0$ and $\boldsymbol{Z}_1$, having the probability mass function

$$f(w; \boldsymbol{\nu}) = \nu_2^{\frac{w - 1/\nu_1}{1 - 1/\nu_1}} (1 - \nu_2)^{\frac{1-w}{1-1/\nu_1}},$$

such that $\mathsf{E}(W^{-1/2}) = \nu_2 + \sqrt{\nu_1}(1 - \nu_2)$ and $\mathsf{E}(W^{-1}) = \nu_2 + \nu_1(1 - \nu_2)$. If $\boldsymbol{\gamma} = W^{-1/2}|\boldsymbol{Z}_0|$, hierarchical representations of the CSN distribution are possible at these three levels:

$$\boldsymbol{Y}|(\boldsymbol{\gamma}, w) \sim \mathsf{N}_p\left(\boldsymbol{\xi} + \boldsymbol{\lambda}\boldsymbol{\gamma}, w^{-1}\boldsymbol{\Sigma}\right), \ \boldsymbol{\gamma}|w \sim \mathsf{HN}(0, w^{-1}\boldsymbol{I}_s), \ W \sim f(w; \boldsymbol{\nu}), \tag{2.4}$$

where $f(w; \boldsymbol{\nu})$ is the pdf of W. This leads to obtain the pdf of $\boldsymbol{Y}$, called CSN and denoted by $\boldsymbol{Y} \sim \mathsf{CSN}_{p,s}(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \boldsymbol{\nu})$, as

$$f_{\text{CSN}}(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \nu_2 f_{\text{SN}}(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}) + (1 - \nu_2) f_{\text{SN}}(\boldsymbol{y}; \boldsymbol{\xi}, \nu_1\boldsymbol{\Sigma}, \sqrt{\nu_1}\boldsymbol{\lambda}). \tag{2.5}$$

The mean vector and covariance matrix of $\boldsymbol{Y}$, derived using the law of iterated expectations on (2.3), are $\mathsf{E}(\boldsymbol{Y}) = \boldsymbol{\xi} + a_{\boldsymbol{\nu}}\boldsymbol{\Delta}\mathbf{1}_s$ and

$$\mathsf{Cov}(\boldsymbol{Y}) = b_{\boldsymbol{\nu}}(\boldsymbol{\Sigma} + (1 - \frac{2}{\pi})\boldsymbol{\lambda}\boldsymbol{\lambda}^\top) + \left[\frac{2}{\pi}b_{\boldsymbol{\nu}} - a_{\boldsymbol{\nu}}^2\right]\boldsymbol{\lambda}\boldsymbol{J}_s\boldsymbol{\lambda}^\top, \tag{2.6}$$

where $a_{\boldsymbol{\nu}} = \sqrt{\frac{2}{\pi}}E(W^{-1/2})$, $b_{\boldsymbol{\nu}} = E(W^{-1})$.

Theorem-derived higher CSN moments, while obtainable via the moment generating function (mgf), demand considerable computational effort.

Theorem 2.1. *If $\boldsymbol{Y} \sim \mathsf{CSN}_{p,s}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \boldsymbol{\nu})$, then the mgf of $\boldsymbol{Y}$ is*

$$M_{\boldsymbol{Y}}(\boldsymbol{t}) = 2\exp\{\boldsymbol{t}^\top\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{t}^\top\boldsymbol{\Omega}\boldsymbol{t}\}\left[\nu_2\Phi_1(\boldsymbol{t}^\top\boldsymbol{\lambda}) + (1-\nu_2)\Phi_1(\sqrt{\nu_1}\boldsymbol{t}^\top\boldsymbol{\lambda})\right].$$

An appealing closure property of CFUCSN under affine transformations, frequently used in our methodology, is shown in the following theorem.

Theorem 2.2. *The CSN distribution is closed under linear transformation, i.e. if $\boldsymbol{Y} \sim \mathsf{CSN}_p(\boldsymbol{\xi}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \boldsymbol{\nu})$, then for any full rank matrix $\boldsymbol{L} \in \mathbb{R}^{k\times p}$, the random vector $\boldsymbol{L}\boldsymbol{Y}$ is distributed by $\mathsf{CSN}_k(\boldsymbol{L}\boldsymbol{\xi}, \boldsymbol{L}\boldsymbol{\Sigma}\boldsymbol{L}^\top, \boldsymbol{L}\boldsymbol{\lambda}, \boldsymbol{\nu})$.*

Proof. Theorem 2.1 and the basic random vector transformation rule directly imply the proof. □

3 The CSN factor analysis model with missing information

For a set of random vectors $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$, the generic form of an CSN factor analysis (CSNFA) model is theoretically expressed as a linear combination of latent factors:

$$\boldsymbol{Y}_j = \boldsymbol{\xi} + \boldsymbol{B}\boldsymbol{X}_j + \boldsymbol{\varepsilon}_j \tag{3.1}$$

where $\boldsymbol{Y}_j \in \mathbb{R}^p$, $\boldsymbol{\xi}$ is a p-dimensional location vector and the $p \times q$ factor loadings matrix $\boldsymbol{B}$. We assume that the joint vector of latent factors and idiosyncratic errors follows a $(p+q)$-variate CSN distribution. Specifically,

$$\begin{pmatrix}\boldsymbol{X}_j\\ \boldsymbol{\varepsilon}_j\end{pmatrix} \sim \mathsf{CSN}_{q+p}\left(\begin{pmatrix}-a_{\boldsymbol{\nu}}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\lambda}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Lambda}^{-1} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{D}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\lambda}\\ \mathbf{0}\end{pmatrix}, \boldsymbol{\nu}\right). \tag{3.2}$$

where $\boldsymbol{\Lambda} = (\boldsymbol{I}_q + (1-\frac{2}{\pi})\boldsymbol{\lambda}\boldsymbol{\lambda}^\top) + \left[\frac{2}{\pi} - \frac{a_{\boldsymbol{\nu}}^2}{b_{\boldsymbol{\nu}}}\right]\boldsymbol{\lambda}\boldsymbol{\lambda}^\top$. We have $\mathsf{E}(\boldsymbol{X}_j) = 0$ and $\mathsf{Cov}(X_j) = b_{\boldsymbol{\nu}}\boldsymbol{I}_q$.

The marginal distribution of $\boldsymbol{Y}_j$ arising from (3.1) is

$$\boldsymbol{Y}_j \sim \mathsf{CSN}_p\left(\boldsymbol{\xi} - a_{\boldsymbol{\nu}}\boldsymbol{\alpha}, \boldsymbol{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\nu}\right), \tag{3.3}$$

where $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{\Lambda}^{-1}\boldsymbol{B}^\top + \boldsymbol{D}$ and $\boldsymbol{\alpha} = \boldsymbol{B}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\lambda}$ is a skewness matrix of size $p \times q$.

To carry out ML estimation of model (3.3), the paper is focusing on establishing a procedure when the observed data $\boldsymbol{Y}_j$ are incomplete. Under this assumption, we partition $\boldsymbol{Y}_i$ into the observed component $\boldsymbol{Y}_j^o \in \mathbb{R}^{p_j^o}$ and the missing component $\boldsymbol{Y}_j^m \in \mathbb{R}^{p_j^m}$, where $p_j^o + p_j^m = p$. For easy of notation and computation, two auxiliary permutation matrices are introduced as $\boldsymbol{O}_j$ $(p_j^o \times p_j)$ and $\boldsymbol{M}_j$ $((p - p_j^o) \times p_j)$ to identify $\boldsymbol{Y}_j^o$ and $\boldsymbol{Y}_j^m$ as

$$\begin{aligned}&\boldsymbol{Y}_j^o = \boldsymbol{O}_j\boldsymbol{Y}_j \quad \text{and} \quad \boldsymbol{Y}_j^m = \boldsymbol{M}_j\boldsymbol{Y}_j, \quad \text{such that} \quad \boldsymbol{Y}_j = \boldsymbol{O}_j'\boldsymbol{Y}_j^o + \boldsymbol{M}_j'\boldsymbol{Y}_j^m\\ &\text{and} \quad \boldsymbol{O}_j'\boldsymbol{O}_j + \boldsymbol{M}_j'\boldsymbol{M}_j = \boldsymbol{I}_p.\end{aligned} \tag{3.4}$$

Thus, an alternative representation of the CSNFA model with incomplete data is

$$\boldsymbol{Y}_j^o \sim \mathsf{CSN}_{p_j^o}\left(\boldsymbol{\xi}_j^o - a_{\boldsymbol{\nu}}\boldsymbol{\alpha}_j^o, \boldsymbol{\Sigma}_j^{oo}, \boldsymbol{\alpha}_j^o, \boldsymbol{\nu}\right), \tag{3.5}$$

where $\boldsymbol{\xi}_j^o = \boldsymbol{O}_j\boldsymbol{\xi}$, $\boldsymbol{\alpha}_j^o = \boldsymbol{O}_j\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}_j^{oo} = \boldsymbol{O}_j\boldsymbol{\Sigma}\boldsymbol{O}_j^\top$. Scaling transformations $\widetilde{\boldsymbol{B}} \triangleq \boldsymbol{B}\boldsymbol{\Lambda}^{-1/2}$ and $\widetilde{\boldsymbol{X}_j} \triangleq \boldsymbol{\Lambda}^{1/2}\boldsymbol{X}_j$ are applied to the matrices of factor loadings $\boldsymbol{B}$ and common factors $\boldsymbol{X}_j$ to simplify the mathematical and computational processes.

In the context of outlier detection, a binary random variable, which we will denote as r_j, may be incorporated to act as an indicator or label for outliers. This variable will take on the value of 1 if the observation $\boldsymbol{Y}_j$ stems from the good component, and it will take on the value of r_j if $\boldsymbol{Y}_j$ originates from the component that is contaminated or anomalous. The random variable r_j exhibits a Bernoulli distribution characterized by a success probability of ν_2, indicated by $r_j \sim \mathsf{Ber}(1, \nu_2)$. A fully hierarchical version of model (3.5) is shown in the theorem that follows.

Theorem 3.1. *A five-level hierarchy forms the basis of the uCSNFA model, as described below:*

$$\begin{aligned}
\boldsymbol{Y}_j^o|(\widetilde{\boldsymbol{X}_j}, r_j) &\sim \begin{cases} \mathsf{N}_p\left(\boldsymbol{\xi}_j^o + \widetilde{\boldsymbol{B}}_j^o\widetilde{\boldsymbol{X}_j}, \boldsymbol{D}_j^{oo}\right) & \text{if} \quad r_j = 1; \\ \mathsf{N}_p\left(\boldsymbol{\xi}_j^o + \widetilde{\boldsymbol{B}}_j^o\widetilde{\boldsymbol{X}_j}, \nu_1\boldsymbol{D}_j^{oo}\right) & \text{if} \quad r_j = 0, \end{cases} \\
\widetilde{\boldsymbol{X}}_{ij}|(\boldsymbol{\gamma}_j, r_j) &\sim \begin{cases} \mathsf{N}_q\left(\boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}}), \boldsymbol{I}_q\right) & \text{if} \quad r_j = 1; \\ \mathsf{N}_q\left(\boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}}), \nu_1\boldsymbol{I}_q\right) & \text{if} \quad r_j = 0, \end{cases} \\
\boldsymbol{\gamma}_j|(r_j) &\sim \begin{cases} \mathsf{HN}_q\left(\boldsymbol{0}, \boldsymbol{I}_q\right) & \text{if} \quad r_{ij} = 1; \\ \mathsf{HN}_q\left(\boldsymbol{0}, \nu_1\boldsymbol{I}_q\right) & \text{if} \quad r_{ij} = 0, \end{cases} \\
r_j &\sim \mathsf{Ber}(1, \nu_2),
\end{aligned}$$

where $\boldsymbol{\xi}_j^{m.o} = \boldsymbol{M}_j\left[\boldsymbol{\xi}_j + \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{X}_j} + \boldsymbol{D}\boldsymbol{C}_j^{oo}(\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{X}_j})\right]$, $\boldsymbol{D}_j^{oo} = \boldsymbol{O}_j\boldsymbol{D}\boldsymbol{O}_j^\top$, $\boldsymbol{C}_j^{oo} = \boldsymbol{O}_j^\top\left(\boldsymbol{O}_j\boldsymbol{D}\boldsymbol{O}_j^\top\right)^{-1}\boldsymbol{O}_j$ *and* $\boldsymbol{D}_j^{mm.o} = \boldsymbol{M}_j\left(\boldsymbol{I}_p - \boldsymbol{D}\boldsymbol{C}_j^{oo}\right)\boldsymbol{D}\boldsymbol{M}_j^\top$.

Utilizing Bayes rule, we can derive and then present in the following theorem a collection of foundational conditional distributions. These are necessary for developing the methods explained later.

Theorem 3.2. *The following conditional distributions are derived from the specifications in* (3.6)*:*

$$\begin{aligned}
\widetilde{\boldsymbol{X}_j}|(\boldsymbol{y}_j^o, \boldsymbol{\gamma}_j, r_j) &\sim \begin{cases} \mathsf{N}_q\left(\boldsymbol{q}_j^o, \boldsymbol{H}_j^{oo}\right) & \text{if} \quad r_j = 1; \\ \mathsf{N}_q\left(\boldsymbol{q}_j^o, \nu_1\boldsymbol{H}_j^{oo}\right) & \text{if} \quad r_j = 0, \end{cases} \\
\boldsymbol{\gamma}_j|(\boldsymbol{y}_j^o, r_j) &\sim \begin{cases} \mathsf{HN}_s\left(\boldsymbol{a}_j^o, (\boldsymbol{I}_q - \boldsymbol{\alpha}_j^\top\boldsymbol{\Gamma}_j^{oo}\boldsymbol{\alpha}_j)\right) & \text{if} \quad r_j = 1; \\ \mathsf{HN}_s\left(\boldsymbol{a}_j^o, \nu_1(\boldsymbol{I}_q - \boldsymbol{\alpha}_j^\top\boldsymbol{\Gamma}_j^{oo}\boldsymbol{\alpha}_j)\right) & \text{if} \quad r_j = 0, \end{cases} \\
r_j|\boldsymbol{y}_j^o &\sim \mathsf{Ber}(1, \tilde{\nu}_2), .
\end{aligned}$$

where $\boldsymbol{q}_j^o = \boldsymbol{H}_j^{oo}\left\{\widetilde{\boldsymbol{B}}^\top\boldsymbol{D}^{-1}(\boldsymbol{y}_j - \boldsymbol{\xi}) + \boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}})\right\}$ *and* $\boldsymbol{a}_j^o = \boldsymbol{\alpha}^\top\boldsymbol{\Gamma}_j^{oo}(\boldsymbol{y}_j - \boldsymbol{\xi} + \sqrt{2/\pi}\boldsymbol{\alpha})$ *with* $\boldsymbol{H}_j^{oo} = (\boldsymbol{I}_q + \widetilde{\boldsymbol{B}}^\top\boldsymbol{C}_j^{oo}\widetilde{\boldsymbol{B}})^{-1}$, $\boldsymbol{\Gamma}_j^{oo} = \boldsymbol{O}_j\boldsymbol{\Omega}_j^{oo\,-1}\boldsymbol{O}_j^\top$ *and* $\tilde{\nu}_2 = \frac{\nu_2 f_{\mathsf{SN}_{p_j^o}}\left(\boldsymbol{\xi}_j^o - a_{\boldsymbol{\nu}}\boldsymbol{\alpha}_j^o, \boldsymbol{\Sigma}_j^{oo}, \boldsymbol{\alpha}_j^o\right)}{f_{\mathsf{CSN}_{p_j^o}}\left(\boldsymbol{\xi}_j^o - a_{\boldsymbol{\nu}}\boldsymbol{\alpha}_j^o, \boldsymbol{\Sigma}_j^{oo}, \boldsymbol{\alpha}_j^o, \boldsymbol{\nu}\right)}$.

4 Parameter estimation via the ECM algorithm

A popular technique for calculating ML parameter estimates from incomplete data is the expectation maximization (EM) algorithm [4]. Iteration of the E- and M-steps continues until the convergence criterion

is reached. By adding CM steps for multiple complete data sets, the ECM algorithm offers a more efficient optimization method than the classical EM algorithm, especially when applied to complex models such as CSNFA.

To implement our feasible EM procedure for learning the CSNFA model, let $\boldsymbol{Y}_c = (\boldsymbol{y}^o, \boldsymbol{y}^m, \widetilde{\boldsymbol{X}}, \boldsymbol{\gamma}, \boldsymbol{r})$ represent the complete data, where $\boldsymbol{y}^m = (\boldsymbol{y}_1^m, \ldots, \boldsymbol{y}_n^m)$ denotes the missing portion of the data, and $\boldsymbol{y}^o = (\boldsymbol{y}_1^o, \ldots, \boldsymbol{y}_n^o)$ denotes the observed portion. Additionally, $\widetilde{\boldsymbol{X}} = \left\{\widetilde{\boldsymbol{X}}_1, \ldots, \widetilde{\boldsymbol{X}}_n\right\}$ represents the set of latent factors. Moreover, $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_n)$, $\boldsymbol{r} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ denotes the unobserved outlier indicator, such that $r_j = 1$ if jth observation be a typical point and $r_j = 0$ if it is an outlier. From (3.6), the log-likelihood function of $\boldsymbol{\Theta}$ for $\boldsymbol{y}_c$, without considering constant terms is

$$
\begin{aligned}
\ell_c(\boldsymbol{\Theta}; \boldsymbol{y}_c) = & \sum_{j=1}^{n} [r_j \log \nu_2 + (1 - r_j) \log(1 - \nu_2)] - \sum_{j=1}^{n} \frac{r_j}{2} \Big[\log |\boldsymbol{D}| + \mathsf{tr}(\boldsymbol{D}^{-1} \boldsymbol{\Upsilon}_j \\
& + \boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q)(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q)^\top \boldsymbol{\lambda}^\top - \boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q) \widetilde{\boldsymbol{X}}_j^\top \\
& - \widetilde{\boldsymbol{X}}_j (\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q)^\top \boldsymbol{\lambda}^\top + \boldsymbol{\gamma}_j \boldsymbol{\gamma}_j^\top + \widetilde{\boldsymbol{X}}_j \widetilde{\boldsymbol{X}}_j^\top) \Big] \\
& - \sum_{j=1}^{n} \frac{(1 - r_j)}{2} \Big[(p + 2q) \log \nu_1 + 1 + \log |\boldsymbol{D}| + \nu_1 \mathsf{tr}(\boldsymbol{D}^{-1} \boldsymbol{\Upsilon}_j \\
& + \boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q)(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q)^\top \boldsymbol{\lambda}^\top - \boldsymbol{\lambda}(\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q) \widetilde{\boldsymbol{X}}_j^\top \\
& - \widetilde{\boldsymbol{X}}_j (\boldsymbol{\gamma}_j - a_{\boldsymbol{\nu}} \mathbf{1}_q)^\top \boldsymbol{\lambda}^\top + \boldsymbol{\gamma}_j \boldsymbol{\gamma}_j^\top + \widetilde{\boldsymbol{X}}_j \widetilde{\boldsymbol{X}}_j^\top) \Big]
\end{aligned} \tag{4.1}
$$

where $\boldsymbol{\Upsilon}_j = (\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{X}}_j)(\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{X}}_j)^\top$. The Q-function calculation necessitates the evaluation of the following conditional expectations:

$$
\begin{aligned}
\hat{r}_j^{(k)} = & E\left(r_j \mid \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \ \hat{\boldsymbol{y}}_j^{(k)} = E\left(\boldsymbol{Y}_j \mid \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \\
\hat{\boldsymbol{\gamma}}_{1j}^{(k)} = & E\left(\boldsymbol{\gamma}_j \mid r_j = 1, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \ \tilde{\boldsymbol{\gamma}}_{1j}^{(k)} = E\left(\boldsymbol{\gamma}_j \mid r_j = 0, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \\
\hat{\boldsymbol{\gamma}}_{2j}^{(k)} = & E\left(\boldsymbol{\gamma}_j \boldsymbol{\gamma}_j^\top \mid r_j = 1, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \ \tilde{\boldsymbol{\gamma}}_{2j}^{(k)} = E\left(\boldsymbol{\gamma}_j \boldsymbol{\gamma}_j^\top \mid r_j = 0, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \\
\hat{\boldsymbol{\zeta}}_j^{(k)} = & E\left(\widetilde{\boldsymbol{X}}_j \mid r_j = 1, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \ \tilde{\boldsymbol{\zeta}}_j^{(k)} = E\left(\widetilde{\boldsymbol{X}}_j \mid r_j = 0, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \\
\hat{\boldsymbol{\kappa}}_j^{(k)} = & E\left(\widetilde{\boldsymbol{X}}_j \boldsymbol{\gamma}_j^\top \mid r_j = 1, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \ \tilde{\boldsymbol{\kappa}}_j^{(k)} = E\left(\widetilde{\boldsymbol{X}}_j \boldsymbol{\gamma}_j^\top \mid r_j = 0, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \\
\widehat{\boldsymbol{\Psi}}_j^{(k)} = & E\left(\widetilde{\boldsymbol{X}}_j \widetilde{\boldsymbol{X}}_j^\top \mid r_j = 1, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right), \ \widetilde{\boldsymbol{\Psi}}_j^{(k)} = E\left(\widetilde{\boldsymbol{X}}_j \widetilde{\boldsymbol{X}}_j^\top \mid r_j = 0, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right).
\end{aligned} \tag{4.2}
$$

The following steps show how the ECM algorithm estimates the ML parameters of the CSNFA model:

- E-step: Given the observed data $\boldsymbol{y}_j^o$ and the current estimate $\widehat{\boldsymbol{\Theta}}^{(k)}$, the E-step calculates the conditional expectation of complete-data log-likelihood function (4.1). This leads to the so-called Q-function:

$$
Q(\boldsymbol{\Theta} \mid \widehat{\boldsymbol{\Theta}}^{(k)}) = \sum_{j=1}^{n} \left[\hat{r}_j^{(k)} \log \eta + (1 - \hat{r}_j^{(k)}) \log(1 - \eta)\right] + Q^{\mathsf{good}}(\boldsymbol{\Theta} \mid \widehat{\boldsymbol{\Theta}}^{(k)}) + Q^{\mathsf{bad}}(\boldsymbol{\Theta} \mid \widehat{\boldsymbol{\Theta}}^{(k)}), \tag{4.3}
$$

  where

$$
Q^{\mathsf{good}}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) = -\frac{1}{2} \sum_{j=1}^{n} \hat{r}_j^{(k)} \left[\log |\boldsymbol{D}| + \mathsf{tr}\left\{ \boldsymbol{D}^{-1} \widehat{\boldsymbol{\Upsilon}}_j^{(k)}(\boldsymbol{\xi}, \widetilde{\boldsymbol{B}}) + \boldsymbol{\lambda} \widehat{\boldsymbol{F}}_j^{(k)} \boldsymbol{\lambda}^\top - \boldsymbol{\lambda} \hat{\boldsymbol{\lambda}}_j^{(k)^\top} - \hat{\boldsymbol{\lambda}}_j^{(k)} \boldsymbol{\lambda}^\top + \widehat{\boldsymbol{\Psi}}_j^{(k)} \right\} \right]
$$

with $\widehat{\boldsymbol{F}}_j^{(k)} = \hat{\boldsymbol{\gamma}}_{2j}^{(k)} - a_{\boldsymbol{\nu}}\hat{\boldsymbol{\gamma}}_{1j}^{(k)} - a_{\boldsymbol{\nu}}\hat{\boldsymbol{\gamma}}_{1j}^{(k)^\top} + a_{\boldsymbol{\nu}}^2$, $\hat{\boldsymbol{\delta}}_j^{(k)} = \hat{\boldsymbol{\kappa}}_j^{(k)} - a_{\boldsymbol{\nu}}\hat{\boldsymbol{\zeta}}_j^{(k)}$,

$$\begin{aligned}\widehat{\boldsymbol{\Upsilon}}_j^{(k)} =& \mathsf{E}\left((\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{X}}_j)(\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{X}}_j)^\top \mid r_j = 1, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right)\\ =& \left(\boldsymbol{I}_p - \hat{\boldsymbol{D}}^{(k)}\hat{\boldsymbol{C}}_j^{oo^{(k)}}\right)\hat{\boldsymbol{D}}^{(k)} + (\hat{\boldsymbol{y}}_j^{(k)} - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\hat{\boldsymbol{\zeta}}_j^{(k)})(\hat{\boldsymbol{y}}_j^{(k)} - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\hat{\boldsymbol{\zeta}}_j^{(k)})^\top\\ & - (\hat{\boldsymbol{A}}_j^{oo^{(k)}} - \widetilde{\boldsymbol{B}})(\widehat{\boldsymbol{\Psi}}_j^{(k)} - \hat{\boldsymbol{\zeta}}_j^{(k)}\hat{\boldsymbol{\zeta}}_j^{(k)^\top})(\hat{\boldsymbol{A}}_j^{oo^{(k)}} - \widetilde{\boldsymbol{B}})^\top\end{aligned}$$

, $\hat{\boldsymbol{A}}_j^{oo^{(k)}} = \left(\boldsymbol{I}_p - \hat{\boldsymbol{D}}^{(k)}\boldsymbol{C}_j^{oo(k)}\right)\widetilde{\boldsymbol{B}}^{(k)}$ and

$$\begin{aligned}Q^{\mathsf{bad}}(\boldsymbol{\Theta} \mid \hat{\boldsymbol{\Theta}}^{(k)}) =& -\frac{1}{2}\sum_{j=1}^{n}(1 - \hat{r}_j^{(k)})\Big[(p+2q)\log\nu_1 + \log|\boldsymbol{D}|\\ & + \nu_1 \mathsf{tr}\Big\{\boldsymbol{D}^{-1}\widetilde{\boldsymbol{\Upsilon}}_j^{(k)}(\boldsymbol{\xi}, \widetilde{\boldsymbol{B}}) + \boldsymbol{\lambda}_i\widetilde{\boldsymbol{F}}_j^{(k)}\boldsymbol{\lambda}^\top - \boldsymbol{\lambda}\tilde{\boldsymbol{\delta}}_{ij}^{(k)^\top} - \tilde{\boldsymbol{\delta}}_j^{(k)}\boldsymbol{\lambda}^\top + \widetilde{\boldsymbol{\Psi}}_j^{(k)}\Big\}\Big]\end{aligned}$$

with $\widetilde{\boldsymbol{F}}_j^{(k)} = \tilde{\boldsymbol{\gamma}}_{2j}^{(k)} - a_{\boldsymbol{\nu}}\tilde{\boldsymbol{\gamma}}_{1j}^{(k)} - a_{\boldsymbol{\nu}}\tilde{\boldsymbol{\gamma}}_{1j}^{(k)^\top} + a^2(\hat{\nu}^{(k)}, \hat{\eta}^{(k)})$, $\tilde{\boldsymbol{\delta}}_j^{(k)} = \tilde{\boldsymbol{\kappa}}_j^{(k)} - a_{\boldsymbol{\nu}}\tilde{\boldsymbol{\zeta}}_j^{(k)}$,

$$\begin{aligned}\widetilde{\boldsymbol{\Upsilon}}_{ij}^{(k)} =& \mathsf{E}\left((\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{X}}_j)(\boldsymbol{y}_j - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{X}}_j)^\top \mid r_j = 0, \boldsymbol{y}_j^o, \widehat{\boldsymbol{\Theta}}^{(k)}\right)\\ =& \left(\boldsymbol{I}_p - \hat{\boldsymbol{D}}^{(k)}\hat{\boldsymbol{C}}_j^{oo^{(k)}}\right)\hat{\boldsymbol{D}}^{(k)} + (\tilde{\boldsymbol{y}}_j^{(k)} - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\tilde{\boldsymbol{\zeta}}_j^{(k)})(\tilde{\boldsymbol{y}}_j^{(k)} - \boldsymbol{\xi} - \widetilde{\boldsymbol{B}}\tilde{\boldsymbol{\zeta}}_j^{(k)})^\top\\ & - (\hat{\boldsymbol{A}}_j^{oo^{(k)}} - \widetilde{\boldsymbol{B}})(\widetilde{\boldsymbol{\Psi}}_j^{(k)} - \tilde{\boldsymbol{\zeta}}_j^{(k)}\tilde{\boldsymbol{\zeta}}_j^{(k)^\top})(\hat{\boldsymbol{A}}_j^{oo^{(k)}} - \widetilde{\boldsymbol{B}})^\top\end{aligned}$$

- CM-step 1: Let $S(a,b) = a + b(1-a)$ and $S(a,b,c,d) = ba + dc(1-a)$. The CM-steps obtained by maximizing (4.3) concerning $\boldsymbol{\xi}, \boldsymbol{\lambda}$, $\widetilde{\boldsymbol{B}}$ and $\boldsymbol{D}$ proceed as follows:

$$\begin{aligned}\hat{\boldsymbol{\xi}}^{(k+1)} =& \frac{\sum_{j=1}^n S(\hat{r}_j^{(k)}, \hat{\boldsymbol{y}}_j^{(k)}, \tilde{\boldsymbol{y}}_j^{(k)}, \hat{\nu}_1^{(k)}) - \widehat{\widetilde{\boldsymbol{B}}}^{(k)}\sum_{j=1}^n S(\hat{r}_j^{(k)}, \hat{\boldsymbol{\zeta}}_j^{(k)}, \tilde{\boldsymbol{\zeta}}_j^{(k)}, \hat{\nu}_1^{(k)})}{\sum_{j=1}^n S(\hat{r}_j^{(k)}, \hat{\nu}_1^{(k)})},\\ \hat{\boldsymbol{\lambda}}^{(k+1)} =& \sum_{j=1}^n S(\hat{r}_j^{(k)}, \hat{\boldsymbol{\delta}}_j^{(k)}, \tilde{\boldsymbol{\delta}}_j^{(k)}, \hat{\nu}_1^{(k)}) / \sum_{j=1}^n S(\hat{r}_j^{(k)}, \widehat{\boldsymbol{F}}_j^{(k)}, \widetilde{\boldsymbol{F}}_j^{(k)}, \hat{\nu}_1^{(k)})\\ \widehat{\widetilde{\boldsymbol{B}}}^{(k+1)} =& \left(\sum_{j=1}^n S(\hat{r}_j^{(k)}, (\hat{\boldsymbol{y}}_j^{(k)} - \hat{\boldsymbol{\xi}}^{(k+1)})\hat{\boldsymbol{\zeta}}_j^{(k)^\top}, (\tilde{\boldsymbol{y}}_j^{(k)} - \hat{\boldsymbol{\xi}}^{(k+1)})\tilde{\boldsymbol{\zeta}}_j^{(k)^\top}, \hat{\nu}_1^{(k)})\right)\\ & \left(\sum_{j=1}^n S(\hat{r}_j^{(k)}, \hat{\boldsymbol{A}}_j^{oo^{(k)}}(\widehat{\boldsymbol{\Psi}}_j^{(k)} - \hat{\boldsymbol{\zeta}}_j^{(k)}\hat{\boldsymbol{\zeta}}_j^{(k)^\top}), \hat{\boldsymbol{A}}_j^{oo^{(k)}}(\widetilde{\boldsymbol{\Psi}}_j^{(k)} - \tilde{\boldsymbol{\zeta}}_j^{(k)}\tilde{\boldsymbol{\zeta}}_j^{(k)^\top}), \hat{\nu}_1^{(k)})\right)^{-1}\\ \hat{\boldsymbol{D}}^{(k+1)} =& \frac{1}{n}\mathrm{Diag}\left(\sum_{j=1}^n S(\hat{r}_j^{(k)}, \widehat{\boldsymbol{\Upsilon}}_j^{(k)}(\hat{\boldsymbol{\xi}}^{(k+1)}, \widehat{\widetilde{\boldsymbol{B}}}^{(k+1)}), \widetilde{\boldsymbol{\Upsilon}}_j^{(k)}(\hat{\boldsymbol{\xi}}^{(k+1)}, \widehat{\widetilde{\boldsymbol{B}}}^{(k+1)}), \hat{\nu}_1^{(k)})\right).\end{aligned}$$

- CM-step 2: For the fix $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}^{(k+1)}$, $\widetilde{\boldsymbol{B}} = \widehat{\widetilde{\boldsymbol{B}}}^{(k+1)}$, $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}^{(k+1)}$ and $\boldsymbol{D} = \hat{\boldsymbol{D}}^{(k+1)}$, $\hat{\boldsymbol{\nu}}^{(k+1)}$ is calculated by maximizing either the Q-function over $\boldsymbol{\nu}$, or the constrained actual observed log-likelihood function

$$\hat{\nu}^{(k+1)} = \arg\max_{\boldsymbol{\nu}}\left\{\sum_{j=1}^n \log f_{\mathrm{CSN}}(\boldsymbol{y}_j; \hat{\boldsymbol{\xi}}_i^{(k+1)} - a_{\boldsymbol{\nu}}\hat{\boldsymbol{\Lambda}}_i^{(k+1)^{-1/2}}\hat{\boldsymbol{\lambda}}_i^{(k+1)}, \hat{\boldsymbol{\Sigma}}_i^{(k+1)}, \hat{\boldsymbol{\alpha}}_i^{(k+1)}, \boldsymbol{\nu})\right\}$$

Table 1: An overview of the Pima data.

Attributes	Number of missing values	Sample skewness	Sample kurtosis	Jarque-Bera (p-value)
X_1	0	1.111	3.668	68.236(<0.01)
X_2	3	0.653	3.643	56.428(0)
X_3	19	0.161	3.973	110.256(0)
X_4	139	0.357	2.660	364.789(0)
X_5	236	2.499	11.843	279.459(0)
X_6	9	0.468	2.983	1102.360(0)
X_7	0	2.025	9.048	28.263(0)
X_8	0	1.567	4.932	24.361(0)

Table 2: Comparison of maximum likelihood estimation results on the Pima data.

Model	q	m	$\ell_{\max}$	AIC	BIC
FA	1	24	-4865.885	9779.770	9880.921
	2	31	-4733.202	9528.404	9659.056
	3	37	-4672.183	9418.366	9574.306
	4	42	-4664.781	9413.561	9590.575
rSNFA	1	25	-4735.014	9520.028	9625.394
	2	33	-4721.152	9508.303	9647.385
	3	40	-4531.454	9142.908	9311.492
	4	46	-4563.573	9219.146	9413.018
CSNFA	1	25	-4722.034	9494.028	9599.393
	2	33	-4689.257	9444.514	9583.596
	3	40	-4507.961	9095.922	9264.506
	4	46	-4510.746	9113.492	9307.364

where $\hat{\boldsymbol{\alpha}}_i^{(k+1)} = \widehat{\widetilde{\boldsymbol{B}}}_i^{(k+1)} \hat{\boldsymbol{\lambda}}_i^{(k+1)}$, $\hat{\boldsymbol{\Sigma}}_i^{(k+1)} = \widehat{\widetilde{\boldsymbol{B}}}_i^{(k+1)} \widehat{\widetilde{\boldsymbol{B}}}_i^{(k+1)^\top} + \hat{\boldsymbol{D}}_i^{(k+1)}$ and

$$\hat{\boldsymbol{\Lambda}}^{(k+1)} = (\boldsymbol{I}_q + (1 - \frac{2}{\pi})\hat{\boldsymbol{\lambda}}^{(k+1)}\hat{\boldsymbol{\lambda}}^{(k+1)^\top}) + \left[\frac{2}{\pi} - \frac{a^2_{\hat{\boldsymbol{\nu}}^{(k)}}}{b_{\hat{\boldsymbol{\nu}}^{(k)}}})\right] \hat{\boldsymbol{\lambda}}^{(k+1)} \boldsymbol{J}_q \hat{\boldsymbol{\Delta}}^{(k+1)^\top}.$$

5 Pima Indians diabetes data

We analyse the Pima Indian women data to illustrate the proposed method. The dataset comprises $p = 8$ measurements related to female patients who were at least 21 years old of Pima Indian heritage and living near Phoenix, Arizona, USA. The data is freely available at "http : //archive.ics.uci.edu/ml". Our subsequent analysis is confined to a subset comprising 500 non-diabetic test participants.

he eight attributes are detailed in Table 1, along with their sample skewness and kurtosis coefficients. Normality is assessed using Jarque and Bera test statistic, also shown in the table. Analysis of the table shows considerable asymmetry and kurtosis present in all attributes. Given the reported p-values, it is concluded that none of the attributes are normally distributed.

To identify the optimal values of q we monitor the Bayesian Information Criterion (BIC) and Akaike information criterion (AIC) scores, defined as $AIC = 2m - 2\ell_{\max}$ and $AIC = m \log n - 2\ell_{\max}$, where $\ell_{\max}$ is the maximized log-likelihood value, and m is the number of free parameters. The model exhibiting a lower AIC or BIC score is superior according to the data. Table 2 presents a comparison of the machine learning estimation results. Analysis of AIC and BIC values indicates that the uCSNFA model ($q = 4$) is the most favorable among the models considered. The FA model's performance was negatively impacted by its inadequate resilience to asymmetric data and outliers.

6 Concluding remarks

It is recommended that the factor analysis model be enhanced through the utilization of compound symmetric normal distributions to model latent factors and error components. This paper introduces CSNFA, a

robust family model designed to handle heavy-tailed data and outliers. We present computationally feasible expectation-maximization algorithms that leverage stochastic representations for the purpose of parameter estimation. Results from experimentation demonstrate the robustness of the proposed CSNFA model in handling datasets with heavy-tailed distributions and outliers. Results demonstrate superior outlier detection performance of CSNFA, suggesting potential advantages over conventional FA and SNFA models regarding data calibration.

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Starlike function involving cardioid domain

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Abstract

In this present research article, we take an univalent function $\psi(z) = 1 + 5/7z + 3/5z^2$ which makes a cardioid domain. Using this specific function, the subclass, $S^*_{ca}(\psi)$ of starlike functions will be defined. For this class, our aim is to find some sharp bound of inequalities that consist of logarithmic coefficients.
Keywords: Univalent Functions, Starlike functions, Subordination, Cardioid.

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} \nu_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. The class of univalent function in $\mathcal{A}$ is denoted by $\mathcal{S}$. That is, f being univalent in Δ means mathematically that $f(z_1) = f(z_2)$ if $z_1 = z_2$ for $z_1, z_2 \in \Delta$ (see [1]).
Ma and Minda in [3] considered the class of Carathéodory functions in Δ with positive real part and taking the form of

$$p(z) = 1 + \sum_{k=1}^{\infty} \theta_k z^k. \tag{1.2}$$

The set of functions is denoted by $\mathcal{P}$. Carathéodory functions is symmetric with respect to real axis and normalized by the condition $p(0) = 1$, $p'(0) > 0$ and $Re\{p(z)\} > 0$.
For analytic functions f and g belonging to $\mathcal{A}$, we say that the function f is subordinate to the function g, and we write $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z)$. If $g(z)$ is univalent in Δ, then the subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ (see [4]).
The logarithmic coefficient Λ_k of f is defined by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{k=1}^{\infty} \Lambda_k z^k, \qquad \log 1 = 0. \tag{1.3}$$

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Making use of equation (1.1), for $f \in \mathcal{S}$, the logarithmic coefficients obtained by some calculation as follow

$$\Lambda_1 = \frac{1}{2}\nu_2, \tag{1.4}$$

$$\Lambda_2 = \frac{1}{2}\left(\nu_3 - \frac{1}{2}\nu_2^2\right), \tag{1.5}$$

$$\Lambda_3 = \frac{1}{2}\left(\nu_4 - \nu_2\nu_3 + \frac{1}{3}\nu_2^3\right), \tag{1.6}$$

$$\Lambda_4 = \frac{1}{2}\left(\nu_5 - \nu_2\nu_4 + \nu_2^2\nu_3 - \frac{1}{2}\nu_3^2 - \frac{1}{4}\nu_2^4\right). \tag{1.7}$$

In 1992 Ma and Minda [3] considered the subordination and introduced the class of starlike and convex functions as follow

$$S^*(p) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec p(z) \right\}.$$

In recent years, some researchers have been studied a number of different subclass of univalent functions in the context of Geometric function theory [3, 4].
The logarithmic coefficient Λ_k of these function satisfies the assertion

$$\sum_{n=1}^{\infty} |\Lambda_k|^2 \leq \frac{\pi^2}{6}.$$

Now, we define the function $\psi : D \to \mathbb{C}$ as follow

$$\psi(z) = 1 + \frac{7}{5}z + \frac{3}{5}z^2. \tag{1.8}$$

If $\psi(z_1) = \psi(z_2)$, we get $(z_1 - z_2)\,(7 - 3\,(z_1 - z_2)) = 0$. Then, $z_1 = z_2$. This show that the function $\psi(z)$ is univalent in Δ. The function $\psi(z)$ map the open disk Δ onto the domain consist of cardioid shaped region (see figure 1). It is clear that $Re(\psi(z)) > 0$.
Let $S^*_{ca}(\psi)$ to be the class of function $f : \Delta \to \mathbb{C}$ that satisfies

$$\frac{zf'(z)}{f(z)} \prec \psi(z) = 1 + \frac{7}{5}z + \frac{3}{5}z^2.$$

Thus, $f \in S^*_{ca}(\psi)$ if and only if

$$f(z) = z \exp \int_0^z \frac{\phi(\zeta) - 1}{\zeta} d\zeta,$$

where $\phi \prec \psi$.

2 Preliminaries Lemma

In the following Lemma, we are taken inequalities from [4, 5].

Lemma 2.1. *Let $0 \leq \tau \leq 1$ and $p \in \mathcal{P}$ and has the series representation (1.4). Then*

$$|\theta_{k+t} - \tau\theta_k\theta_t| \leq 2\max(1, |2\tau - 1|), \tag{2.1}$$

$$|\theta_k| \leq 2 \quad for \quad k \geq 1, \tag{2.2}$$

$$|\theta_{k+t} - \tau\theta_k\theta_t| \leq 2. \tag{2.3}$$

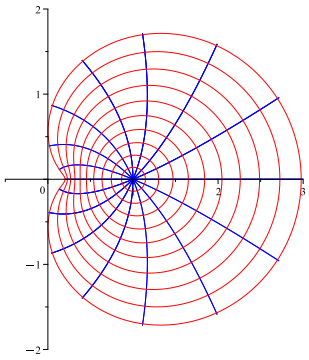


Figure 1: Image Δ under the function $\psi(z) = 1 + \frac{7z}{5} + \frac{3}{5}z^2$

Lemma 2.2. *Let $M, N, K \in N$ and $p \in \mathcal{P}$ and has the series representation (1.4). Then*

$$\frac{1}{2}|M\theta_1^3 - K\theta_1\theta_2 + N\theta_3| \leq |M| + |K-2| + |K-M+N|.$$

3 Coefficient inequalities

In this section determine the logarithmic bounds for $|\Lambda_k|$ in the class $S^*_{ca}(\psi)$.

Theorem 3.1. *Let $f(z)$ be the analytic function as in (1.1) and suppose $f \in S^*_{ca}(\psi)$. Then, we have*

$$|\Lambda_1| \leq \frac{7}{10}, \tag{3.1}$$

$$|\Lambda_2| \leq \frac{7}{20}, \tag{3.2}$$

$$|\Lambda_3| \leq \frac{29}{60}, \tag{3.3}$$

$$|\Lambda_4| \leq \frac{12399}{20000}. \tag{3.4}$$

Some inequalities obtained in Theorem 3.1 are sharp. For example, we have

$$f_1(z) = z\exp\int_0^z \left(\frac{7}{5} + \frac{3}{5}\zeta\right)d\zeta = z + \frac{7}{5}z^2 + ...,$$

$$f_2(z) = z\exp\int_0^z \left(\frac{7}{5}\zeta + \frac{3}{5}\zeta^3\right)d\zeta = z + \frac{7}{10}z^3 + ...,$$

$$f_3(z) = z\exp\int_0^z \left(\frac{7}{5}\zeta^2 + \frac{3}{5}\zeta^5\right)d\zeta = z + \frac{7}{15}z^4 + ...,$$

$$f_4(z) = z\exp\int_0^z \left(\frac{7}{5}\zeta^3 + \frac{3}{5}\zeta^7\right)d\zeta = z + \frac{7}{20}z^5 + ...$$

In view of figure 2, it is clear that the functions $f_i(z),\ i = 1, 2, 3, 4$ are starlike functions.

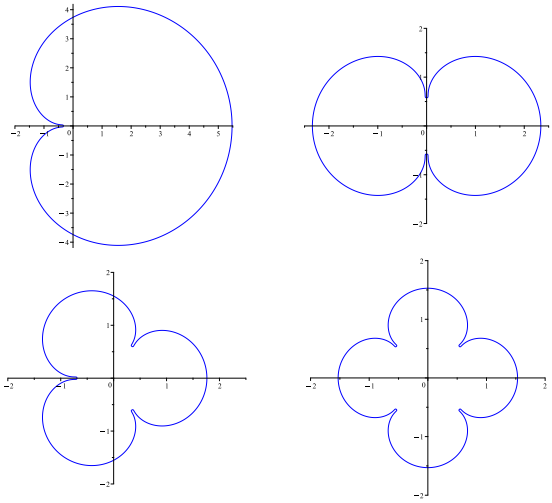


Figure 2: Image Δ under the functions $f_i(z),\ i = 1, 2, 3, 4$.

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Result on Differential subordination associated with integral operator

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Abstract

In this paper, we discuss various properties of analytic univalent functions by using differential subordination. We also introduced an integral operator by applying convolutions. Over and above, we derive interesting results involving differential subordination as well.

Keywords: Analytic functions, Subordination, Gaussian hypergeometric function, Loewner chain.

1 Introduction

Let $\mathcal{A}$ consisting of anlytic functions in the unite disc $D = \{z : z \in \mathbb{C}, |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Therefore, the function $f \in \mathcal{A}$ having the expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The class of univalent functions in $\mathcal{A}$ denoted by S. In particular, S^* is the class of starlike functions and K is the class of convex functions [1].
Let the function $f(z)$ given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. The convolution or Hadamard product of two functions $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in D.$$

The analytic function f is subordinate to analytic function g, written $f \prec g$, if there exist Schwartz function $w(z)$ in D, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, for $z \in D$ [2].
The function $L(z;t)$ with $z \in D$ and $t \geq 0$ is a subordination (or a loewner) chain, if $L(.,t)$ is analytic and univalent in D for all $t \geq 0$, and $L(z,.)$ is continuously differentiable on $\mathbb{R}^+$ for all $z \in D$, and $L(z;s) \prec L(z;t)$ for $0 \leq s \leq t$ [2].
Let $n \in \mathbb{N}$ and v is a positive integer. We define the integral operator $\mathfrak{I}_v^n : \mathcal{A} \to \mathcal{A}$ as follow

$$\mathfrak{I}_v^n f(z) = F_v^n(z) * f(z),$$

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where $F_v^n(z)$ is defined by the relation

$$F_v^n(z) * \frac{z}{(1-z)^{n+v}} = \frac{z}{(1-z)^v}.$$

So, If f is of the form (1.1), then

$$\mathfrak{I}_v^n f(z) = z + \sum_{k=2}^{\infty} \frac{(k)_n (v)_n}{(v+k-1)_n} a_k z^k, \qquad z \in D,\ n \in \mathbb{N}, \tag{1.2}$$

where $(\alpha)_k$ the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} 1, & k = 0, \\ \alpha(\alpha+1)...(\alpha+k-1), & k \in \mathbb{N}. \end{cases} \tag{1.3}$$

By simple computation, we obtain that $\mathfrak{I}_1^1 f(z) = f(z)$.
If $\beta, \gamma \in \mathbb{C}$ with $Re\{\beta\} > 0$ and $Re\{\gamma\} \geq 0$, the set $K_{\beta,\gamma}$ is defined in [2] as follow

$$K_{\beta,\gamma} = \begin{cases} H(D), & \text{if } \beta = 1,\ \gamma \neq 0, \\ \{f \in H(D) : f(0) = 0\}, & \text{if } \beta = 1,\ \gamma = 0, \\ \{f \in H(D) : f(z) = z^j h(z), h(z) \neq 0, z \in D, j \geq 1\}, & \text{if } \frac{1}{\beta} \in \mathbb{N} - \{1\}, \\ \{f \in H(D) : f(0) = 0, f'(0) \neq 0, Re\{\beta \frac{z f'(z)}{f(z)} + \gamma\} > 0, z \in D\}, & \text{otherwise}, \end{cases}$$

where $H(D)$ be a set of complex-valued functions on D.
Let a, b and c be complex number with $c \neq -1, -2, -3, ...$
The classical Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ is defined by

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.4}$$

where $(\alpha)_k$ is given by (1.3) [4].

2 Fundamental Lemma

The following lemmas are very useful in our investigation.

Lemma 2.1. *[1] Let the function $p \in \mathcal{P}$ be given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$$

Then the coefficient estimate is given by

$$|p_n| \leq 2, \qquad n \in \mathbb{N}.$$

Lemma 2.2. *[2] Let $t \geq 0$. The function*

$$L(z;t) = a_1(t)z + a_2(t)z^2 + ..., \quad a_1(t) \neq 0, \quad \lim_{t \longrightarrow \infty} |a_1(t)| = \infty$$

is a subordinate chain if

$$Re\{z \frac{\partial L/\partial z}{\partial L/\partial t}\} > 0, \quad z \in D, \quad t \geq 0.$$

Lemma 2.3. *[2] Let $\beta \in \mathbb{C}$ with $Re\{\beta\} > -\frac{1}{2}$, $h \in \mathcal{A}$ with $h'(0) \neq 0$ and*

$$J(\beta, z) = (\beta - 1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)}. \tag{2.1}$$

If

$$Re\ J(1, h) > -\frac{1}{2}, \qquad z \in D,$$

or

$$Re\ J(\beta, h(z) > -\frac{1}{2}, \quad \beta \neq 0, \ and\ h(0) = 0,$$

then the function $h(z)$ is univalent in D.

3 Subordination

In this section, we establish the following results, which gives a interesting differential subordination for defined analytic functions.

Theorem 3.1. *Let $\beta, \gamma \in \mathbb{C}$ with $Re\beta > 0$, $Re\gamma > 0$ and $v, n \in \mathbb{N}$. Let*

$$\delta = \min\{Re\gamma, \delta_0\},$$

where

$$\delta_0 = \frac{2Re\beta Re\gamma}{|\beta + \gamma| + |\beta - \bar{\gamma}|^2}.$$

If $\mathfrak{I}_v^n f(z), \mathfrak{I}_v^n g(z) \in K_{\beta,\gamma}$, with $g'(0) \neq 0$ and

$$Re\left[(\beta - 1)\frac{z\left(\mathfrak{I}_v^n g(z)\right)'}{\mathfrak{I}_v^n g(z)} + 1 + \frac{z\left(\mathfrak{I}_v^n g(z)\right)''}{\left(\mathfrak{I}_v^n g(z)\right)'}\right] > -\delta, \qquad z \in D.$$

Then $\mathfrak{I}_v^n f(z) \prec \mathfrak{I}_v^n g(z)$ implies that

$$I_{\beta,\gamma}^{n,\nu} f(z) \prec I_{\beta,\gamma}^{n,\nu} g(z), \qquad z \in D,$$

where the integral operator $I_{\beta,\gamma}^{n,\nu} f(z)$ is defined by

$$I_{\beta,\gamma}^{n,\nu} f(z) = \left[\frac{\beta + \gamma}{z^\gamma}\int_0^z \left(\mathfrak{I}_v^n f(z)\right)^\beta t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \tag{3.1}$$

and $\mathfrak{I}_v^n f(z)$ is given by (1.2).

Put $n = 1$ and $\nu = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $\beta, \gamma \in \mathbb{C}$ with $Re\beta > 0$, $Re\gamma > 0$ and let*

$$\delta = \min\{Re\gamma, \delta_0\},$$

where

$$\delta_0 = \frac{2Re\beta Re\gamma}{|\beta+\gamma|+|\beta-\bar{\gamma}|^2}.$$

If $f, g \in K_{\beta,\gamma}$, *with* $g'(0) \neq 0$ *and*

$$Re\left[(\beta-1)\frac{zg'(z)}{g(z)}+1+\frac{zg''(z)}{g'(z)}\right] > -\delta, \qquad z \in D,$$

then $f(z) \prec g(z)$ *implies that*

$$I_{\beta,\gamma}f(z) \prec I_{\beta,\gamma}g(z), \qquad z \in D.$$

where the integral operator $I_{\beta,\gamma}f(z)$ *is defined by*

$$I_{\beta,\gamma}f(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z (f(z))^\beta t^{\gamma-1}dt\right]^{\frac{1}{\beta}}. \tag{3.2}$$

Theorem 3.3. *Let* $\beta \in \mathbb{C}$ *with* $Re\{\beta\} > -\frac{1}{2}$ *and let* $\mathfrak{I}_v^n f(z)$ *be given by* (1.2). *If*

$$Re\{J(1, \mathfrak{I}_v^n f(z))\} > -\frac{1}{2}, \qquad z \in D,$$

or

$$Re\{J(\beta, \mathfrak{I}_v^n f(z))\} > -\frac{1}{2}, \qquad z \in D, \ \text{if } \beta \neq 1,$$

where $J(\beta, h)$ *is given by* (2.1). *Then* $\mathfrak{I}_v^n f(z)$ *is univalent in* D.

Theorem 3.4. *Let* $\beta > 0$, $\beta+\gamma > 0$ *and let*

$$-\beta < \delta \leq \min\left\{\frac{1}{2}, \gamma, \frac{\gamma-\beta+1}{2}\right\},$$

such that

$${}_2F_1\left(1, 2(\beta+\delta), \beta+\gamma+1; \frac{1}{2}\right) > 0,$$

and

$$\beta+\gamma \geq {}_2F_1\left(1, 2(\beta+\delta), \beta+\gamma+1; \frac{1}{2}\right),$$

where ${}_2F_1$ *is represented by* (1.4). *If* $\mathfrak{I}_v^n f, \mathfrak{I}_v^n g \in K_{\beta,\gamma}$ *with* $g'(0) \neq 0$ *and*

$$Re\left[(\beta-1)\frac{z\left(\mathfrak{I}_v^n g(z)\right)'}{\mathfrak{I}_v^n g(z)}+1+\frac{z\left(\mathfrak{I}_v^n g(z)\right)''}{\left(\mathfrak{I}_v^n g(z)\right)'}\right] > -\delta, \qquad z \in D.$$

Then $\mathfrak{I}_v^n f(z) \prec \mathfrak{I}_v^n g(z)$ *implies that*

$$I_{\beta,\gamma}^{n,\nu} f(z) \prec I_{\beta,\gamma}^{n,\nu} g(z), \qquad z \in D,$$

where the integral operator $I_{\beta,\gamma}^{n,\nu}$ *is defined in* (3.1).

Put $n = 1$ and $\nu = 1$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.5. *Let $\beta > 0$, $\beta + \gamma > 0$ and let*

$$-\beta < \delta \leq \min\left\{\frac{1}{2}, \gamma, \frac{\gamma - \beta + 1}{2}\right\},$$

such that

$${}_2F_1\left(1, 2(\beta + \delta), \beta + \gamma + 1; \frac{1}{2}\right) > 0,$$

and

$$\beta + \gamma \geq {}_2\ F_1\left(1, 2(\beta + \delta), \beta + \gamma + 1; \frac{1}{2}\right),$$

where ${}_2F_1$ is represented by (1.4). If $f, g \in K_{\beta,\gamma}$ with $g'(0) \neq 0$ and

$$Re\left[(\beta - 1)\frac{zg'(z)}{g(z)} + 1 + \frac{zg''(z)}{g'(z)}\right] > -\delta.$$

Then $f(z) \prec g(z)$ implies that

$$I_{\beta,\gamma} f(z) \prec I_{\beta,\gamma} g(z),$$

where the integral operator $I_{\beta,\gamma}$ is defined in (3.2).

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Sharp coefficients bounds for Starlike functions

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Abstract

This paper explores sharp coefficient bounds for starlike functions, specifically focusing on the subclass of Ma-Minda starlike functions. We introduce these functions through the generalized subordination condition and investigate their properties. Key results on the coefficient bounds and implications for the subclasses of univalent functions are presented, providing insights into the behavior of these analytic functions.

Keywords: Univalent functions, Starlike function, Hankel determinant, Logarithmic coefficients.

1 Introduction

The class $\mathcal{A}$ consists of analytic functions f in the open unit disk $\mathbb{D}$ that are normalized by the conditions $f(0) = 0$ and $f'(0) - 1 = 0$. Within $\mathcal{A}$, the subclass $\mathcal{S}$ includes functions that are univalent in $\mathbb{D}$.

An important concept is the subordination of analytic functions. Specifically, f is said to be subordinate to g (written $f \prec g$) if there exists another analytic function w mapping $\mathbb{D}$ into itself with $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent, then $f(0) = g(0)$ and the image $f(\mathbb{D})$ is contained within $g(\mathbb{D})$ (see [1]).

Ma and Minda [3], generalized several classes of starlike and convex functions by replacing the subordinate function $\frac{1+z}{1-z}$ with a more general function ψ that has a positive real part and is normalized by $\psi(0) = 1$, $\psi'(0) > 0$ and ψ maps $\mathbb{D}$ onto univalently a region starlike with respect to 1 and symmetric with respect to the real axis. They introduced the class:

$$\mathcal{ST}[\psi] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\}.$$

Functions in this class are referred to as Ma-Minda starlike functions. A specific case occurs for the Janowski starlike functions, denoted as $\mathcal{ST}[A, B]$, which are defined for parameters $-1 \leq B < A \leq 1$ using the function:

$$\psi(z) = \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D}.$$

The class $\mathcal{ST}(\beta)$ of starlike functions of order β $(0 \leq \beta < 1)$ defined by taking

$$\psi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad z \in \mathbb{D}$$

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Note that $\mathcal{ST} = \mathcal{ST}(0)$ is the classical class of starlike functions. Let $\mathcal{P}$ be a class of the analytic functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad z \in \mathbb{D}. \tag{1.1}$$

satisfying $\Re\{p(z)\} > 0$ in the unit disc $\mathbb{D}$. We are examining the function

$$\begin{aligned} \Psi(z) &:= -\frac{\ln(1-z)}{z} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n+1} \quad (|z| < 1) \end{aligned} \tag{1.2}$$

defined on the open unit disk $\mathbb{D}$. We introduce the class

$$\mathcal{ST}_H = \left\{ f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \prec \Psi(z) \right\}.$$

The graphic representation of $\Psi(\mathbb{D})$ 1 is provided below. The following lemmas is needed for the main results.

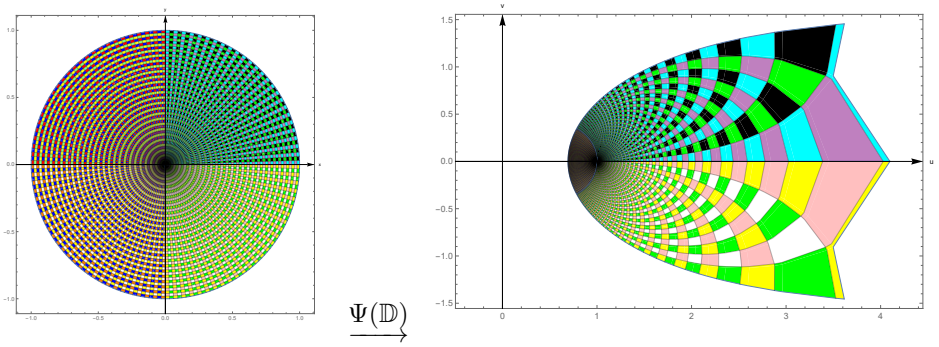


Figure 1: The image of $\mathbb{D}$ under Ψ.

Lemma 1.1. *[2] If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{1.3}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)\left(1 - |x|^2\right) y, \tag{1.4}$$

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Lemma 1.2. *[2] If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$, then*

$$|p_n| \leq 2 \quad (n \geq 1), \tag{1.5}$$

and if $Q \in [0, 1]$ and $Q(2Q - 1) \leq R \leq Q$, then

$$\left|p_3 - 2Qp_1 p_2 + Rp_1^3\right| \leq 2. \tag{1.6}$$

Also

$$\begin{aligned} |p_{n+k} - \mu p_n p_k| &\leq 2\max\{1, |2\mu - 1|\} \\ &= 2\begin{cases} 1, & \text{for } 0 \leq \mu \leq 1, \\ |2\mu - 1|, & \text{otherwise.} \end{cases} \end{aligned} \tag{1.7}$$

Lemma 1.3 ([4]). *Let $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and convex univalent in $\mathbb{D}$. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{D}$ and satisfies the subordination $p(z) \prec q(z)$, then*

$$|A_n| \leq |B_1| \qquad n = 1, 2, \ldots.$$

2 Main Results

Finding the upper bound for coefficients have been one of the central topic of research in geometric function theory as it gives several properties of functions. Therefore, we will be interested in the following problem in this section.

The Hankel determinants [1] $H_q(n)$ $(n = 1, 2, ...,\ q = 1, 2, ...)$ of the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

are defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$. For example, we know that the functional $H_2(1) := a_3 - a_2^2$ is known as the Fekete-Szegö functional and they consider the further generalized functional $a_3 - \mu a_2^2$, where μ is some real number. Estimating for the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem.

Associated with each $f \in \mathcal{S}$ (see [1]) is well defined function

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} 2\gamma_n z^n \qquad z \in \mathbb{D},$$

and γ_n are called logarithmic coefficients of the function f.

In this paper, we seek sharp upper bounds for the coefficients a_n $(n = 2, 3, 4, 5)$ and functional $H_2(2)$ for functions f belonging to the class $\mathcal{ST}_H$.

Theorem 2.1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{ST}_H$. Then*

$$|a_{n+1}| \leq \frac{1}{2n} \quad (n = 1, 2, 3, 4).$$

The first four results are sharp for the function $f : \mathbb{D} \to \mathbb{C}$ given by

$$\begin{aligned} f_n(z) &= z \exp\left(\int_0^z \frac{\Psi(t^n) - 1}{t} dt\right) \\ &= z \exp\left(\frac{1}{n} - \frac{\ln(1 - z^n)}{n} + \frac{\ln(1 - z^n)}{nz^n}\right) \\ &= z + \frac{z^{n+1}}{2n} + \frac{(4n + 3)z^{2n+1}}{24n^2} + \frac{\left(4n^2 + 4n + 1\right) z^{3n+1}}{48n^3} + \cdots. \end{aligned} \tag{2.1}$$

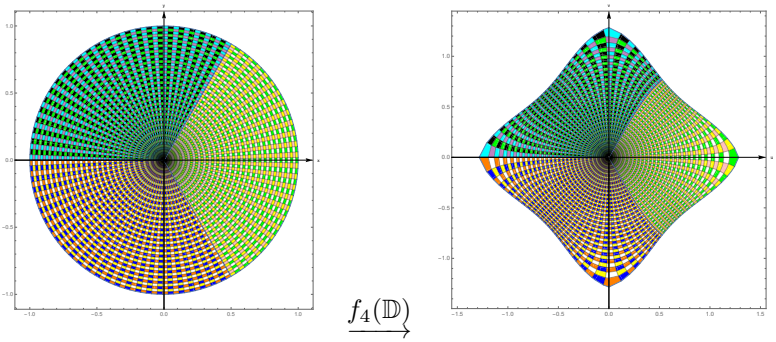


Figure 2: The image of $\mathbb{D}$ under $f_4(z)$.

Theorem 2.2. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{ST}_H$. Then, the following sharp estimates holds:*

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \max\left\{1, \left|\mu - \frac{7}{6}\right|\right\} \quad (\mu \in \mathbb{C}),$$

The inequalities are sharp for the functions

$$f(z) = \begin{cases} f_2(z) & for \quad |\mu - \frac{7}{6}| \leq 1, \\ f_1(z) & for \quad |\mu - \frac{7}{6}| > 1. \end{cases}$$

Theorem 2.3. *Let $f \in \mathcal{A}$. Then the logarithmic coefficients of f satisfy*

$$|\gamma_n| \leq \frac{1}{4n} \qquad n \geq 1.$$

All the inequalities are sharp for function f given by (2.1).

3 Conclusion

Conclusion of the paper should be short.

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An Enhanced Version of Becker's Univalence Condition

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Abstract

In this paper we have derived new variations of Becker's criteria for univalence in the open unit disk for the integral $\mathscr{H}_\beta(\xi)$, where the function h belongs to the class $\mathcal{A}$ and β is a complex number with $\Re(\beta) > 0$.

Keywords: Löewner chain, analytic function, integral operator, univalence criteria.

1 Introduction

Let $\mathcal{A}$ be the collection of analytic functions h in the open unit disk $\Delta = \{\xi \in \mathbb{C}: |\xi| < 1\}$ such that $h'(0) - 1 = h(0) = 0$. Let $\mathbb{C}_+$ refers to the set of all complex numbers ξ such that the real part of ξ is greater than 0.

A function h that belongs to the set of analytic functions $\mathcal{A}$ is said to be univalent if, for any $\xi_1, \xi_2 \in \Delta$ with $\xi_1 \neq \xi_2$, we have $h(\xi_1) \neq h(\xi_2)$. The subset of $\mathcal{A}$ consisting only of univalent functions h in Δ is represented by $\mathcal{S}$.

In the paper by Becker [1], it was demonstrated that if $0 < r \leq 1$ and h is an analytic function on the open unit disk Δ, which satisfies the inequality

$$\left(1 - |\xi|^2\right) \left|\frac{\xi h''(\xi)}{h'(\xi)}\right| \leq r \quad \text{for} \quad \xi \in \Delta,$$

then h is univalent in Δ.

An extension of Becker's criterion, developed by N. N. Pascu [2] guarantees the univalence of an integral operator.

Theorem 1.1. *Let $\beta \in \mathbb{C}_+$. If $h \in \mathcal{A}$ satisfy the following inequality:*

$$\frac{1 - |\xi|^{2\Re(\beta)}}{\Re(\beta)} \left|\frac{\xi h''(\xi)}{h'(\xi)}\right| \leq 1 \quad \textit{for} \quad \xi \in \Delta,$$

then the integral operator

$$\mathscr{H}_\beta(\xi) = \left[\beta \int_0^\xi \tau^{\beta-1} h'(\tau) d\tau\right]^{1/\beta} \tag{1.1}$$

is analytic and univalent in Δ.

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V. Pescar [3] obtain an univalence criterion which is a generalization of Ahlfors' and Becker's criterion of univalence and is given in the following theorem.

Theorem 1.2. *Let $\beta \in \mathbb{C}_+$ and $c \in \overline{\Delta}\backslash\{-1\}$. If $h \in \mathcal{A}$ satisfy the following inequality:*

$$|c||\xi|^{2\Re(\beta)} + \frac{1-|\xi|^{2\Re(\beta)}}{\Re(\beta)}\left|\frac{\xi h''(\xi)}{h'(\xi)}\right| \leq 1 \quad for \quad \xi \in \Delta,$$

then the function $\mathscr{H}_\beta(\xi)$ defined by (1.1) belongs the $\mathcal{S}$.

In this paper, we will propose a criterion for determining the univalence of the integral function defined by equation (1.1). Recently, Masih et al. obtained results related to the univalence of functions $h \in \mathcal{A}$ belonging to the class of analytic functions.

The following theorem [4] establishes conditions for the univalence of the function $\mathscr{H}_\beta(\xi)$ defined by (1.1) under certain constraints:

Theorem 1.3. *Let $\alpha, \beta \in \mathbb{C}_+$ such that $\Re(\beta(1+\alpha)) > 0$ and $c \in \overline{\Delta}\backslash\{-1\}$. If*

$$\left|c + \frac{1-\alpha}{2}\right| < \left|\frac{1+\alpha}{2}\right| \tag{1.2}$$

and $h \in \mathcal{A}$ satisfy the following inequality:

$$\left|c|\xi|^{\beta(1+\alpha)} + \frac{1-|\xi|^{\beta(1+\alpha)}}{\beta}\cdot\frac{\xi h''(\xi)}{h'(\xi)} + \frac{1-\alpha}{2}\right| \leq \left|\frac{1+\alpha}{2}\right| \quad for \quad \xi \in \Delta\backslash\{0\}, \tag{1.3}$$

then the function $\mathscr{H}_\beta(\xi)$ defined by (1.1) belongs the $\mathcal{S}$.

With the above theorem, we can now apply both the previously mentioned univalence criteria as well as new criteria in our analysis.

2 Particular cases

In the following, we first state the main univalent condition involving the integral operator given by (1.1).

Theorem 2.1. *Let $\alpha, \beta \in \mathbb{C}_+$ such that $\Re(\beta(1+\alpha)) > 0$ and $c \in \overline{\Delta}\backslash\{-1\}$. If*

$$\left|c + \frac{1-\alpha}{2}\right| < \left|\frac{1+\alpha}{2}\right| \tag{2.1}$$

and $h \in \mathcal{A}$ satisfy the following inequality:

$$|c||\xi|^{\Re(\beta(1+\alpha))} + |1+\alpha|\cdot\frac{1-|\xi|^{\Re(\beta(1+\alpha))}}{\Re(\beta(1+\alpha))}\cdot\left|\frac{\xi h''(\xi)}{h'(\xi)}\right| + \left|\frac{1-\alpha}{2}\right| \leq \left|\frac{1+\alpha}{2}\right|. \tag{2.2}$$

Then the function $\mathscr{H}_\beta$ defined by (1.1) belongs to the class $\mathcal{S}$.

By applying Theorem 2.1, we can deduce the following corollaries.

Corollary 2.2. *Let α be a positive real number and β be a complex number with $\Re(\beta) > 0$. If $h \in \mathcal{A}$ satisfies the inequality for all $\xi \in \Delta\backslash\{0\}$:*

$$\frac{1-|\xi|^{(1+\alpha)\Re(\beta)}}{\Re(\beta)}\cdot\left|\frac{\xi h''(\xi)}{h'(\xi)}\right| \leq \begin{cases}\alpha & : \quad 0 < \alpha \leq 1,\\ 1 & : \quad \alpha \geq 1,\end{cases}$$

then the function $\mathscr{H}_\beta(\xi)$ defined by (1.1) belongs to the class $\mathcal{S}$.

Theorem 2.3. *Let α be a positive real number and β be a complex number with $\Re(\beta) > 0$ and $c \in \Delta$. If $h \in \mathcal{A}$ satisfies the inequality for all $\xi \in \Delta$:*

$$\left|\frac{\xi h''(\xi)}{h'(\xi)}\right| \leq \frac{F(\alpha,\beta)}{1+\alpha},$$

where

$$F(\alpha,\beta) := \begin{cases} \alpha[1+(1+\alpha)\Re(\beta)]^{\frac{1+(1+\alpha)\Re(\beta)}{(1+\alpha)\Re(\beta)}} & \text{for} \quad 0<\alpha\leq 1, \\ [1+(1+\alpha)\Re(\beta)]^{\frac{1+(1+\alpha)\Re(\beta)}{(1+\alpha)\Re(\beta)}} & \text{for} \quad \alpha \geq 1, \end{cases} \tag{2.3}$$

then the function $\mathscr{H}_\beta$ defined by (1.1) belongs to the class $\mathcal{S}$.

Corollary 2.4. *In Theorem 2.3, we observe that the following maximum and minimum values hold:*

1. $$\max_{\alpha>0}\frac{F(\alpha,1)}{1+\alpha} = \begin{cases} \max\limits_{0<\alpha\leq 1}\dfrac{\alpha(2+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{1+\alpha} \\ \max\limits_{\alpha\geq 1}\dfrac{(2+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{1+\alpha} \end{cases} = \frac{3\sqrt{3}}{2}$$

2. $$\min_{\Re(\beta)>0}\frac{1}{2}F(1,\beta) = \frac{e}{2}.$$

3. *If m is a natural number, then*

$$\min_{\Re(\beta)\geq m}\frac{1}{2}F(1,\beta) = \frac{(1+2m)\sqrt[2m]{1+2m}}{2}.$$

Example 2.5. Suppose that β is a complex number satisfying $0 < \Re(\beta)$ and $|\beta| \leq e/2$. For $\xi \in \Delta$, consider the function

$$h_\beta(\xi) = \frac{e^{\beta\xi}-1}{\beta}.$$

Then, the function (see Figure 1)

$$\mathscr{H}_\beta(\xi) = \left[\beta\int_0^\xi \tau^{\beta-1}e^{\beta\tau}d\tau\right]^{1/\beta} = z\left[\frac{\beta}{1+\beta}{}_1F_1(\beta,1+\beta;\beta\xi)\right]^{1/\beta},$$

where ${}_1F_1(\cdot,\cdot;\cdot)$ denotes a confluent (or Kummer) hypergeometric function, belongs to the class $\mathcal{S}$.

Theorem 2.6. *Let $f \in \mathcal{A}$, β be a complex number such that $\Re(\beta) \geq m$ and λ be a real number such that $\lambda > 1$. If*

$$|\xi f'(\xi)| \leq \lambda \tag{2.4}$$

and

$$\lambda|\beta| \leq \frac{(1+2m)\sqrt[2m]{1+2m}}{2}, \tag{2.5}$$

then the function

$$F_\beta(\xi) = \left[\beta\int_0^\xi \tau^{\beta-1}\left(e^{f(\tau)}\right)^\beta d\tau\right]^{1/\beta} \tag{2.6}$$

is in the class $\mathcal{S}$.

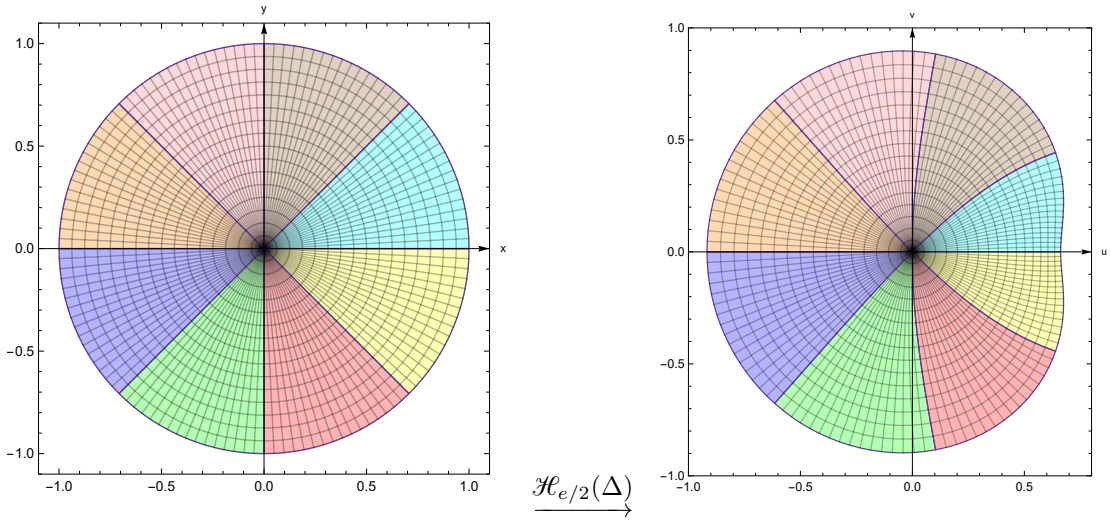


Figure 1: The image of Δ under $\mathscr{H}_\beta(\xi)$ for $\beta = e/2$.

Theorem 2.7. *Let $f \in \mathcal{A}$, β be a complex number such that $\Re(\beta) \geq m$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(1+2m)\sqrt[2m]{1+2m}}{2|\beta|}, \tag{2.7}$$

then the function

$$M_\beta(\xi) = \left[\beta \int_0^\xi \frac{(f(\tau))^\beta}{\tau} d\tau\right]^{1/\beta} \tag{2.8}$$

is in the class $\mathcal{S}$.

Theorem 2.8. *Let $f \in \mathcal{A}$, β be a complex number such that $\Re(\beta) > 0$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{e}{2|\beta|}, \tag{2.9}$$

then the function $M_\beta(\xi)$ given by (2.8) is in the class $\mathcal{S}$.

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Multi-objective Fractional Programming with Fuzzy Interval Coefficients

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Abstract

This paper presents a methodology for solving Multi-objective Fractional Programming (MOFP) problems where the objective functions involve fractional forms and coefficients are represented as fuzzy intervals. These fuzzy interval coefficients model the uncertainty and imprecision commonly found in real-world data. The proposed approach first transforms the fuzzy interval fractional objectives into equivalent crisp forms by applying defuzzification techniques, followed by solving the resulting multi-objective problem using conventional optimization techniques. A numerical example illustrates the step-by-step application of the method. The results show the feasibility and computational effectiveness of the proposed strategy in handling such complex problems.

Keywords: Fractional programming; Fuzzy set; Interval-value fuzzy number.

1 Introduction

In real-world decision-making scenarios, conflicting objectives and imprecise data frequently appear together. Multi-objective programming (MOP) has emerged as a vital tool for balancing trade-offs among several objectives. When uncertainty in parameters exists, fuzzy logic, introduced by Zadeh [3] , offers a robust modeling framework. Bellman and Zadeh[1] extended this to decision-making under fuzziness. Zimmermann[4] later integrated fuzzy logic with multi-objective programming. While traditional MOP handles linear or nonlinear objectives with crisp data, Multi-objective Fractional Programming (MOFP) addresses ratios of functions, providing a more flexible and meaningful representation in economics, engineering, and management. However, fractional models with fuzzy interval coefficients pose additional computational challenges. To overcome these, this paper presents a systematic transformation of the fuzzy fractional model into an equivalent crisp Multi-objective Linear Programming (MOLP) format using defuzzification techniques, followed by optimization.

2 Methodology

The method developed in this study is designed to handle fractional objective functions with fuzzy interval coefficients. It includes the following main steps:

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1. Formulation of MOFP with Fuzzy Interval Coefficients: Each objective function in the problem is a fractional expression whose numerator or denominator contain fuzzy interval coefficients. The constraints are also expressed with fuzzy intervals.
2. Defuzzification of Fuzzy Interval Coefficients: To make the problem solvable by classical optimization tools, fuzzy intervals are converted into crisp numbers. The most common method is the -cut technique, which considers intervals with a given membership level . Other techniques like centroid-based and possibility-based ranking may also be used.

Definition 1. *Let* $\tilde{A} = [(p, q, s, r; \lambda), (e, q, s, h; \rho)] \in F_{IV}(\lambda, \rho)$. *The signed distance of* $\tilde{A}$ *from* $\tilde{0}$ *is*

$$\begin{aligned} d(\tilde{A}, \tilde{0}) &= \frac{1}{\lambda}\int_0^{\lambda} d([A_l^U(\alpha)_\alpha, A_l^L(\alpha)_\alpha] \cup [A_r^L(\alpha)_\alpha, A_r^U(\alpha)_\alpha], \tilde{0})d\alpha \\ &+ \frac{1}{\rho-\lambda}\int_{\lambda}^{\rho} d([A_l^U(\alpha)_\alpha, A_r^U(\alpha)_\alpha], \tilde{0})d\alpha \\ &= \frac{1}{8}\left(3(q+s) + p + r + 4e + 4h + 3(q+s-e-h)\frac{\lambda}{\rho}\right) \end{aligned}$$

The trapezoidal fuzzy representation of intervals $\tilde{A}_i^t = [\tilde{A}_i^{tL}, \tilde{A}_i^{tU}]$ and $\tilde{B}_i^t = [\tilde{B}_i^{tL}, \tilde{B}_i^{tU}]$ for all $i = 1, 2, ..., n$ and $t = 1, 2, ..., k$ in $(\lambda - \rho)$-level are as follows

$$\begin{aligned} \tilde{A}_i^{tL} &= (p_i^t, q_i^t, s_i^t, r_i^t; \lambda) \quad \text{and} \quad \tilde{A}_i^{tU} = (e_i^t, q_i^t, s_i^t, h_i^t; \rho) \\ \tilde{B}_i^{tL} &= (p_i^{'t}, q_i^{'t}, s_i^{'t}, r_i^{'t}; \lambda) \quad \text{and} \quad \tilde{B}_i^{tU} = (e_i^{'t}, q_i^{'t}, s_i^{'t}, h_i^{'t}; \rho) \end{aligned}$$

Such that $e_i^t < p_i^t < q_i^t < s_i^t < r_i^t < h_i^t$ and $e_i^{'t} < p_i^{'t} < q_i^{'t} < s_i^{'t} < r_i^{'t} < h_i^{'t}$.
Also $d(\tilde{A}, 0)$ are defined as below

$$d(\tilde{A}_i^t, \tilde{0}) = \frac{1}{8}\left(3(q_i^t + s_i^t) + p_i^t + r_i^t + 4e_i^t + 4h_i^t + 3(q_i^t + s_i^t - e_i^t - h_i^t)\frac{\lambda}{\rho}\right) \tag{2.1}$$

$$d(\tilde{B}_i^t, \tilde{0}) = \frac{1}{8}\left(3(q_i^{'t} + s_i^{'t}) + p_i^{'t} + r_i^{'t} + 4e_i^{'t} + 4h_i^{'t} + 3(q_i^{'t} + s_i^{'t} - e_i^{'t} - h_i^{'t})\frac{\lambda}{\rho}\right) \tag{2.2}$$

3. Transformation of the Fractional Objective: Using the Charnes-Cooper transformation or other linearization techniques, the fractional objectives are transformed into linear or non-fractional forms. This step is critical for ensuring that standard linear programming methods can be applied.
4. Formulation of the Equivalent MOLP: Once the fractional objectives are linearized and the fuzzy parameters are defuzzified, the problem is reduced to a Multi-objective Linear Programming (MOLP) problem.
5. Solution of MOLP: To solve the MOLFP problem, we firstly solve it's each objective function separately by any existing methods in the literature to obtain the optimal objective values corresponding to each $Z^t(X) = \frac{N_{Z^t(X)}}{D_{Z^t(X)}}$ for $t = 1, 2, ..., k$[2].

$$Max \quad \sum_{t=1}^{k} N_{Z^t(X)} - Z_{opt}^t D_{Z^t(X)} \qquad S.t \quad h_j(x) \leq 0 \qquad j = 1, ..., m \tag{2.3}$$

Now on solving the LP problem in (2.3). we get the final optimum solution to the problem.

3 Example Problem and Solution

To demonstrate the methodology, the authors present a multi-objective fractional programming problem with fuzzy interval coefficients. The fuzzy parameters are represented using LR fuzzy numbers. The original fuzzy objective functions are defined as: Let us consider MOLFP problem with (IV) fuzzy numbers as coefficients and three objective functions as follows

$$Max \quad Z_1(X) = [(0.5,1,1.5,2;0.6),(0.2,1,1.5,2.5;0.8)]\ x_1$$

$$Max \quad Z_2(X) = \frac{[(0.5,0.75,1.25,2;0.6),(0,0.75,1.25,2.5;08)]}{[(0.5,1,1.5,2;0.6),(0.2,1,1.5,2.5;0.8)]\ x_1 + [(0.5,0.75,1.25,2;0.6),(0,0.75,1.25,2.5;08)]}$$

$$Max \quad Z_3(X) = \frac{[(-1.5,-1,-0.5,0;0.6),(-2,-1,-0.5,0.5;08)]x_2}{[(0.5,1,1.5,2;0.6),(0.2,1,1.5,2.5;0.8)]\ x_1 - [(6,7,8,9;0.6),(5,7,8,9.5;08)]}$$

$$\begin{aligned}
&S.t\\
&[(-1.5,-1,-0.5,0;0.6),(-2,-1,-0.5,0.5;08)]x_1\\
&+[(2,2.5,3,3.5;0.6),(1.5,2.5,3,4;0.8)]x_2 \le [(4,6,6.5,7;0.6),(3,6,6.5,8;0.8)]\\
&[(0.5,1,1.5,2;0.6),(0.2,1,1.5,2.5;0.8)]\ x_1\\
&+[(2,2.5,3,3.5;0.6),(1.5,2.5,3,4;0.8)]x_2 \le [(10.5,11,12,13.5),(10,11,12,13.5;0.8)]\\
&[(0.5,1,1.5,2;0.6),(0.2,1,1.5,2.5;0.8)]\ x_1 \ge [(7,7.5,8,9;0.6),(6.5,7.5,8,9.5;0.8)]\\
&\quad x_1\ ,\ x_2 \ge 0
\end{aligned}$$

According to (2.1) and (2.2), we convert the interval-fuzzy number coefficients into crisp coefficients.

$$\begin{aligned}
&d([(0.5,1,1.5,2;0.6),(0.2,1,1.5,2.5;0.8)],\tilde{0}) = 2.54\\
&d([(0.5,0.75,1.25,2;0.6),(0,0.75,1.25,2.5;08)],\tilde{0}) = 2.17\\
&d([(-1.5,-1,-0.5,0;0.6),(-2,-1,-0.5,0.5;08)],\tilde{0}) = -1.5\\
&d([(6,7,8,9;0.6),(5,7,8,9.5;08)],\tilde{0}) = 14.89\\
&d([(2,2.5,3,3.5;0.6),(1.5,2.5,3,4;0.8)],\tilde{0}) = 5.5\\
&d([(4,6,6.5,7;0.6),(3,6,6.5,8;0.8)],\tilde{0}) = 11.98\\
&d([(10.5,11,12,13.5),(10,11,12,13.5;0.8)],\tilde{0}) = 23.17\\
&d([(7,7.5,8,9;0.6),(6.5,7.5,8,9.5;0.8)],\tilde{0}) = 15.67
\end{aligned}$$

on using the definition the above MOLFP is converted to the following MOLP.

$$\begin{aligned}
&Max \quad Z_1(X) = 2.54\ x_1\\
&Max \quad Z_2(X) = \frac{2.17}{2.54\ x_1 + 2.17}\\
&Max \quad Z_3(X) = \frac{-1.5x_2}{2.54\ x_1 - 14.89}\\
&S.t \quad -1.5x_1 + 5.5x_2 \le 11.98, \quad 2.54\ x_1 + 5.5x_2 \le 23.17, \quad 2.54\ x_1 \ge 15.67, \quad x_1, x_2 \ge 0
\end{aligned}$$

We can obtain the following optimum solutions on solving each of the objective functions indivisualy.

$$Z_1(opt) = 1.67\,,\ Z_2(opt) = 0.12\,,\ Z_3(opt) = 0$$

Now on using the method in (2.3) the above MOLP problem is finally converted to the following linear programming.

$$\begin{aligned} Max \quad & Z(X) = 2.24x_1 - 1.5x_2 + 1.9 \\ & S.t \\ & -1.5x_1 + 5.5x_2 \leq 11.98 \\ & 2.54\ x_1 + 5.5x_2 \leq 23.17 \\ & 2.54\ x_1 \geq 15.67 \\ & x_1\ ,\ x_2 \geq 0 \end{aligned}$$

After obtaining the decision variables $x_1 = 9.12$ and $x_2 = 0$ from the above linear programming and substitute them in $Z_1 = 23.16$, $Z_2 = 0.086$ and $Z_3 = 0$ we get the finall optimum solution to the problem for each objective functions seperately.

4 Discussion

The proposed defuzzification and transformation approach ensures robustness and flexibility in solving MOLFP problems. Nonetheless, the choice of defuzzification method influences the solution quality and needs careful consideration. The paper also notes that extending the method to large-scale problems may require hybrid or metaheuristic methods.

5 Conclusion and Future Work

This study successfully develops a method to address fuzzy interval multi-objective fractional programming by reducing it to an equivalent crisp MOLFP model. The approach simplifies solving such problems and enables use of existing linear programming solvers. Future work may explore adaptive fuzzy ranking strategies or integration with artificial intelligence techniques to enhance scalability and accuracy.

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Some remarks on isosimple and co-isosimple modules

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Abstract

A non-zero module M is called isosimple (resp., co-isosimple), if it is isomorphic to all its nonzero submodules (resp., quotients). We show that any isosimple (resp., co-isosimple) module is a critical (resp., an atomic) module. As an application of this result, we show that every isoartinian module with finite Goldie dimension is λ-finitely embedded for some ordinal number λ. Finally, we study morphic isosimple (resp., co-isosimple) modules.
Keywords: Isosimple modules, Isoartinian modules, Co-isosimple modules, λ-finitely embedded modules, Morphic modules.

1 Introduction and preliminaries

Throughout in this paper all rings are associative and with unity and all modules are unital right modules. We begin with the central definition of this work.

Definition 1. *Let M be a non-zero module.*

- *([3]) M is called isosimple, if every non-zero submodule of M is isomorphic to M.*
- *([1]) M is called co-isosimple, if every non-zero factor module of M is isomorphic to M.*

A module M is called isoartinian, if for every descending chain $N_1 \supseteq N_2 \supseteq \cdots$ of submodules of M, there exists an index $k \geq 1$ such that N_k is isomorphic to N_i for every $i \geq k$. Also, a ring R is called right isoartinian, if R_R is isoartinian, see [3]. Note that any isoartinian module contains an isosimple submodule, see [3, Lemma 2.1]. Several authors have studied isosimple and co-isosimple modules. We recall some key properties of these modules, which will be used later (see [3, 1]).

Proposition 1.1. *1. Any is isosimple module is uniform and Noetherian.*

2. Any co-isosimple module is uniserial and Artinian.

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The notion of Krull dimension of a module M, denoted by M and quantifying its deviation from being Artinian, was first proposed by Gabriel and Rentschler (for finite ordinals) in 1967. The dual of this dimension, which measures the deviation of an R-module from Noetherianness was first introduced and studied by Lemonnier and called by him codeviation. Karamzadeh also extensively studied this dimension in his Ph.D. thesis and called it Noetherian dimension (denoted by (M)). It is convenient that, when we are dealing with Krull (resp., Noetherian) dimension, we may begin our list of ordinals with -1.

Definition 2. *An R-module M is called α-critical, where α is an ordinal, if $(M) = \alpha$ and $(\frac{M}{N}) < \alpha$ for any non-zero submodule N of M. M is said be critical if it is α-critical for some α.*
Dually, a module M is called α-atomic, if $(M) = \alpha$ and $(N) < \alpha$ for any proper submodule N of M. M is said be atomic if it is α-atomic for some α, see [5].

The following results are also needed, see [6, 5].

Proposition 1.2. *Let M be a module.*

1. *Every Noetherian module has Krull dimension.*
2. *Every Artinian module has Noetherian dimension.*
3. *If M has Krull dimension, then M has a non-zero critical submodule.*
4. *If M has Noetherian dimension, then M has a non-zero atomic factor module.*

Recall that a module M is called finitely embedded, briefly f.e., if its socle is essential and finitely generated. The authors of [4], introduced a generalization of finitely embedded modules as follows:

Definition 3. *Let M be an R-module. For each ordinal α, we define $S_\alpha = \bigoplus_{i\in I} C_i$, where $\{C_i\}_{i\in I}$ is a maximal independent set of α-critical submodules of M. Then S_α is called an α-critical socle of M. Now, a critical socle of M is defined to be a submodule S of M with $S = \sum_{\alpha<\lambda} S_\alpha$, where λ is the least ordinal such that each critical submodule is α-critical for some $\alpha \leq \lambda$. If for some ordinal α, there is no α-critical submodule, then we put $S_\alpha = (0)$. Now, M is called λ-finitely embedded (briefly, λ-f.e.), if λ is the least ordinal such that each critical submodule of M is α-critical for some $\alpha \leq \lambda$ and M contains a finitely generated essential critical socle. Note that 0-f.e. modules are precisely f.e. modules.*

Definition 4. *An R-module M is called prime, in case $M \neq 0$ and $\operatorname{ann}_R(M) = \operatorname{ann}_R(N)$ for every nonzero submodule N of M. On the other hand, an R-module M will be called a coprime module provided $M \neq 0$ and $\operatorname{ann}_R(M) = \operatorname{ann}_R(M/N)$ for every proper submodule N of M.*

It is well-known that the annihilator of a prime module is a prime Ideal. Also, we have the following result.

Lemma 1.3. *([2, Lemma 2.2]) If M is a coprime module, then $\operatorname{ann}_R(M)$ is a prime ideal of R.*

2 Main results

In this section, we present the main results of this paper. It is well-known that any isosimple module is uniform. We begin by a generalization of this fact.

Proposition 2.1. *Every isosimple module is critical.*

Proof. Let M be an isosimple module. Then, by Proposition 1.1(1), M is Noetherian and so M has Krull dimension, by Proposition 1.2(1). It follows that M has a non-zero critical submodule N, by Proposition 1.2(3). But, by our assumption, M is isosimple and hence $N \cong M$. Consequently, M is a critical module. □

By [6, Lemma 6.2.12], in a particular case of the above proposition, we get:

Corollary 2.2. *Every isosimple module is uniform.*

By Proposition 1.2(3), every module with Krull dimension has a non-zero critical submodule. Although isoartinian modules need not have Krull dimension (even finite Goldie dimension), we prove in the next result they nevertheless satisfy this property.

Corollary 2.3. *Any isoartinian module contains a non-zero critical submodule.*

Proof. It follows by [3, Lemma 2.1] and the previous proposition. □

We note that isoartinian (resp., isosimple) modules are not f.e. in general. For example, $\mathbb{Z}_{\mathbb{Z}}$ is isosimple (in particular, isoartinian), but it is not finitely embedded. In the next result, we show that a wide class of isoartinian modules are λ-finitely embedded for some ordinal number λ.

Theorem 2.4. *Every isoartinian module with finite Goldie dimension (e.g., $\mathbb{Z}_{\mathbb{Z}}$) is λ-finitely embedded for some ordinal number λ.*

Proof. Let M be an isoartinian module. Then M contains a non-zero critical submodule, by Corollary 2.3. Hence, M contains a non-zero critical socle. Now, let N be a non-zero submodule of M. By Corollary 2.3, there exists a non-zero critical submodule C of M with $C \subseteq N$. Thus, we have $C \cap S \neq \emptyset$, so $N \cap S \neq \emptyset$. It follows that S is essential. On the other hand, since M has finite Goldie dimension, S must be finitely generated. Consequently, M is a λ-finitely embedded module for some ordinal number λ. □

Corollary 2.5. *Let R be a semiprime right isoartinian ring. The following are equivalent.*

1. *R is right Goldie.*
2. *R_R has finite Goldie dimension.*
3. *R is a finite direct sum of isosimple right ideals.*
4. *R is right Noetherian.*
5. *R_R is λ-f.e. for some ordinal number λ.*
6. *R_R has Krull dimension.*

Proof. (1) ⇒ (2) Clear.
(2) ⇒ (3) Follows from [3, Theorem 4.6].
(3) ⇒ (4) Since any isosimple module is Noetherian, it is clear.
(4) ⇒ (5) Let R be a right Noetherian ring. Then R_R has finite Goldie dimension, so the result follows from the previous theorem.
(5) ⇒ (6) Follows from [4, Theorem 2.18].
(6) ⇒ (1) It is well-known that any semiprime ring with right Krull dimension is right Goldie, see [6, Proposition 6.3.5]. □

In the rest of the paper, we provide some results about co-isosimple modules. We begin by a dual version of Proposition 2.1.

Proposition 2.6. *Every co-isosimple module is atomic.*

Proof. Let M be a co-isosimple module. Then, by Proposition 1.1(2), M is Artinian and so M has Noetherian dimension, by Proposition 1.2(2). It follows that M has a non-zero atomic factor module $\frac{M}{N}$, by Proposition 1.2(4). But, by our assumption, M is co-isosimple and hence $M \cong \frac{M}{N}$. Consequently, M is an atomic module. □

Corollary 2.7. *Let R be a ring. The following are equivalent.*

1. *R_R is co-isosimple.*
2. *R_R is atomic.*
3. *R is a division ring.*

Proof. It follows by the previous proposition and [5, Corollary 2.3]. □

For the next result, we need the following elementary lemma.

Lemma 2.8. *Let M, N be two R-modules. If $M \cong N$, then $\operatorname{ann}_R(M) = \operatorname{ann}_R(N)$.*

Proposition 2.9. *Every co-isosimple module is a coprime module.*

Proof. Let N be a proper submodule of M. Then $M \cong M/N$. Thus, $\operatorname{ann}_R(M) = \operatorname{ann}_R(M/N)$, by the above lemma. Hence, M is a coprime module. □

By a similar argument, we can show that any isosimple module is a prime module.

Corollary 2.10. *If M is a co-isosimple (resp., an isosimple) R-module, then $\operatorname{ann}(M)$ is a prime ideal of R.*

Proof. Follows by the previous proposition and Lemma 1.3. □

According to [7], a module M is called morphic, if $\frac{M}{\varphi(M)} \cong \ker(\varphi)$ for every endomorphism φ of M.

Lemma 2.11. *([7, Theorem 5]) The following are equivalent for a module M:*

1. *M is morphic.*
2. *If $\frac{M}{K} \cong N$ where K and N are submodules of M, then $\frac{M}{N} \cong K$.*

Proposition 2.12. *Let M be a morphic R-module. The following statements hold.*

1. *M is isosimple if and only if it is simple.*
2. *M is simple, or M is not a co-isosimple module. In other words, any co-isosimple non-simple module is never a morphic module.*

Proof. (1) Suppose that M is isosimple and N is a non-zero submodule of M. Then $N \cong M \cong \frac{M}{0}$. Now, in view of the previous lemma, we have $\frac{M}{N} \cong 0$ that is $N = M$, which implies that M is simple. The converse is clear.
(2) Assume that M is a non-simple module. We have to show that M is co-isosimple. If M is zero, then we are done. So, we may suppose that there exists a non-zero proper submodule N of M. Now, if $\frac{M}{N} \cong M$, then we have $\frac{M}{M} \cong N$ by the previous lemma. This implies $N = 0$, which is a contradiction. Hence, M is not co-isosimple. □

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Nonparametric Bayesian optimal designs for exponentiated Shanker regression model

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Abstract

Nonlinear regression models find extensive application across numerous scientific fields. Accurately fitting an optimal nonlinear model is essential, particularly with consideration for biases in Bayesian optimal design. This study presents optimal designs for exponentiated Shanker regression model, utilizing both Bayesian and Nonparametric Bayesian methods.
Keywords: D-optimal design, Bayesian optimal design, information matrix, Dirichlet process, Nonparametric Bayesian.

1 Introduction

As discussed in our previous work [28], in experimental design, the term "optimal design" refers to a specific class of designs categorized according to particular statistical criteria. It is widely acknowledged that a well-designed experiment can significantly enhance the accuracy of statistical analyses, specifically the accuracy of the parameters corresponding to the size of the parametric confidence region. Consequently, numerous researchers have dedicated their efforts to address the challenge of constructing optimal designs for nonlinear regression models. Experimental design plays a pivotal role in scientific research domains, including but not limited to biomedicine and pharmacokinetics. Its application in these fields enables researchers to conduct rigorous investigations and yield valuable insights.

Optimal designs are sought using optimality criteria, typically based on the information matrix. In nonlinear models, the presence of unknown parameters introduced complexities in the design problem, as the optimality criteria depends on these unknown parameters [3, 6]. To address this challenge, researchers proposed various solutions, including local optimal designs [1, 8,19, 31], sequential optimal designs, minimax optimal designs, Bayesian optimal designs [29, 21-25], and pseudo-Bayesian designs [26]. Chernoff [8] proposed to use a suitable guess of the parameter value which leads to locally optimal design. This approach aimed to overcome the difficulties associated with the dependence of the design problem on unknown

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parameters in nonlinear models. It's important to note that local designs for nonlinear models are derived subsequent to an initial linearization of the model, using the parameter set as a reference point.

The selection of unknown parameters in local designs is typically obtained from previous studies or experiments specifically conducted for this purpose. The sensitivity of the locally optimal design with respect to the initial guess of parameter value is demonstrated by Dette, Melas, and Shpilev [12]. A locally optimal design do not incorporate any uncertainty in the parameter values. One of the proposed approaches, which deals with the uncertainty in parameter values, is the Bayesian procedure that assumes a prior distribution for the parameter. In the Bayesian method, the first step is to represent the available information in the form of a probability distribution for the model parameter, known as the prior distribution. So that, the idea of the Bayesian optimal design is to use the informative and/or historical knowledge of the unknown parameters as prior distribution.A Bayesian optimal design aims to maximize the relevant optimality criterion over this prior distribution. Nevertheless, it is crucial to acknowledge that the selection of the prior distribution within the Bayesian framework can be problematic and may potentially lead to erroneous results. The choice of the prior distribution is subjective, relying on the researcher's beliefs, and it significantly influences the final outcome. Unfortunately, the Bayesian approach lacks a definitive method for selecting the prior distribution. Subjective specification of a prior distribution which captures all existing information about the uncertainty of the parameter values was employed by many authors such as Chaloner and Larentz [11], Chaloner and Duncan [9], Burghaus and Dette [5], Chaloner and Vardinelli [10], Pronzato and Walter [30], Mukhopadhyay and Haines [26], Dette and Ngobauer [13,14], Fedorov [15,16], and Firth and Hinde [18] that have contributed extensively to this field. Chapter 18 of Atkinson et al.'s book [3] provides further reading on this topic. Often the choice of the prior is a compromise of a representation of uncertainty of the parameter values that may not be correct and can lead to misleading decision. Moreover, using the custom priors is a restrictive choice of priors, and it is possible to distort the actual prior distribution of the parameter. Indeed, no single probability distribution can model ignorance satisfactory, therefore large classes of distributions are needed. In this regards, we show that such a class is attained by considering infinite dimensional families of prior distributions; We consider the prior distribution function as an unknown distribution, which belongs to a class of distribution functions. Thus, from a Bayesian point of view, we need to construct a prior distribution on the space of all distribution functions. Several methods were introduced to construct a prior for random distribution. In this paper, we apply a nonparametric Bayesian approach that puts a prior on such families. In nonparametric Bayesian analysis, the prior process in terms of a probability measure P instead of the corresponding distribution function is discussed. The Dirichlet process, which is defined by Ferguson [17], plays a central role in nonparametric Bayesian methods. Dirichlet process (DP) involved two desirable properties of a prior; First, DP is defined on an arbitrary probability space. Second, it belonged to a conjugate family of priors. The aim of this paper is to apply Dirichlet process priors to obtain the Bayesian D-optimal design.

We organized this paper as follows. In section 2, the exponentiated Shanker distribution is presented. In Section 3, the Introduction to optimal designs in nonlinear modelsl is presented. Also the Bayesian and nonparametric Bayesian D-optimal design is introduced in section 4. Finally, Section 5 concludes the paper with some closing remarks.

2 The exponentiated Shanker distribution

Modeling lifetime data is very important in many applied sciences like engineering, medicine, and finance. Recently, several lifetime distributions have been introduced, that have some advantages and disadvantages in modeling lifetime data.

The probability density function (pdf) and the cumulative distribution function (cdf) of the Shanker

distribution is as followes:

$$f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}, \qquad x>0, \quad \theta>0, \tag{2.1}$$

and

$$F(x;\theta) = 1 - \frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}, \qquad x>0, \quad \theta>0, \tag{2.2}$$

respectively.

Shanker showed that the relation (2.1) is a two-component mixture of an exponential distribution and a gamma distribution. Shanekr has also presented various mathematical and statistical characteristics of this distribution. The Shanker and Lindley distributions have increasing hrfs. The purpose of this paper is to introduce a new two-parameter distribution, called the exponentiated Shanker distribution, that can be established by the relationship (2.1) and has increasing, decreasing and bathtub-shaped hrfs and it is a very flexible model in lifetime phenomena. The exponentiated Shanker ("E-Sh" for short) distribution can work better than some other distributions for modeling lifetime data.

As discussed in our previous work if the random variable X has an E-Sh distribution, then its cdf is given by

$$F(x) = \left[1 - \frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha}, \qquad x>0, \quad \alpha,\theta>0. \tag{2.3}$$

The pdf of the E-Sh distribution is obtained by differentiating (3.1) and therefore we have

$$f(x) = \frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1 - \frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1}, \qquad x>0, \quad \alpha,\theta>0. \tag{2.4}$$

The hrf of the new distribution is given by

$$h(x) = \frac{\frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1 - \frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1}}{1-\left[1 - \frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha}}.$$

We plotted the pdfs of the new distribution for selected values of the parameters in Figure 1. We see that the pdf of this distribution can be either decreasing or unimodal depending on the values of the parameters. The plots of the hrfs of the E-Sh distribution for selected values of the parameters are given in Figure 2. From Figure 2, we observe that the hrf is decreasing, increasing or bathtub-shaped.

3 Introduction to optimal designs in nonlinear models

In the realm of nonlinear experimental design, a common scenario arises where in the relationship between the response variable y and the independent variable x is given by the equation $y = \eta(x,\boldsymbol{\theta}) + \epsilon$ where $x \in \chi \subseteq \mathrm{R}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is the unknown parameter vector and ϵ is a normally distributed residual value with mean 0 and known variance $\sigma^2 > 0$. For simplicity, we assume $\sigma^2 = 1$ in this problem. If $\eta(x,\boldsymbol{\theta})$ is differentiable with respect to $\boldsymbol{\theta}$ then, the information matrix $I(x,\boldsymbol{\theta})$ at a given point x can be represented as follows:

$$I(x,\boldsymbol{\theta}) = \frac{\partial}{\partial\boldsymbol{\theta}}\eta(x,\boldsymbol{\theta})\frac{\partial}{\partial\boldsymbol{\theta}^T}\eta(x,\boldsymbol{\theta}). \tag{3.1}$$

There exist several optimality criteria used to obtain the optimal design. the most common of which are D-optimality and A-optimality. These criteria are based on functions of the information matrix and can be expressed as follows:

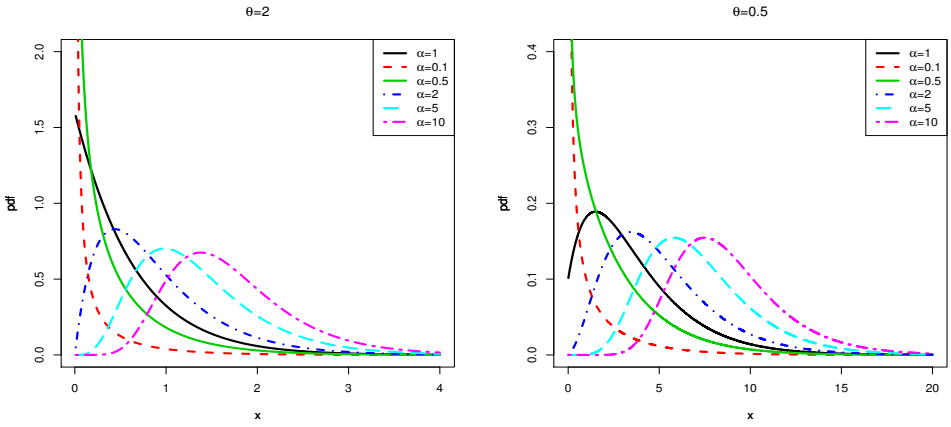


Figure 1: Pdfs of the E-Sh distribution for selected values of α and θ.

$$\Psi_D(\xi,\boldsymbol{\theta}) = -\log(\det(M(\xi,\boldsymbol{\theta}))) \ , \ \Psi_A(\xi,\boldsymbol{\theta}) = tr(M^{-1}(\xi,\boldsymbol{\theta})),$$

where ξ denotes a design with two components; the first component represents specific values from the design space χ while the second component corresponds to the weights assigned to these values, so that design ξ can be defined as follows:

$$\xi = \left\{ \begin{array}{cccc} x_1 & x_2 & \dots & x_\ell \\ w_1 & w_2 & \dots & w_\ell \end{array} \right\} \in \boldsymbol{\Xi}, \tag{3.2}$$

where

$\boldsymbol{\Xi} = \{\xi \mid 0 \leq w_j \leq 1; \sum_{j=1}^{\ell} w_j = 1, x \in \boldsymbol{\chi}\}$, $p \leq \ell \leq \dfrac{p(p+1)}{2}$[25].

For a discrete probability measure ξ with finite support, the information function of ξ can be expressed as follows [3]:

$$M(\xi,\boldsymbol{\theta}) = \sum_{j=1}^{\ell} w_j I(x_j,\boldsymbol{\theta}). \tag{3.3}$$

In generalized linear and non-linear models, due to the dependence of the information matrix $M(\xi,\boldsymbol{\theta})$ to the unknown parameter $\boldsymbol{\theta}$, one approach to address this issue is to employ the Bayesian method and incorporate a prior distribution of the parameter vector instead of the local method to achieve a better result. The Bayesian D-optimality criterion can be formulated as follows:

$$\Psi_\Pi(\xi) = E(\psi(\xi;\boldsymbol{\theta})) = \int_\Theta \psi(\xi;\boldsymbol{\theta}) d\Pi(\boldsymbol{\theta}) = \int_\Theta -\log(\det(M(\xi,\boldsymbol{\theta}))) d\Pi(\boldsymbol{\theta}), \tag{3.4}$$

where Π represents the prior distribution for $\boldsymbol{\theta}$ and the Bayesian D-optimal design is attained by minimizing (3.4). In certain situations, specifying a prior distribution on the parameter space $\boldsymbol{\Theta}$ can be challenging for the experimenter. In such cases, an alternative approach involves considering an unknown prior distribution Π for the parameter $\boldsymbol{\theta}$. In this condition, Π is treated as a parameter itself. Consequently, equation (3.4) becomes a random functional, and it becomes necessary to determine its distribution or

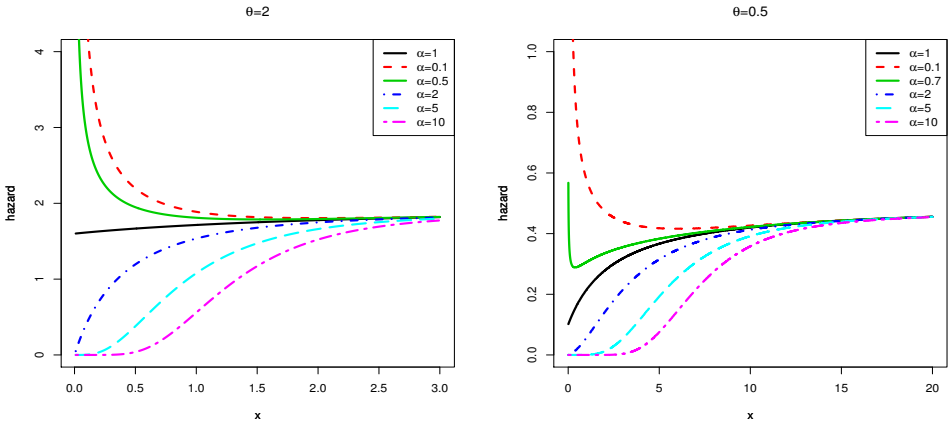


Figure 2: Hrfs of the E-Sh distribution for selected values of α and θ.

approximation. From a Bayesian perspective, we construct a prior distribution on the space of all distribution functions to address this issue. To achieve this objective, Ferguson introduced the concept of the Dirichlet Process (DP) [17].

3.1 Nonparametric Bayesian D-optimal design

Nonparametric models constitute an approach to model selection and fitting, where the size of the models is allowed to grow with the size of the data. It is unlike parametric models that use a fixed number of parameters. In this section, we introduce the nonparametric Bayesian optimal design. In the nonparametric Bayesian framework, it is assumed that $\theta \mid P \sim P$, where P is a random probability distribution and $P \sim \Pi$. General method of construction a random measure is to start with the stochastic processes. Ferguson [17] formulated the requirements which must be imposed on a prior distribution and proposed a class of prior distributions, named DP. One of the main argument in using the Dirichlet distribution in practical applications is based on the fact that this distribution is a good approximation of many parametric probability distributions. Below, the concept of the DP will be peresented.

3.1.1 Dirichlet Process

To have a random distribution G distributed according to a DP, its marginal distributions must follow a Dirichlet distribution. Specically, let H be a distribution over $\boldsymbol{\Theta}$ and α be a positive real number. For any finite measurable partition $A_1, A_2, ..., A_r$ of $\boldsymbol{\Theta}$ the vector $(G(A_1), G(A_2), ..., G(A_r))$ is random since G is random. We say G is DP distributed with base distribution H and concentration parameter α, written $G \sim \text{DP}(\alpha,H)$, if the following conditions hold:

$$(G(A_1), G(A_2), ..., G(A_r)) \sim Dir(\alpha H(A_1), ..., \alpha H(A_r)), \tag{3.5}$$

for every finite measurable partition $A_1, A_2, ..., A_r$ of $\boldsymbol{\Theta}$. The parameters of H and α play intuitive roles in the definition of the DP. The base distribution H represents the mean of the DP, such that for any measurable set $A \subset \boldsymbol{\Theta}$ we have $E[\text{G(A)}] = H(A)$. On the other hand, the concentration parameter α can be viewed as an inverse variance: $V[G(A)]=H(A)(1 - H(A))/(\alpha + 1)$. The larger α is, the smaller the

variance, and the DP will concentrate more of its mass around the mean. The concentration parameter is also referred to as the strength parameter, referring to the strength of the prior when using the DP as a nonparametric prior in Bayesian nonparametric models, It can be interpreted as the amount of mass or sample size associated with the observations. It is worth noting that α and H only appear as their product in the definition of the DP (equation 3.5). Consequently, some authors treat $\tilde{\mathrm{H}}$=αH, as the same as the single (positive measure) parameter of the DP, writing DP($\tilde{\mathrm{H}}$) instead of DP(α, H). This parametrization can be notationally convenient, but loses the distinct roles that α and H play in describing the DP.

As the concentration parameter α increases, the mass of the DP becomes more concentrated around its mean. Consequently, when α approaches infinity ($\alpha \to \infty$), $G(A)$ approaches $H(A)$ for any measurable set A, indicating weak or pointwise convergence of G to H. However, it's important to note that this does not imply a direct convergence of G to H as a whole. In fact, as we will explore later, samples drawn from a DP will typically be discrete distributions with probability one, even if the base distribution H is smooth. Therefore, G and H may not be absolutely continuous with respect to each other [31]. Despite this, some authors still utilize the DP as a nonparametric extension of a parametric model represented by H. However, if one aims at to assure the smoothness, it is possible to extend the DP by convolving G with kernels, resulting in a random distribution with a density function. An alternative definition of the DP is proposed by Ferguson [17] that defined a random probability measure which is a DP on ($\boldsymbol{\Theta}$, B($\boldsymbol{\Theta}$)), as:

$$P(.) = \sum_{i=1}^{\infty} p_i \delta_{\theta_i}(.), \tag{3.6}$$

where θ_i ($i > 1$) is a sequence of *i.i.d.* random variables with common distribution Q, δ_{θ_i} represents a probability measure that is degenerate at θ that δ_{θ_i}=1 if $\theta_i \in A$ and 0 otherwise, and p_i's are the random weights satisfying $p_i \neq 0$ and $\sum_{i=1}^{\infty} p_i$=1. The random distribution P is discrete with probability one. Several authors have proposed alternative series representations of the DP. Bondesson [7], Sethuraman [32], and Zarepour and Al Labadi [33] are among those who have contributed to this area.

4 Bayesian and Nonparametric Bayesian D-optimal design for exponentiated Shanker regression model

In the following section, we will discuss Bayesian and Nonparametric Bayesian D-optimal designs for exponentiated Shanker regression model. Bayesian optimal designs for the exponentiated Shanker regression model with a truncated normal distribution as Prior Distribution focus on developing an efficient sampling strategy to best estimate model parameters. In this context, "optimal design" aims to select data points or intervals that maximize the amount of information gathered about the model while minimizing effort or resources required.
Suppose we have the following regression model:

$$E(y|x) = \eta(x;\theta,\alpha) = \frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1 - \frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1}, \qquad x>0, \theta>0, \alpha>0. \tag{4.1}$$

The Bayesian optimal designs are found by optimizing an optimality criterion averaged over a user-specified prior distribution for the unknown parameters [9,10]. Strictly speaking, the latter are not fully Bayesian because they do not involve computing a posterior distribution. Instead, they borrow the concept of having prior distributions to find robust designs for the frequentists [5,19]. Accordingly, they are sometimes referred to as "pseudo" Bayesian designs [17]. In the optimal design literature, Bayesian optimal designs found under

Table 1: Bayesian D-optimal designs with respect to the truncated normal prior distribution for E-Sh regression model.

Parameters	*Design*	*One − point*	*Two − point*	
$\mu = 50$, $\sigma = 30$	x	1.00000	0.10526	1.0000
	w		0.00001	0.99999
$\mu = 150$, $\sigma = 90$	x	1.00000	0.10560	1.0000
	w		0.00001	0.99999
$\mu = 500$, $\sigma = 1000$	x	1.00000	0.10570	1.0000
	w		0.00001	0.99999

Table 2: Nonparametric Bayesian D-optimal designs with truncated normal base distribution and concentration parameter when α=1. (with stick breaking method)

Parameters	*Design*	*Two points*		*Three points*		
$\mu = 50$, $\sigma = 30$	x	0.09959735	0.9978526	0.09999	0.3250	0.98031
	w	0.00000004	0.99999996	0.000002	0.013006	0.986992
$\mu = 150$, $\sigma = 90$	x	0.0997456	1.000000	0.20015	0.30086	0.98031
	w	0.00000006	0.99999994	0.04428	0.08642	0.86930

Table 3: Nonparametric Bayesian D-optimal designs with truncated normal base distribution and concentration parameter when α=5. (with stick breaking method)

Parameters	*Design*	*Two points*		*Three points*		
$\mu = 50$, $\sigma = 30$	x	0.0991659	0.9995066	0.0999960	0.330326	1.00000
	w	0.00000004	0.99999996	0.000003	0.004786	0.995211
$\mu = 150$, $\sigma = 90$	x	0.0991451	1.0000000	0.099999	0.311274	0.985680
	w	0.00000006	0.99999994	0.035744	0.073102	0.891154

a discrete prior distribution are usually referred to as robust or optimum-on-average designs [14]. For an overview of optimal designs for nonlinear models, see [15,16].

In Table 1, the Bayesian D-optimal design is shown for the exponentiated Shanker regression model (4.1) with respect to a truncated normal prior distribution for different values of ℓ(ℓ is the number of design points) and for various parameter spaces [0, B] for a special case $\alpha = 1$. According to our results, the largest point has the greatest weight.

In the following, a nonparametric Bayesian optimal design is obtained for different selections of DP parameters. we consider the truncated normal distribution as the base measure in DP. We use the stick breaking [32] and the Zarepour [33] method respectively. Since the weights produced by the stick breaking process dont follow a decreasing trend, therefore, the DP can be simulated in a way where the weights are produced in a decreasing manner [33]. We tabulate the results for two different values of α=1, 5, 10. Without loss of generality, we consider a bounded design space χ=[0,1]. Tables 2-4 depict the results as the concentration parameter and uncertainty in the base measure increase. Based on the findings, within the investigated range, the results show that in the class of two-point designs, the largest weight corresponds to the largest point. Additionally, when considering the truncated normal distribution as the base measure, in the class of two-point designs, the weight of one point is zero and in fact a one-point design is obtained.

Table 4: Nonparametric Bayesian D-optimal designs with truncated normal base distribution and concentration parameter when α=10. (with stick breaking method)

Parameters	*Design*	*Two points*		*Three points*		
$\mu = 50$, $\sigma = 30$	x	0.0992247	1.0000000	0.0999985	0.3310138	1.0000000
	w	0.00000004	0.99999996	0.0000031	0.003340	0.9966559
$\mu = 150$, $\sigma = 90$	x	0.0989508	1.0000000	0.2000857	0.3064402	0.9789505
	w	0.00000006	0.99999994	0.0520376	0.1028740	0.8450884

Table 5: Nonparametric Bayesian D-optimal designs with truncated normal base distribution and concentration parameter when α=1. (with Zarepour method)

Parameters	*Design*	*Two points*		*Three points*		
$\mu = 50$, $\sigma = 30$	x	0.09955440	1.000000	0.09999	0.30467	1.00000
	w	0.00000004	0.99999996	0.000002	0.000088	0.99999
$\mu = 150$, $\sigma = 90$	x	0.0992588	1.000000	0.10000	0.30006	0.99069
	w	0.00000006	0.99999994	0.02670	0.05232	0.92098

Table 6: Nonparametric Bayesian D-optimal designs with truncated normal base distribution and concentration parameter when α=5. (with Zarepour method)

Parameters	*Design*	*Two points*		*Three points*		
$\mu = 50$, $\sigma = 30$	x	0.09918473	0.9995066	0.099995	0.33769	1.000000
	w	0.00000004	0.99999996	0.000003	0.007772	0.992225
$\mu = 150$, $\sigma = 90$	x	0.09910911	1.0000000	0.0999993	0.3080848	0.983440
	w	0.00000006	0.99999994	0.035649	0.073140	0.891211

Table 7: Nonparametric Bayesian D-optimal designs with truncated normal base distribution and concentration parameter when α=10. (with Zarepour method)

Parameters	*Design*	*Two points*		*Three points*		
$\mu = 50$, $\sigma = 30$	x	0.0991404	1.0000000	0.1000008	0.3293693	1.000000
	w	0.00000004	0.99999996	0.0000033	0.0043486	0.9956481
$\mu = 150$, $\sigma = 90$	x	0.9897642	1.0000000	0.1000003	0.3127211	0.9840623
	w	0.00000006	0.99999994	0.0289682	0.0587282	0.9123036

5 Concluding Remarks And Future Works

Nonlinear regression models are widely used in various scientific fields, and the Bayesian method is commonly employed to obtain optimal designs in such models. However, one of the challenges in the Bayesian framework is the subjective selection of the prior distribution, which can potentially lead to incorrect results. The choice of the prior distribution is often based on the researcher's beliefs, and it strongly influences the final outcome. Unfortunately, the Bayesian approach lacks a systematic method for selecting the prior distribution. To overcome these limitations and reduce reliance on restrictive parametric assumptions, nonparametric Bayesian methods are pursued. In this study, we consider the prior distribution as an unknown parameter and utilize the DP to derive nonparametric Bayesian D-optimal designs. Specifically, we focus on a nonlinear model, namely the exponentiated Shanker distribution. We investigate the Bayesian D-optimal design for the exponentiated Shanker regression model (equation 4.1) for a special case $\alpha = 1$, for using a truncated normal

prior distribution and examining various parameter values. By adopting a nonparametric Bayesian approach and utilizing the DP, we aim to address the challenges associated with selecting the prior distribution in Bayesian optimal design construction. This allows us to account for uncertainty and mitigate the impact of restrictive parametric assumptions, providing more flexible and robust designs for nonlinear regression models.

In this investigation, our emphasis is on employing truncated normal distribution as the base distribution in the DP. To provide a comprehensive understanding the influence of the concentration parameter α, we present the results in tables for different values of α. These tables provide valuable insights into the nonparametric Bayesian optimal designs and show the weights and support points. Through the analysis of results for different values of α, we can better understand the impact of this parameter on the design outcomes. This approach allows us to explore and evaluate the performance of the nonparametric Bayesian optimal designs across different levels of the concentration parameter α.

It is crucial to highlight that the approach employed in this study holds applicability to other optimality criteria and diverse models featuring two or more parameters. For instance, extending nonparametric Bayesian optimal designs to encompass A- or E-optimality criteria for the discussed nonlinear model, in conjunction with a Dirichlet process prior, presents avenues for future research. We anticipate presenting new findings in this domain in the near future.

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Locally, Bayesian and Nonparametric Bayesian optimal designs for Gumbel regression model with Respect to Prior Processes(with Polya Urn Scheme as the base measure)

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Abstract

This study presents optimal designs for Gumbel regression model, utilizing both Bayesian and nonparametric Bayesian methods. A nonparametric Bayesian approach was applied by introducing a Dirichlet Process (DP) prior over the space of distribution functions. We employed the D-optimality criterion considering the Dirichlet process as a functional tool with the aim of identification of the Nonparametric Bayesian optimal design.
Keywords: D-optimal design, Bayesian optimal design, Gumbel regression model, information matrix, Dirichlet process, Nonparametric Bayesian.

1 Introduction

As discussed in our previous work [26, 27, 28], in experimental design, the term "optimal design" refers to a specific class of designs categorized according to particular statistical criteria. It is widely acknowledged that a well-designed experiment can significantly enhance the accuracy of statistical analyses, specifically the accuracy of the parameters corresponding to the size of the parametric confidence region. Consequently, numerous researchers have dedicated their efforts to address the challenge of constructing optimal designs for nonlinear regression models. Experimental design plays a pivotal role in scientific research domains, including but not limited to biomedicine and pharmacokinetics. Its application in these fields enables researchers to conduct rigorous investigations and yield valuable insights.

Optimal designs are sought using optimality criteria, typically based on the information matrix. In nonlinear models, the presence of unknown parameters introduced complexities in the design problem, as the optimality criteria depends on these unknown parameters [3, 5]. To address this challenge, researchers proposed various solutions, including local optimal designs [6,17], sequential optimal designs, minimax optimal designs, Bayesian optimal designs [1, 29, 19-23], and pseudo-Bayesian designs [25]. Chernoff [7] proposed to use a suitable guess of the parameter value which leads to locally optimal design. This approach aimed to overcome the difficulties associated with the dependence of the design problem on unknown

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parameters in nonlinear models. It's important to note that local designs for nonlinear models are derived subsequent to an initial linearization of the model, using the parameter set as a reference point.

The selection of unknown parameters in local designs is typically obtained from previous studies or experiments specifically conducted for this purpose. The sensitivity of the locally optimal design with respect to the initial guess of parameter value is demonstrated by Dette, Melas, and Shpilev [11]. A locally optimal design do not incorporate any uncertainty in the parameter values. One of the proposed approaches, which deals with the uncertainty in parameter values, is the Bayesian procedure that assumes a prior distribution for the parameter. In the Bayesian method, the first step is to represent the available information in the form of a probability distribution for the model parameter, known as the prior distribution. So that, the idea of the Bayesian optimal design is to use the informative and/or historical knowledge of the unknown parameters as prior distribution.A Bayesian optimal design aims to maximize the relevant optimality criterion over this prior distribution. Nevertheless, it is crucial to acknowledge that the selection of the prior distribution within the Bayesian framework can be problematic and may potentially lead to erroneous results. The choice of the prior distribution is subjective, relying on the researcher's beliefs, and it significantly influences the final outcome. Unfortunately, the Bayesian approach lacks a definitive method for selecting the prior distribution. Subjective specification of a prior distribution which captures all existing information about the uncertainty of the parameter values was employed by many authors such as Chaloner and Larentz [10], Chaloner and Duncan [8], Burghaus and Dette [4], Chaloner and Vardinelli [9], Pronzato and Walter [30], Mukhopadhyay and Haines [24], Dette and Ngobauer [12,13], Fedorov [14], and Firth and Hinde [17] that have contributed extensively to this field. Chapter 18 of Atkinson et al.'s book [3] provides further reading on this topic. Often the choice of the prior is a compromise of a representation of uncertainty of the parameter values that may not be correct and can lead to misleading decision. Moreover, using the custom priors is a restrictive choice of priors, and it is possible to distort the actual prior distribution of the parameter. Indeed, no single probability distribution can model ignorance satisfactory, therefore large classes of distributions are needed. In this regards, we show that such a class is attained by considering infinite dimensional families of prior distributions; We consider the prior distribution function as an unknown distribution, which belongs to a class of distribution functions. Thus, from a Bayesian point of view, we need to construct a prior distribution on the space of all distribution functions. Several methods were introduced to construct a prior for random distribution. In this paper, we apply a nonparametric Bayesian approach that puts a prior on such families. In nonparametric Bayesian analysis, the prior process in terms of a probability measure P instead of the corresponding distribution function is discussed. The Dirichlet process, which is defined by Ferguson [16], plays a central role in nonparametric Bayesian methods. Dirichlet process (DP) involved two desirable properties of a prior; First, DP is defined on an arbitrary probability space. Second, it belonged to a conjugate family of priors. The aim of this paper is to apply Dirichlet process priors to obtain the Bayesian D-optimal design.

We organized this paper as follows. First, we will introduce the Gumbel distribution in section 2. In section 3, an introduction to optimal designs in nonlinear models is presented. In Section 4, the Bayesian D-optimal Design for the Gumbel model is calculated. Also the nonparametric Bayesian D-optimal design is introduced in this section . Finally, Section 5 concludes the paper with some closing remarks.

2 The Gumbel distribution

The Gumbel distribution is used to model the distribution of the maximum (or the minimum) of a number of samples of various distributions. This distribution might be used to represent the distribution of the maximum level of a river in a particular year if there was a list of maximum values for the past ten years. It is useful in predicting the chance that an extreme earthquake, flood or other natural disaster will occur. The potential applicability of the Gumbel distribution to represent the distribution of maxima relates to extreme

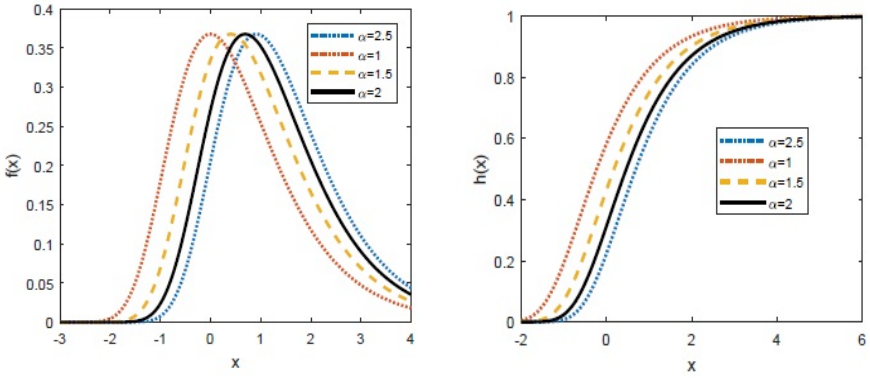


Figure 1: Plot of (a) pdf and (b) hrf of Gumbel distribution for $\theta = 1$.

value theory, which indicates that it is likely to be useful if the distribution of the underlying sample data is of the normal or exponential type.

The Gumbel distribution is a continuous distribution in statistics and probability theory. The cumulative distribution function of the Gumbel distribution (cdf) is given by:

$$F(x \mid \theta) = \exp(-\exp(-\theta x));\ x \in \mathbb{R}, \theta > 0, \tag{2.1}$$

and probability density function of this distribution is as follows:

$$f(x \mid \theta) = \theta \exp(-\theta x) \exp(-\exp(-\theta x));\ x \in \mathbb{R}, \theta > 0, \tag{2.2}$$

Now, we will introduce the two-parameter Gumbel distribution. The probability density function of the exponentiated Gumbel distribution is as follows:

$$f(x \mid \theta, \alpha) = \theta\alpha[\exp(-\exp(-\theta x))]^{\alpha} \exp(-\theta x);\ x \in \mathbb{R}, \theta, \alpha > 0, \tag{2.3}$$

In Figure 1, the pdf and the hrf of this distribution are graphed for various values of the parameters. According to this figure, it is evident that the pdf exhibits an unimodal trend for various values of the parameters and the hazard rate function of this distribution is also increasing.

3 Introduction to optimal designs in nonlinear models

In the realm of nonlinear experimental design, a common scenario arises where in the relationship between the response variable y and the independent variable x is given by the equation $y = \eta(x, \boldsymbol{\theta}) + \epsilon$ where $x \in \chi \subseteq \mathrm{R}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is the unknown parameter vector and ϵ is a normally distributed residual value with mean 0 and known variance $\sigma^2 > 0$. For simplicity, we assume $\sigma^2 = 1$ in this problem. If $\eta(x, \boldsymbol{\theta})$ is differentiable with respect to $\boldsymbol{\theta}$ then, the information matrix $I(x, \boldsymbol{\theta})$ at a given point x can be represented as follows:

$$I(x, \boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \eta(x, \boldsymbol{\theta}) \frac{\partial}{\partial \boldsymbol{\theta}^T} \eta(x, \boldsymbol{\theta}). \tag{3.1}$$

There exist several optimality criteria used to obtain the optimal design. the most common of which are D-optimality and A-optimality. These criteria are based on functions of the information matrix and can be expressed as follows:

$$\Psi_D(\xi,\boldsymbol{\theta}) = -\log(\det(M(\xi,\boldsymbol{\theta}))) \ , \ \Psi_A(\xi,\boldsymbol{\theta}) = tr(M^{-1}(\xi,\boldsymbol{\theta})),$$

where ξ denotes a design with two components; the first component represents specific values from the design space χ while the second component corresponds to the weights assigned to these values, so that design ξ can be defined as follows:

$$\xi = \left\{ \begin{array}{cccc} x_1 & x_2 & \dots & x_\ell \\ w_1 & w_2 & \dots & w_\ell \end{array} \right\} \in \boldsymbol{\Xi}, \tag{3.2}$$

where

$\boldsymbol{\Xi} = \{\xi \mid 0 \leq w_j \leq 1; \sum_{j=1}^{\ell}, \ w_j = 1, \ x \in \boldsymbol{\chi}\}$, $p \leq \ell \leq \dfrac{p(p+1)}{2}$ [23].

For a discrete probability measure ξ with finite support, the information function of ξ can be expressed as follows [3]:

$$M(\xi,\boldsymbol{\theta}) = \sum_{j=1}^{\ell} w_j I(x_j,\boldsymbol{\theta}). \tag{3.3}$$

In generalized linear and non-linear models, due to the dependence of the information matrix $M(\xi,\boldsymbol{\theta})$ to the unknown parameter $\boldsymbol{\theta}$, one approach to address this issue is to employ the Bayesian method and incorporate a prior distribution of the parameter vector instead of the local method to achieve a better result. The Bayesian D-optimality criterion can be formulated as follows:

$$\Psi_\Pi(\xi) = E(\psi(\xi;\boldsymbol{\theta})) = \int_\Theta \psi(\xi;\boldsymbol{\theta}) d\Pi(\boldsymbol{\theta}) = \int_\Theta -\log(\det(M(\xi,\boldsymbol{\theta}))) d\Pi(\boldsymbol{\theta}), \tag{3.4}$$

where Π represents the prior distribution for $\boldsymbol{\theta}$ and the Bayesian D-optimal design is attained by minimizing (3.4). In certain situations, specifying a prior distribution on the parameter space $\boldsymbol{\Theta}$ can be challenging for the experimenter. In such cases, an alternative approach involves considering an unknown prior distribution Π for the parameter $\boldsymbol{\theta}$. In this condition, Π is treated as a parameter itself. Consequently, equation (3.4) becomes a random functional, and it becomes necessary to determine its distribution or approximation. From a Bayesian perspective, we construct a prior distribution on the space of all distribution functions to address this issue. To achieve this objective, Ferguson introduced the concept of the Dirichlet Process (DP) [16].

4 Locally, Bayesian and Nonparametric optimal designs for Gumbel regression with polya urn scheme as the base measure

In the following section, we will discuss Locally and Bayesian optimal designs for Gumbel regression model in a special case $\theta = 1$. Bayesian optimal designs for the Gumbel regression model with a Uniform Prior Distribution focus on developing an efficient sampling strategy to best estimate model parameters. In this context, "optimal design" aims to select data points or intervals that maximize the amount of information gathered about the model while minimizing effort or resources required.

The locally optimal designs are found by replacing the unknown parameters with some initial estimates from a pilot or previous studies [6]. Locally optimal designs usually become inefficient when the initial estimates are far from their true values. Without loss of generality, we consider a bounded design space as $\chi = [0, 1]$ and we are focusing on continuous designs, specifically seeking designs to infer the pdf from data. In Table

Table 1: Locally D-optimal design for Gumbel regression model.

θ	0.01	0.1	0.5	0.9	1	10
x	0.0000	0.0000	0.2692	0.8570	0.9624	1.00000

2, locally D-optimal designs are shown for the Gumbel regression model (4.1). Now suppose we have the following regression model:

$$E(y|x) = \eta(x,\theta) = f(x \mid \theta,\alpha) = \theta\alpha[\exp(-\exp(-\theta x))]^{\alpha}\exp(-\theta x);\ x \in \mathbb{R}, \theta, \alpha > 0. \tag{4.1}$$

As is clear from Table 1, only one-point design is available due to the one-parameter nature of the Gumbel regression model. Also, we can see that the increase of the θ parameter leads the support point to 1. These designs are usually the minimum of support points and equal to the number of parameters. It should be noted that, they are often used to obtain other designs such as Bayesian, sequential and minimax optimal designs that are called saturated designs. The Bayesian optimal designs are found by optimizing an optimality criterion averaged over a user-specified (continuous) prior distribution for the unknown parameters [9,10]. Strictly speaking, the latter are not fully Bayesian because they do not involve computing a posterior distribution. Instead, they borrow the concept of having prior distributions to find robust designs for the frequentists [5,19]. Accordingly, they are sometimes referred to as "pseudo" Bayesian designs [17]. In the optimal design literature, Bayesian optimal designs found under a discrete prior distribution are usually referred to as robust or optimum-on-average designs [14]. For an overview of optimal designs for nonlinear models, see [17,18].

Now, let's consider the regression model (4.1), therefore, the Bayesian D-optimality criterion, denoted as $\Psi_{\Pi}(\xi)$ can be expressed as follows:

$$\begin{aligned}\Psi_{\Pi}(\xi) =& E(\psi(\xi;\boldsymbol{\theta}))\\ &= \int_{\Theta}\psi(\xi;\theta)d\Pi(\boldsymbol{\theta}) = \int_{\Theta} -\log(\sum_{j=1}^{\ell} w_j[\exp(-x_j)((\exp(-\theta\exp(-x_j)))\\ &+\theta(-\exp(-x_j))(\exp(-\theta\exp(-x_j))))]^2)d\Pi(\theta)\end{aligned} \tag{4.2}$$

where Π is the prior distribution for θ. The Bayesian D-optimal design is attained by minimizing equation (4.2).

In the nonparametric Bayesian framework, we consider $P \sim \mathrm{DP}(\alpha,P_0)$ and its collective representation as $P(.) = \sum_{i=1}^{\infty} p_i\ \delta_{\theta_i}(.)$. In this context, the optimality criterion can be expressed as follows:

$$\Psi_{\Pi}(\xi) = \sum_{i=1}^{\infty} p_i(-\log(\sum_{j=1}^{\ell} w_j[\exp(-x_j)((\exp(-\theta_i\exp(-x_j))) + \theta_i(-\exp(-x_j))(\exp(-\theta_i\exp(-x_j))))]^2)). \tag{4.3}$$

Equation (4.2) is a stochastic function of the DP. According to Ferguson's definition of the DP, the calculation (4.2) is not easily possible, so to solve this problem in obtaining the optimal nonparametric Bayesian criterion, methods such as the stick breaking process are used to approximate this criterion. Another method has been presented by Zarepour and Ellabadi [33] whose simulation speed and accuracy is much higher than the stick breaking process [32].

In Table 2, the Bayesian D-optimal design is shown for the Gumbel regression model (4.1) with respect to a uniform prior distribution for different values of ℓ(ℓ is the number of design points) and for various

Table 2: Bayesian D-optimal designs with respect to a uniform prior distribution on the interval [0, B] for Gumbel regression model (4.1).

Prior distribution	*Design*	*two − point*		*Three − point*		
$U[0,50]$	x	0.04178	1.00000	–	–	–
	w	0.01583	0.98417	–	–	–
$U[0,100]$	x	0.03857	0.99999	0.03927	0.04025	0.99999
	w	0.00442	0.99558	0.00117	0.00324	0.99559
$U[0,300]$	x	0.07912	0.99999	0.07926	0.08001	0.99999
	w	0.00140	0.99860	0.00112	0.00028	0.99860

Table 3: Nonparametric Bayesian D-optimal design when α=1 with Polya Urn Scheme as the base measure.

Prior distribution	*Design*	*Two − point*		*Three − point*		
$U[0,50]$	x	0.1624775	0.9997862	0.1000585	0.2152788	0.9962394
	w	0.0000005	0.9999995	0.0000003	0.0000004	0.9999993
$U[0,100]$	x	0.1424469	0.9976339	0.0997903	0.2043154	0.9971559
	w	0.0050516	0.9949484	0.0000008	0.0000008	0.9999984
$U[0,300]$	x	0.1077194	0.9998708	0.1001532	0.2053924	0.9989359
	w	0.0000036	0.9999964	0.0000028	0.0000026	0.9999946
$U[0,1000]$	x	0.1000005	1.0000000	0.1000000	0.2000000	1.000000
	w	0.0000030	0.9999970	0.0000068	0.0000081	0.9999991

Table 4: Nonparametric Bayesian D-optimal design when α=5 with Polya Urn Scheme as the base measure.

Prior distribution	*Design*	*Two − point*		*Three − point*		
$U[0,50]$	x	0.1622163	0.9751743	0.1002404	0.2149980	0.9977200
	w	0.0298513	0.9701487	0.0000003	0.0000003	0.9999994
$U[0,100]$	x	0.1237927	0.9996191	0.1002267	0.2095751	0.9983523
	w	0.0000011	0.9999989	0.0000008	0.0000011	0.9999981
$U[0,300]$	x	0.1059501	1.0000000	0.1000385	0.2018666	0.9996617
	w	0.0000035	0.9999965	0.0000026	0.0000025	0.9999949
$U[0,1000]$	x	0.1000003	1.0000000	0.1000003	0.1999899	1.000000
	w	0.0000052	0.9999948	0.0000044	0.0000047	0.9999909

parameter spaces [0, B]. According to our results, the largest point has the greatest weight and this weight increases by increasing the concentration parameter. Also, for three-point design, the largest support point has the greatest weight.

Tables 3-6 represent the results when the concentration parameter and uncertainty in the base measure increase. According to the results, when the value of α increases, the support points in two-point design do not significantly change and the largest point has the greater weight. This weight almost increases or remains fixed by increasing the concentration parameter. Additionally, in the case of a three-point design, the largest support point has the greatest weight.

5 Concluding Remarks And Future Works

Nonlinear regression models are widely used in various scientific fields, and the Bayesian method is commonly employed to obtain optimal designs in such models. However, one of the challenges in the Bayesian framework is the subjective selection of the prior distribution, which can potentially lead to incorrect results. The

Table 5: Nonparametric Bayesian D-optimal design when α=10 with Polya Urn Scheme as the base measure.

Prior distribution	*Design*	*Two − point*		*Three − point*		
$U[0,50]$	x	0.1410844	0.9978502	0.1000047	0.2192015	0.9921956
	w	0.0000005	0.9999995	0.0000003	0.0000004	0.9999993
$U[0,100]$	x	0.1238141	0.9980651	0.09988979	0.2116378	0.9977390
	w	0.0030780	0.9969220	0.0000007	0.0000008	0.9999985
$U[0,300]$	x	0.1058878	0.9997465	0.09989644	0.2053006	0.9977421
	w	0.0000032	0.9999968	0.0000026	0.0000026	0.9999948
$U[0,1000]$	x	0.2396256	1.0000000	0.1000004	0.2000000	1.0000000
	w	0.0000010	0.9999990	0.0000043	0.0000045	0.9999912

Table 6: Nonparametric Bayesian D-optimal design when α=50 with Polya Urn Scheme as the base measure.

Prior distribution	*Design*	*Two − point*		*Three − point*		
$U[0,50]$	x	0.1494640	0.9959386	0.09985050	0.2294167	0.9936747
	w	0.0080651	0.9919349	0.0000003	0.0000003	0.9999994
$U[0,100]$	x	0.1166416	0.9998919	0.1000399	0.2131274	0.9966916
	w	0.0000012	0.9999988	0.0000008	0.0000008	0.9999984
$U[0,300]$	x	0.1030530	1.0000000	0.1003115	0.2057020	0.9998710
	w	0.0000033	0.9999967	0.0000024	0.0000025	0.9999951
$U[0,1000]$	x	0.1000004	1.0000000	0.1000002	0.2000000	1.0000000
	w	0.0000038	0.9999962	0.0000035	0.0000036	0.9999929

choice of the prior distribution is often based on the researcher's beliefs, and it strongly influences the final outcome. Unfortunately, the Bayesian approach lacks a systematic method for selecting the prior distribution. To overcome these limitations and reduce reliance on restrictive parametric assumptions, nonparametric Bayesian methods are pursued. In this study, we consider the prior distribution as an unknown parameter and utilize the DP to derive nonparametric Bayesian D-optimal designs. Specifically, we focus on a nonlinear model with one-parameter, namely the Gumbel distribution. We investigate the Bayesian D-optimal design for the Gumbel regression model (equation 4.1) using a uniform prior distribution and examining various parameter values. By adopting a nonparametric Bayesian approach and utilizing the DP, we aim to address the challenges associated with selecting the prior distribution in Bayesian optimal design construction. This allows us to account for uncertainty and mitigate the impact of restrictive parametric assumptions, providing more flexible and robust designs for nonlinear regression models. To provide a comprehensive understanding the influence of the concentration parameter α, we present the results in tables for different values of α. These tables provide valuable insights into the nonparametric Bayesian optimal designs and show the weights and support points. Through the analysis of results for different values of α, we can better understand the impact of this parameter on the design outcomes.

The results reveal interesting findings. The results show that in the class of two- and three-point designs, the largest weight corresponds to the largest point. Moreover, as the uncertainty in the base measure and the concentration parameter in the Dirichlet process increase, the support points in the two- and three-point designs do not undergo significant changes. The weight of the largest point increases rapidly, and it becomes the point with the highest weight. This weight tends to either increase or remain relatively stable with an increase in the concentration parameter.

It is crucial to highlight that the approach employed in this study holds applicability to other optimality criteria and diverse models featuring two or more parameters. For instance, extending nonparametric Bayesian optimal designs to encompass A- or E-optimality criteria for the discussed nonlinear model, in

conjunction with a Dirichlet process prior, presents avenues for future research. We anticipate presenting new findings in this domain in the near future.

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Some topological properties of π-regular rings

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Abstract

A commutative and unital ring R is said to b π-regular if for every $r \in R$, there exists a natural number n and $s \in R$ such that $r^n = r^{2n}s$. In this paper considering a π-regular ring R we study some properties of Zariski topological space Spec(R).

Keywords: π-regular ring, Boolean ring, Zariski topology

1 Introduction

Regular rings has important rule in ring theory. A commutative and unital ring R is said to be regular if for every $r \in R$ there exists $s \in R$ such that $r = r^2s$[4]. on the other hand a commutative and unital ring R is said to b π-regular if for every $r \in R$, there exists a natural number n and $s \in R$ such that $r^n = r^{2n}s$[1,2]. So if we set $n = 1, s = 1_R$ then π-regular and regular rings are the same. One of the most important examples of π-regular rings are boolean rings. A commutative and unital ring R is a boolean ring if for every $r \in R$, we have $r^2 = r$. Boolean rings are very important specially in cryptography and coding theory.

Example 1.1.

1. Let $R = (\mathbb{Z}, +_2, .2)$ then $\bigoplus_{i\in I} R$ and $\prod_{i\in I} R$ are Boolean rings for every non empty index set I.
2. every field is a π-regular ring. For this if $r \in R$ then it is enough to considering $n = 1$ and $s = r^{-1}$
3. Let X is a set and $P(X)$ is the power set of X. Then $(P(X), \oplus, \odot)$ is a Boolean ring in which for every $A, B \in P(X)$ we have
$$A \oplus B = (A \cup B) - (A \cap B), A \odot B = A \cap B$$

Definition 1. *Let X is a set and $P(X)$ is the power set of X. Then $\Sigma \subseteq P(X)$ is called a topology on X and X is called a topological space if*

1. $\emptyset, X \in \Sigma$
2. *for every* $A_1, ..., A_n \in \Sigma, A_1 \cap ... \cap A_n \in \Sigma$
3. *for every* $\{A_i\}_{\in I} \subseteq \Sigma, \cup_{i\in I} A_i \in \Sigma$

Definition 2. 1. *If Σ is a topology on a set X then every $A \in \Sigma$ is called an open.*

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2. *A topological space X is called Hausdorff if for every $x, y \in X$, there exists $A, B \in \Sigma$ such that $A \cap B = \emptyset, x \in A, y \in B$.*

Definition 3. *Let R is a commutative and unital ring. for every ideal I of R we define*

$$V(I) = \{P \in Spec(R) \mid I \subseteq P)\}$$

in which $Spec(R)$ is the set of all prime ideal of R.

Lemma 1.2. *Let Let R is a commutative and unital ring. Then*

$$\Sigma = \{V^c(I) \mid I \trianglerighteq R\}$$

is a topology on $Spec(R)$, called Zariski topology[3].

Proof. We have $Spec(R) = V^c(R), \emptyset = V^c(0) \in \Sigma$. on the other hand if $I_1, ..., I_n$ are ideals of R then

$$V^c(I_1) \cap ... \cap V^c(I_n) = (V(I_1) \cup ... \cup V(I_n))^c = V^c(I_1....I_n) \in \Sigma$$

in addition if $\{I_i\}_{i\in\Lambda}$ is a subset of Σ then

$$\cup_{i\in\Lambda} V^c(I_i) = (\cap_{i\in\Lambda} V(I_i))^c = V^c(\cap_{i\in\Lambda} I_i) \in \Sigma$$

□

Example 1.3. let $p_1^{\alpha_1}...p_n^{\alpha_n}$ is the prime decomposition of a natural number n. Then in the ring $\mathbb{Z}$ we have

$$V(n\mathbb{Z}) = \{p_1\mathbb{Z}, ..., p_n\mathbb{Z}\}$$

2 π-regular rings and zariski topology

Theorem 2.1. *Let R is a π-regular ring. Then every prime ideal of R is a maximal ideal.*

Proof. let P is a prime ideal of R and $r \notin P$. Then by definition there exists a natural number n and $s \in R$ such that $r^n = r^{2n}s$. So $r^n(1 - r^n s) = 0 \in P$, $r \notin P$ Therefor $1 - r^n s \in P$. It means that $R = P + rR$ and hence P is a maximal ideal. □

By the above theorem if R is a π-regular ring and $P \in Spec(R)$, then $V(P) = \{P\}$.

Theorem 2.2. *Let R is a π-regular ring then $Spec(R)$ with the zariski topology is a Hausdorff topological space.*

Proof. Let R is a π-regular ring and $P \neq Q \in Spec(R)$, then $V(P) = \{P\}, V(Q) = \{Q\}$. So $P \in V^c(Q), Q \in V^c(P), V^c(P) \cap V^c(Q) = \emptyset$ and hence Spec(R) is a Hausdorff topological space. □

Theorem 2.3. *Let R is a π-regular ring and $r \in R$. Then we set*

$$U_r = \{P \in Spec(R) \mid r \notin P\}$$

then U_r in the topological space $Spec(R)$ is either open and closed.

Proof. we have

$$U_r = \{P \in Spec(R) \mid r \notin P\} = Spec(R)\ V(rR) = V^c(rR)$$

So U_r is an open set. On the other hand R is a π-regular ring so we have a natural number n and $s \in R$ such that $r^n = r^{2n}s$ or in the other word $r^n(1 - r^n s) = 0$. So for every $P \in Spec(R)$ we have $r^n(1 - r^n s) \in P$. Thus for every prime ideal P of R we have $r \in P$ or $1 - r^n s \in P$. If either $r, 1 - r^n s \in P$, then $P = R$ a contradiction. Thus $r \in P, 1 - r^n s \notin P$ or $r \notin P, 1 - r^n s \in P$. So

$$U_r \cap U_{1-r^n s} = \emptyset, U_r \cup U_{1-r^n s} = Spec(R)$$

iy means that

$$U_r = Spec(R)\ U_{1-r^n s} = (U_{1-r^n s})^c$$

So U_r is a closed set. □

Theorem 2.4. *Let Let R is a π-regular ring and $r \in R$,*

$$U_r = \{P \in Spec(R) \mid r \notin P\}$$

then if $r_1, ..., r_n \in R$ there is an $r \in R$ such that

$$U_{r_1} \cup ... \cup U_{r_n} = U_r$$

Proof. It is enough to prove if $r, r' \in R$ then $U_r \cup U_{r'} = U_{r+r'-rr'}$. For this let $P \in Spec(R), P \in U_r \cup U_{r'}$. Let $P \in U_r$ so $r \notin P$ and hence $1-r^n s \in P$ for some natural number n and some $s \in R$. Now if $P \notin U_{r+r'-rr'}$ then $r + r' - rr' \in P$ and hence $r \in P$ a contradiction. So $P \in U_{r+r'-rr'}$ and hence $U_r \cup U_{r'} \subseteq U_{r+r'-rr'}$. Conversely if $P \in Spec(R), P \in U_{r+r'-rr'}$ then $r + r' - rr' \notin P$. Now if $P \notin U_r, P \notin U'_r$ then $r \in P$ or $r' \in P$ so $r + r' - rr' \in P$ a contradiction. So $P \in U_r$ or $P \in U_{r'}$ thus $U_{r+r'-rr'} \subseteq U_r \cup Ur_{r'}$. So

$$U_r \cup U_{r'} = U_{r+r'-rr'}$$

□

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On α-SN-modules

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Abstract

We introduce and study the concept of α-SN-modules (a 0-SN-module is just a SN-module, i.e., there exists a small submodule S of M such that $\frac{M}{S}$ is Noetherian). Using this concept, we extend some of the basic results of SN-modules to α-SN-modules. We show that a module with only finitely many non-zero small submodules, which is an α-SN, has Noetherian dimension less than or equal to α. In particular, we prove that if M is a quotient finite dimensional module such that all of its submodules are α-SN, then $M \leq \alpha$.

Keywords: Noetherian dimension, small submodules, α-SN-dmoules.

1 Introduction

In [8], Lemonnier introduced the concept of the deviation of an arbitrary poset $(E, \leq)$, similar to the concept of Krull dimension of modules. Lemonnier also defined the concept of the dual Krull dimension of E which he named the codeviation of E, as being the Krull dimension (i.e., the deviation) of E^0, the opposite poset of E. We remind the reader that the dual Krull dimension of modules, that is Noetherian dimension, measures the deviation of a module from being Noetherian. For more details and some basic facts about the deviation and the codeviation of poset is referred to [2, 6, 7, 8, 9]. Let us denote the Noetherian dimension of a module M by M. In this article, we introduce and study the concept of α-SN-modules. An R-module M is called α-SN-module if there exists a small submodule S of M such that $\frac{M}{S} \leq \alpha$ and α is the least ordinal number with this property. Using this concept we extend some basic results SN-modules (i.e., there exists a small submodule S of M such that $\frac{M}{S}$ is Noetherian, see [4]) to α-SN-modules. We should remind the reader that by a quotient finite dimensional (briefly, qfd) module M we mean for any submodule N of M, $\frac{M}{N}$ has finite Goldie dimension. We prove that if M is qfd such that all of its submodules are α-SN, then $M \leq \alpha$. Further, if α is a countable ordinal, then every submodule of M is countably generated. An R-module M with only finitely many non-zero small submodules is said to be an fs-module. We show that an fs-module M is α-SN if and only if $\frac{M}{Rad(M)}$ is β-SN, for some $\beta \leq \alpha$. Also, we prove that every fs-module, which is an α-SN-module, has the Noetherian dimension at most α. Throughout this paper, all rings are associative with $1 \neq 0$, and all modules are unital right modules. It is convenient, when we are dealing with the above dimensions, to begin our list of ordinals with -1.

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2 preliminaries

We begin with the following definition.

Definition 1. *Let M be an R-module. A proper submodule S of M is small in M if $S + N \neq M$ for every proper submodule N of M. We will indicate that S is a small submodule of M by notation $S \ll M$.*

In the following, we recall some basic properties of small submodules, see [10].

Lemma 2.1. *Let M be a module with submodules $K \subseteq N \subseteq M$ and $P \subseteq M$. Then*

1. *$N \ll M$ if and only if $K \ll M$ and $\frac{N}{K} \ll \frac{M}{K}$.*
2. *$P + K \ll M$ if and only if $P \ll M$ and $K \ll M$.*
3. *If $K \ll N$, then $K \ll M$.*
4. *$Rad(M)$ is the sum of all the small submodules of M.*

Definition 2. *A right R-module M with only finitely many non-zero small submodules is said to be an fs-module.*

We note that if M is an fs-module, then $S \ll M$ iff $S \subseteq Rad(M)$.

Definition 3. *Let M be an R-module and $S \ll M$, $\frac{M}{S}$ is called small quotient module of M.*

The following result shows that the class of all fs-modules are closed under sumodules with non-zero small submodule and small quotient modules.

Proposition 2.2. *The following are equivalent for any R-module M.*

1. *M is a fs-module.*
2. *Every submodule of M, with non-zero small submodule, is a fs-module.*
3. *Every small quotient module of M is a fs-modules.*

3 α-SN-modules and their properties

According to [4], an R-module M is called an SN -module if $\frac{M}{S}$ is Noetherian, for some small submodule S of M. In this section, we generalize this concept and introduce the concept of α-SN -modules. We extend the basic results of SN-modules to α-SN-modules.

Next, we give our definition of α-SN-modules.

Definition 4. *An non-zero R-module M is called α-SN-module if it has a small submodule S such that $\frac{M}{S} \leq \alpha$ and α is the least ordinal number with this property.*

Clearly each 0-SN-modules is just an SN-module.

Remark 3.1. If M is an R-module with $M = \alpha$, then M is β-SN for some $\beta \leq \alpha$.

Proposition 3.2. *Let M be an α-SN-module. Then every factor module of M is β-SN, for some $\beta \leq \alpha$.*

Proof. Let N is a proper submodule of M. Since M is α-SN, it has a submodule S such that $\frac{M}{S} \leq \alpha$. Clearly $\frac{S+N}{N} \ll \frac{M}{N}$ and $\frac{M/N}{(N+S)/N} \simeq \frac{M}{N+S} \simeq \frac{M/S}{(N+S)/S}$. Thus $\frac{M/N}{(N+S)/N} = \frac{M}{N+S} = \frac{M/S}{(N+S)/S} \leq \frac{M}{S} \leq \alpha$, therefore we are done. □

Theorem 3.3. *The following statements are equivalent for every R-module M and every ordinal α.*
M is an α-SN-module.
$\frac{M}{S}$ is an α-SN-module, for every small quotient $\frac{M}{S}$.
$\frac{M}{S}$ is an α-SN-module, for some small quotient $\frac{M}{S}$.

Proof. (1) $\Longrightarrow$ (2) Suppose that M is an α-SN-module and $S \ll M$. Thus $\frac{M}{S}$ is β-SN, for some $\beta \leq \alpha$, by the previous proposition. That is, there exists a small submodule $\frac{N}{S}$ of $\frac{M}{S}$ such that $\frac{M}{N} = \frac{M/S}{N/S} \leq \beta$. But Lemma 2.1(1) implies that $N \ll M$ and by definition $\alpha \leq \beta$ and so $\alpha = \beta$.
(2) $\Longrightarrow$ (3) It is clear.
(3) $\Longrightarrow$ (1) By assumption, there exists a small submodule $\frac{N}{S}$ of $\frac{M}{S}$ such that $\frac{M/S}{N/S} \leq \alpha$. Since $S \ll M$, $\frac{N}{S} \ll \frac{M}{S}$, thus $N \ll M$, by Lemma 2.1(1) and we have $\frac{M}{N} \leq \alpha$ It follows that M is β-SN for some $\beta \leq \alpha$. On the other hand $\alpha \leq \beta$, By Proposition 3.2. This shows that $\alpha = \beta$ and we are done. □

Corollary 3.4. *Let M be a R-module and $Rad(M) \ll M$. Then M is an α-SN if and only if $\frac{M}{Rad(M)}$ is α-SN.*

In view of Proposision 3.2 and Theorem 3.3, we conclude the following result.

Corollary 3.5. *Let $0 \to N \to M \to K \to 0$ be a short exact sequence of R-modules where $N \ll M$. Then:*

1. *If M is an α-SN, then K is β-SN, for some $\beta \leq \alpha$.*
2. *If K be α-SN, then M is α-SN.*

In view of the definition of fs-modules, if M is an fs-module, then $Rad(M) \ll M$. Thus, we have the following result.

Corollary 3.6. *Let M be an fs-module. Then M is α-SN if and only if $\frac{M}{Rad(M)}$ is β-SN, for some $\beta \leq \alpha$.*

Lemma 3.7. *Let M be an R-module and $Rad(M) = 0$. Then M is α-SN if and only if $M \leq \alpha$.*

Lemma 3.8. *Let M be an R-module and $Rad(M) \ll M$. Then $\frac{M}{Rad(M)}$ is α-SN if and only if $\frac{M}{Rad(M)} \leq \alpha$.*

Proof. It suffices to show that zero is the only small submodule of $\frac{M}{Rad(M)}$. If $\frac{N}{Rad(M)}$ is a small submodule of $\frac{M}{Rad(M)}$, by Lemma 2.1(1), (4), N is a small submodule of M and $N \subseteq Rad(M)$. So $N = Rad(M)$ and therefore $\frac{N}{Rad(M)} = 0$. □

We recal that M is quotient finite dimensional (briefly, qfd) if for any submodule N of M, $\frac{M}{N}$ has finite Goldie dimension.
The following proposition, which is in [8, Theorem 2.4] and [1, Proposition 2.2], is needed.

Proposition 3.9. *The following statements are equivalent for any R-module M and any ordinal $\alpha \geq 0$:*
(1) M is a qfd module and for any submodules $N \subset P \subseteq M$, there exists a submodule X with $N \subseteq X \subset P$ such that $\frac{P}{X} \leq \alpha$.
(2) $M \leq \alpha$.

Theorem 3.10. *Let M be a qfd-module such that all of its submodules are γ-SN, for some ordinal γ. Then M has Noetherian dimension and $M \leq \alpha$, for $\alpha = Sup\{\gamma | there\ exists\ a\ \gamma - SN\ submodule\ of\ M\ \}$.*

Proof. It suffices to prove that M satisfies part (1) of the Proposition 3.9. For each $N \subset P \subseteq M$, $B = \frac{P}{N}$ has small quotient module, $\frac{B}{C}$ say, such that $\frac{B}{C} \leq \alpha$, by Lemma 3.2 and we are done. □

In [7], it is shown that every submodule of modules with countable Noetherian dimension is countably generated. Thus, the following result is immediate.

Corollary 3.11. *Let M be a qfd-module such that all of its submodules are α-SN which α is a countable ordinal. Then every submodule of M is countably generated.*

Lemma 3.12. *[5, Lemma 6] or [3] Let M be $AB5^*$. Then M is a qfd-module if and only if every submodule of M has finite hollow dimension.*

The following result is now immediate.

Corollary 3.13. *Let M be $AB5^*$ and all of its submodules are α-SN with finite hollow dimension. Then $M \leq \alpha$.*

Since each Artinian module is $AB5^*$ with finite hollow dimension, thus for the Artinian modules we have the following result.

Proposition 3.14. *Let M be an Artinian module and all of its submodules are α-SN. Then $M \leq \alpha$.*

Proposition 3.15. *Let M be an fs-module which is α-SN. Then $M \leq \alpha$.*

Proof. If M is an α-SN-module, then there exists a small submodule S such that $\frac{M}{S} \leq \alpha$. By Proposition 1, S is Noetherian. We infer that $(M) = sup\{\frac{M}{S}, S\} = \frac{M}{S} \leq \alpha$.

□

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Mathematical Approach to Latency Optimization in NZDS Fiber Applied to IoT Optical Networks

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Abstract

Due to ever-increasing demands for optical fibers with low-latency used in IoT optical fiber networks, in this theoretical study, a non-zero dispersion shifted (NZDSF) with a particular refractive index which had a minimum latency at 1352 nm is designed. In comparison to a commercial NZDSF, the latency has improved by 0.002 μs in the designed NZDSF with an effective area of 102 μm^2 and macro bending loss of 7.31×10^{-51} dB/km. The dispersion of this fiber is found to be 1.10625 ps/nm.km which is comparable to the commercial NZDSF fiber. We have designed an optical fiber network used in Internet of Things (IoT) networks in order to evaluate the quality of the received signals from 50 km of our designed NZDSF fiber. The results have shown that without using optical amplifier and DCF, the quality factor and minimum bit-error rate are obtained as 8.17 and 10^{-16} respectively.

Keywords: Latency, NZDSF, Optical networks, Optimization

1 Introduction

In recent years, dense wavelength division multiplexing (DWDM) networks have been implemented as a result of increasing demands for a massive bandwidth in long-haul systems, IoT optical fiber networks. During the light transmission, there are many parameters which effect on the quality of the received optical signals. One of these parameters is the nonlinear effects [1]. To overcome nonlinearity effects, large effective area fibers (LEAF) have been employed [2]. At the first generation, zero-dispersion shifted fibers (ZDSF) were used to achieve minimum loss and dispersion; but by increasing the number of signal wavelengths in DWDM networks, the effect of four-wave mixing (FWM) was observed which induces inter-channel crosstalk during the light transmission in optical links [3-4]. Besides ZDSFs, non-zero dispersion shifted fibers (NZDSF) have been designed to avoid the phase matching condition and reduce the effects of FWM in optical networks [1,4]. By using NZDSF, we may obtain a large effective area which minimizes the nonlinear effects [2]; however, larger effective area makes fiber more sensitive to bending loss. So, the choice of optimum parameters during the design of an optical fiber are very important. Using optical fibers with very large effective area may cause a very high mode field diameter (MFD) [5]. In order to avoid this outcome, effective area must be limited and optimized. In a study, the upper limit of effective area in a single mode fiber was reported around $370 \mu m^2$ [6]. In a study, an NZDSF with effective area of 95 μm^2 dispersion slope of 0.1 and bending loss of 0.005 dB has been designed [7]. An NZDSF with effective area of 102 μm^2, dispersion of 4 $ps/nm^2.km$, dispersion slope of 0.06 $ps/nm^2.km$ and bending loss of 0.0013 dB/km at the wavelength 1550 nm was designed in a report [8].

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One dopant has been employed in the core region of this optical fiber. In another study, these amounts have been optimized by using three dopants regions with different refractive index in the core; then the effective area was increased up to 120 μm^2 and the bending loss was decreased down to 1.4×10^{-14} dB/km [9]. At the wavelength of 1550 nm, the dispersion, effective nonlinear refractive index and 1st order PMD were 5.78 ps/nm.km, $1.41 \times 10^{-16} cm^2/w$ and 8.77×10^{-2} ps, respectively. Another important parameter in the next generation transmission communication systems is the latency. As it is reported before, the typical group delay in a conventional single mode fiber is about 5 $\mu s/km$ [10]. Regarding optical network latency due to physical layer, much works have not been reported in the literatures. In our study, we have designed an NZDSF with positive dispersion with a minimal latency and an optimum effective area.

2 Design Procedure

We have started our study by using the Corning conventional NZDSF with a dispersion of 4 ps/nm.km, effective area of 72 μm^2 and effective group refractive index of 1.4693. By assuming above values, we can calculate the latency ζ of the corresponding optical fiber by the following expressions at wavelength 1550 nm:

$$V_{1550} = c/n_{eff} = 299792.458/1.4693 = 204037.608 \ km/s \qquad [1]$$

$$\zeta_{1550} = 1_{km}/V_{1550} = 1/204037.608 = 4.901 \ \mu s \qquad [2]$$

where is the phase velocity of light in a specific medium, is the velocity of light in a vacuum, n_{eff} is the effective group refractive index, and ζ is the latency of optical signals at wavelength of 1550 nm. As we know, the latency of light in a vacuum is 3.336 μs; so we should find a way to minimize the latency of light in optical fiber used as a transmission medium in an optical fiber network. For this purpose, we have designed an NZDSF with a core of exponentially changing refractive n index profile. By adding Germanium or Fluorine atoms would increase or reduce the refractive index of pure silica, respectively. The refractive index n of m mole-percentage doped material can be defined as follows [11]:

$$n = \sqrt{n_o^2 + m/m(n_1^2 - n_0^2)} \qquad [3]$$

where n_1 is the refractive index of the host material and n_1 is the refractive index of doped material. Thus, the exponential profile function of the core can be shown as follows [11]:

$$n(x) = [e/(e-1)][n(0) - n(w)]exp(-x/w) + [en(w) - n(0)]/(e-1) \qquad [4]$$

where $n(0)$ and $n(w)$ are the refractive indices at $x = 0$ and $x = w$, respectively, and $e = 1.6 \times 10^{-16}$.

In order to calculate the latency of this fiber, we should find the group delay of light with refractive index of n for the distance of L by the first frequency-derivation of the propagation constant as follows [12]:

$$l = L\frac{d\beta}{d\omega} = z\frac{d\lambda}{d\omega}\frac{d(nk_0)}{d\lambda} = \frac{-2\pi cL}{\omega^2}(k_0\frac{dn}{d\lambda} + n\frac{dn}{d\lambda}) = \frac{L}{c}(n - \lambda\frac{dn}{d\lambda}) \qquad [5]$$

The total dispersion of light is calculated by the following expression:

$$D_{total} = -\frac{L}{c}\lambda\frac{d^2N_{eff}}{d\lambda^2} \qquad [6]$$

It's known the more effective area would decrease the none-linearity effects; so it is important to minimize the none-linearity of fiber in our design. Assume that $E(x, y)$ is the optical mode field distribution, then the effective area can be defined by using the following expressing [13]:

$$A_{eff} = \frac{(\int\int |E(x,y)|^2 dxdy)^2}{\int\int |E(x,y)|^4 dxdy} \qquad [7]$$

Also, the nonlinear effective index can be calculated in a fiber as follows:

$$n_{eff} = \frac{\int\int n_2(x,y)|F(x,y)|^4 dxdy}{\int\int |F(x,y)|^2 dxdy} \qquad [8]$$

where $n(x,y)$ is the user-defined spatially dependent nonlinear refractive index of the various layers of the fiber and $F(x,y)$ is the normalized mode field pattern. We have started our study by employing the profile of an optical fiber which was used to achieve a high negative dispersion in a DCF [14]. Since our first purpose was to design a positive dispersion NZDSF, we have changed the core profile into an exponential function. There are some literatures that showed the exponential profile would decrease the loss level [15, 16]. The core with a radius of 3 μm has a refractive index changing linearly from 1.4615 to 1.44692, marked as region 1. Regions 3, 5, and 7 have refractive indices of 1.455, 1.45, and 1.449 with the radius of 2, 3.7, and 52.58 μm, respectively. The refractive index of Regions 4 and 6 are 1.45 with radii of 1.0 and 0.2 μm, respectively. Also, Region 2 is formed with refractive index of 1.4448 and radius of 0.02 μm.

We have found that the minimum latency of 1.487 μs is obtained at the wavelength of 1352 nm. Using Eq. 3, the group index of the above profile is found as 1.46888. Replacing this amount in Eqs. 1 and 2, the estimated latency is obtained as 4.899 μs. So, we have 0.002 μs tolerance between the estimated and practical latencies. We found dispersion of the designed profile. As it's noted, total dispersion and dispersion slope are 1.10625 $ps/nm.km$ and 0.09112 $ps/nm^2.km$ respectively, at the wavelength of 1352 nm. The effective area of the profile is calculated as 102 μm^2. Because of the large effective area, there will be a limitation in bending condition. The macro- and micro-bending losses in this profile are obtained as 7.31×10^{-51} and 0.031 dB/km, respectively, at the wavelength of 1352 nm.

For the next step, we have employed our designed NZDSF in an optical network to evaluate how far the light will be transmitted and can be detected with desirable quality without using EDFA and DCF. For this aim, we have designed an optical network based on an optical transmitter and a receiver with the bit rate of 10 Gb/s.

The received signal at a distance of 50 km was monitored by using the eye-diagram, The quality factor and minimum bit-error rate of the received signal are 8.17 and 10^{-16} respectively, which are desirable amounts for this transmission distance [17].

3 Conclusion

In this study, we have designed an NZDSF with core refractive index of 1.46888 which had a minimum latency at 1352 nm. In comparison to Corning NZDSF, the latency has improved by 0.002 μs The effective area is found to be 102 μm^2 and macrobending loss is 7.31×10^{-51} dB/km. The dispersion of this fiber is 1.10625 ps/nm.km which is comparable to Corning NZDSF fiber. We have designed an optical fiber network in order to evaluate the quality of the received signals from 50 km of our fiber. The results have shown that without using optical amplifier and DCF, the quality factor and minimum bit-error rate are obtained as 8.17 and 10^{-16} respectively.

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A POD/DEIM Based Direct Shooting Method for Distributed Optimal Control of the Viscous Burgers Equation

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Abstract

This paper introduces an efficient computational approach for solving distributed optimal control problems governed by the viscous Burgers equation. The proposed method combines a direct shooting framework with model order reduction techniques to significantly enhance computational performance. The key idea involves replacing the conventional finite element (FE) solution of the governing partial differential equation (PDE) with a Proper Orthogonal Decomposition (POD) Galerkin approximation at each iteration step. Moreover, in the presence of the nonlinear term, to reduce the order of the nonlinear calculations, a Discrete Empirical Interpolation Method (DEIM) is applied.We show that the presented POD/DEIM direct shooting methods dramatically reduce the CPU time compared to the direct shooting method, whereas, there is no significant difference between the accuracy of the reduced and full indirect shooting methods.

Keywords: distributed optimal control, viscous Burgers equation, Proper Orthogonal Decomposition (POD), Discrete Empirical Interpolation Method (DEIM), shooting method

1 Introduction

The viscous Burgers equation, a simplified model of the Navier-Stokes equations, has attracted significant attention due to its ability to capture key features of fluid dynamics. As a result, extensive analytical and numerical studies have been devoted to investigating its properties and solutions [1].
In recent years, optimal control problems governed by the Burgers equation have emerged as an active area of research [2, 3].These problems serve as a foundational step toward developing advanced control methodologies for more complex fluid flows. The Burgers equation provides a tractable framework for testing numerical algorithms and theoretical approaches before their application to higher-dimensional flow systems.

In the distributed control problems for the viscous Burgers equation, the primary objective is to determine a control function $u(t,x)$, and a corresponding state function $y(t,x)$ such that the following Burgers equation

$$\frac{\partial}{\partial t}y(t,x) + \tfrac{1}{2}\frac{\partial}{\partial x}y^2(t,x) - \nu\frac{\partial^2}{\partial x^2}y(t,x) = h(t,x) + u(t,x), \tag{1.1}$$

with initial condition

$$y(0,x) = f(x), \tag{1.2}$$

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and boundary conditions

$$y(t,0) = 0, \tag{1.3}$$
$$y(t,1) = 0, \tag{1.4}$$

are satisfied and furthermore the following cost functional is minimized

$$\mathcal{J}[u] = \frac{1}{2}\int_0^T \int_0^1 \left(y(t,x) - z(t,x)\right)^2 dxdt + \frac{\sigma}{2}\int_0^T \int_0^1 u^2(t,x)dxdt. \tag{1.5}$$

Here, $z(t,x)$ is the desired state, and $\sigma > 0$ is the regularization parameter.

Due to physical realizability and technical limitations in practical applications, the control input $u(t,x)$ must often satisfy certain constraints. A common restriction can be expressed as:

$$u_a \leq u(t,x) \leq u_b. \tag{1.6}$$

Note that the numerical properties of (1.1)-(1.4) is strongly influenced by the viscosity parameter ν. Specifically, for small values of ν, the nonlinear term influences the numerical solution more and the PDE is called stiff.

2 Main results

Suppose that the functions $\varphi_1(x), ..., \varphi_m(x)$ be m linearly independent nodal basis functions and

$$X^h = \text{span}\{\varphi_1(x), ..., \varphi_m(x)\} \subset L^2(0,1). \tag{2.1}$$

We approximate the state function $y(t,x)$ and control function $u(t,x)$ by $y^h(t,x)$ and $u^h(t,x)$ as:

$$y(t,x) \simeq y^h(t,x) = \sum_{j=1}^{m} \mathrm{y}_j(t)\varphi_j(x), \tag{2.2}$$

$$u(t,x) \simeq u^h(t,x) = \sum_{j=1}^{m} \mathrm{u}_j(t)\varphi_j(x). \tag{2.3}$$

where $\mathrm{y}_j(t)$ and $\mathrm{u}_j(t)$, $j = 1, ..., m$ are unknown finite element coefficients and we set:

$$\mathbf{y}(t) = [\mathrm{y}_1(t), ..., \mathrm{y}_m(t)]^T, \quad \mathbf{u}(t) = [\mathrm{u}_1(t), ..., \mathrm{u}_m(t)]^T. \tag{2.4}$$

By the finite element spatial discretization, the distributed optimal control problem (1.1)-(1.6) is transcribed to the following optimal control problem governed by ordinary differential equations:

$$\min \quad J^h[u] = \tfrac{1}{2}\int_0^T \left(\mathbf{y}^T\mathbf{M}\mathbf{y} - \mathbf{z}^T\mathbf{y}\right) \mathrm{d}t + \tfrac{\lambda}{2}\int_0^T \left(\mathbf{u}^T\mathbf{M}\mathbf{u}\right) \mathrm{d}t \tag{2.5a}$$
$$\text{s.t.} \quad \mathbf{M}\dot{\mathbf{y}}(t) = -\tfrac{1}{2}\mathbf{B}\mathbf{y}^{\mathbf{2}}(t) - \nu\mathbf{C}\mathbf{y}(t) + \mathbf{h} + \mathbf{u}, \tag{2.5b}$$
$$\mathbf{M}\mathbf{y}(0) = \mathbf{f}, \tag{2.5c}$$
$$u_a \leq \mathbf{u}(t) \leq u_b. \tag{2.5d}$$

where

$$[\mathbf{M}]_{ij} = \int_0^1 \varphi_i \varphi_j dx, \quad [\mathbf{B}]_{ij} = \int_0^1 (\varphi_i)_x \varphi_j dx, \quad [\mathbf{C}]_{ij} = \int_0^1 (\varphi_i)_x (\varphi_j)_x dx, \tag{2.6a}$$

$$[\mathbf{h}(t)]_i = \int_0^1 h\varphi_i dx, \quad [\mathbf{f}]_i = \int_0^1 f\varphi_i dx, \quad [\mathbf{z}(t)]_i = \int_0^1 z\varphi_i dx. \tag{2.6b}$$

We propose a shooting method to numerically solve the optimal control problem (2.5). In this approach, the control function is parametrized, and during each iteration of the optimization process, the state variables are obtained by solving the system of equations (2.5b)-(2.5c).
To parametrize the control function, we employ Lagrange interpolation polynomials constructed over Gauss-Lobatto (GL) nodes within the interval $[0, T]$. Let $\{t_i\}_{i=1}^r$ denote the set of GL nodes in $[0, T]$, which are non-uniformly distributed and exhibit higher density near the endpoints of the interval.
Specifically, we utilize Legendre-Gauss-Lobatto (LGL) nodes, which are derived from the roots of the first derivative of the Legendre polynomial of degree $r-1$, augmented by the endpoints -1 and 1. These nodes are then mapped to the desired time interval $[0, T]$. Furthermore, let $\{\zeta_i(t)\}_{i=1}^r$ represent the corresponding Lagrange basis polynomials associated with the LGL nodes $\{t_i\}_{i=1}^r$. We consider

$$\mathbf{u}(t) = \left[\sum_{j=1}^r \nu_{1j}\zeta_j(t), ..., \sum_{j=1}^r \nu_{mj}\zeta_j(t)\right]^T$$

if we set $\mathbf{V} = [\nu_{ij}] \in \mathbb{R}^m \times \mathbb{R}^r$, then we have $\mathbf{u}(t) = \mathbf{u}(\mathbf{t}; \mathbf{V})$. By substiuting $u(t; \mathbf{V})$ in (2.5b)-(2.5c) we will have $\mathbf{y}(t; \mathbf{V})$ as solution of (2.5b)-(2.5c) in term of $\mathbf{V}$. In this case, the optimal control problem (2.5) express as following optimization problem

$$\min_{u_a \le \mathbf{v} \le u_b} \bar{J}^h(\mathbf{V}) = \frac{1}{2}\int_0^T \left((\mathbf{y}(t; \mathbf{V}))^T \mathbf{M}(\mathbf{y}(t; \mathbf{V})) - \mathbf{z}^T \mathbf{y}(t; \mathbf{V}) \right) \mathrm{d}t \tag{2.7}$$

$$+ \frac{\lambda}{2}\int_0^T \left((\mathbf{u}(t; \mathbf{V}))^T \mathbf{M}(\mathbf{u}(t; \mathbf{V})) \right) \mathrm{d}t. \tag{2.8}$$

By discretization integrals in (2.7), we will have finite dimensional optimiztion problem and use the Matlab function `fmincon` to solve it. In each iteration of this function, the solution of (2.5b)-(2.5c) is needed. Since the single shooting technique requires many iterations for obtaining optimal solution, we need to solve the system of ODEs (2.5b)-(2.5c) many times. This is very time consuming and making the method unattractive. Therefore, instead of solving the full PDE (1.1)-(1.4) with an expensive FE method, this high dimensional calculation will be substituted by the POD-Galerkin method. In POD-Galerkin method, we find a small Galerkin basis that well expresses the main properties of the underlying system. The main steps of POD-Galerkin method can be summerized as follow:

step 1. We compute the state $y_{\tilde{u}}$ for an arbitrary control $\tilde{u}$. If $0 \le t_1 < t_2 < ... < t_n \le T$ be a partition of $[0, T]$, then $y_j^h \approx y(t_j, x)$, $j = 1, ..., n$, that's called snapshots, can be computed by $y_{\tilde{u}}$.

step 2. Define $\mathcal{V} = span\{y_1^h, ..., y_n^h\}$, let $d = dim(\mathcal{V})$, fix $\ell \in \mathbb{N}, \ell \le d$. In our tests, we took $\ell = 3, 4, 5$. The POD basis of rank ℓ is the solution of the minimization problem:

$$\min_{\phi_1,...,\phi_l} \sum_{j=1}^n \alpha_j \left\| y_j^h - \sum_{i=1}^l \langle y_j^h, \phi_i \rangle \phi_i \right\|^2_{L^2(0,1)} \quad s.t \quad \langle \phi_i, \phi_j \rangle = \delta_{ij} \quad 1 \le i, j \le l \tag{2.9}$$

with certain weights $\alpha_j > 0$. We refer to [4] for solving this optimization problem.

	Direct Shooting Method		POD/DEIM Direct Shooting method	
m	$\mathcal{J}(y,u)$	$Time(s)$	$\mathcal{J}(y,u)$	$Time(s)$
50	0.199593	1257	0.199593	15
100	0.200111	6251	0.200185	33
200	0.2002082	10753	0.2002084	371
400	-	-	0.2002840	453

Table 1: Comparison of values of the cost functional and CPU time for Direct Shooting method and POD/DEIM Direct Shooting method.

step 3. With the small Galerkin basis $\{\phi_i\}_{i=1}^{\ell}$, we apply a Galerkin scheme based on this basis for determining the approximation $y^{\ell} \in \mathcal{V}^{\ell} = span\{\phi_1, ..., \phi_{\ell}\}$

$$y^{\ell}(t,x) = \sum_{i=1}^{\ell} \eta_i(t)\,\phi_i(x) \tag{2.10}$$

as solution of the weak form of (1.1)-(1.4).

As it has already been pointed out, the nonlinear term in reduced order model is still of large dimension m. By noting that, m is the number of the finite element basis functions and generally is a large number. As a result, the proposed algorithm is less efficient in the presence of nonlinearity. To overcome this drawback, we utilize DEIM, which is an efficient technique, with low computational complexity, for approximating the nonlinear functions [5].

3 Numerical results

We report on one of our numerical tests. In this example a distributed optimal control problem of Burgers equation with inequality control constraints is considered [3]. In the problem (1.1)-(1.6), let $T = 1$,$\sigma = 0.01$, $\nu = 0.1$ and

$$\begin{aligned} h(t,x) &= 0, \\ z(t,x) &= \sin(13x), \\ f(x) &= \sin(13x), \end{aligned}$$

where the following constraint on the control is considered

$$u(t,x) \leq 0.3.$$

For snapshots, the state equation is solved by the finite element scheme with 200 degrees of freedom. For time discretization an explicite Runge-Kutta scheme is applied. The POD-basis are determined by 200 snapshots of solution of state equation with arbitrary control $u_0(t,x) = 0$. In Figure 1, the obtained solutions with $m = 200$ and $r = 50$ are plotted.

Table 1 presents the results for different values of m and $r = 50$. The results clearly show that the POD/DEIM technique provides a significant improvement in computation speed. Furthermore, for $m = 400$, the POD/DEIM-based direct shooting method successfully computes the solution within a reasonable time, in contrast, the standard direct shooting method fails to produce results within an acceptable computational time. This demonstrates that the POD/DEIM approach is much more efficient for larger-scale problems.

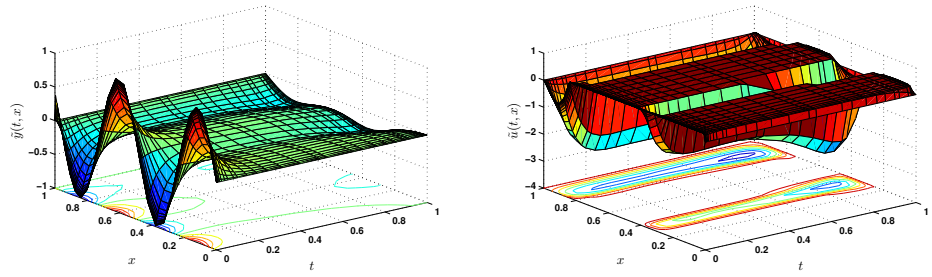


Figure 1: Optimal state and control functions obtained by the POD/DEIM Direct Shooting method with $m = 200$ and $r = 50$.

4 Conclusion

In this study, we integrated the POD technique into the shooting method to solve the distributed optimal control problem governed by the Burgers equation. Our results demonstrate that POD significantly enhances the computational efficiency of the shooting method while maintaining solution accuracy. Furthermore, to address the computational challenges posed by the nonlinear term in the Burgers equation, we employed the DEIM strategy. This approach effectively reduces the complexity of nonlinear calculations, leading to additional improvements in solution time. Based on our findings, we conclude that the direct shooting method, when combined with POD and DEIM techniques, becomes a more efficient for solving optimal control problems involving partial differential equations.

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Remarks on coupled best proximity points for cyclic contractive maps and their applications

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Abstract

We improve the known results about coupled fixed points and coupled best proximity points. We generalize the notion of ordered pairs of cyclic contraction maps and show that previous results on existence and uniqueness and priori and posteriori error estimates for the coupled fixed points and for the coupled best proximity points still hold.

Keywords: Coupled best proximity point, coupled fixed point, Cyclic contraction ordered pair, priori and posteriori error estimates.

1 Introduction

In this section, we recall some definitions and facts will be used hereafter.

Let A and B be nonempty subsets of a metric space (X, d). The map $T : A\cup B \to A\cup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. The cyclic map $T : A\cup B \to A\cup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. The cyclic map $T : A\cup B \to A\cup B$ is called a cyclic contraction map if there holds the inequality $d(Tx, Ty) \leq \lambda d(x, y) + (1 - \lambda)d(A, B)$ for some $\lambda \in (0, 1)$ and all $x \in A$ and $y \in B$. A point $\xi \in A\cup B$ is called a best proximity point for T if $d(\xi, T\xi) = d(A, B)$; see [1] and references therein. If the sets A and B have a nonempty intersection, then every best proximity point of T is a fixed point of T.

Definition 1. *[4, Definition 2.1] Let A_x, A_z, B_x and B_z be nonempty subsets of the metric space (X, d). Let $F : A_x \times A_z \to B_x$ and $f : A_x \times A_z \to B_z$, an order pair $(x^*, z^*) \in A_x \times A_z$ is called a coupled best proximity point of (F, f) if*

$$d(x^*, F(x^*, z^*)) = d(A_x, B_x) \quad \text{and} \quad d(z^*, f(x^*, z^*)) = d(A_z, B_z).$$

Definition 2. *[4, Definition 2.6] Let A_x, A_z, B_x and B_z be nonempty subsets of the metric space (X, d). Let $F : A_x \times A_z \to B_x$, $f : A_x \times A_z \to B_z$, $G : B_x \times B_z \to A_x$ and $g : B_x \times B_z \to A_z$. For any pair $(x, z) \in A_x \times A_z$, we defne the sequences $\{x_n\}$ and $\{z_n\}$ by $x_0 = x$, $z_0 = z$ and*

$$x_{2n+1} := F(x_{2n}, z_{2n}), \quad x_{2n+2} := G(x_{2n+1}, z_{2n+1}),$$
$$z_{2n+1} := f(x_{2n}, z_{2n}), \quad z_{2n+2} := g(x_{2n+1}, z_{2n+1}).$$

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Definition 3. *[4, Definition 2.4] Let A_x, A_z, B_x and B_z be nonempty subsets of the metric space (X,d). Let $F : A_x \times A_z \to B_x$, $f : A_x \times A_z \to B_z$, $G : B_x \times B_z \to A_x$ and $g : B_x \times B_z \to A_z$. The ordered pair of orderer pairs $((F,f),(G,g))$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers $\alpha,\beta,\alpha',\beta'$ such that $\lambda := \max\{\alpha+\alpha',\beta+\beta'\} < 1$ and there holds the inequality*

$$\begin{aligned} d(F(x,z),G(u,v)) + d(f(x',z'),g(u',v')) & \\ \leq \alpha d(x,u) + \alpha' d(x',u') + (1-(\alpha+\alpha'))d(A_x,B_x) & \\ + \beta d(z,v) + \beta' d(z',v') + (1-(\beta+\beta'))d(A_z,B_z), & \end{aligned}$$

for all $(x,z),(x',z') \in A_x \times A_z$ and $(u,v),(u',v') \in B_x \times B_z$.

2 Main results

In this section, we first improve Definition 3 as follows:

Definition 4. *Let A_x, A_z, B_x and B_z be nonempty subsets of the metric space (X,d). Let $F : A_x \times A_z \to B_x$, $f : A_x \times A_z \to B_z$, $G : B_x \times B_z \to A_x$ and $g : B_x \times B_z \to A_z$. The ordered pair of orderer pairs $((F,f),(G,g))$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers α,β such that $\lambda := \max\{\alpha,\beta\} < 1$ and there holds the inequality*

$$\begin{aligned} d(F(x,z),G(u,v)) + d(f(x,z),g(u,v)) & \\ \leq \alpha d(x,u) + (1-\alpha)d(A_x,B_u) + \beta d(z,v) + (1-\beta)d(A_z,B_v), & \end{aligned} \tag{2.1}$$

or equivalently

$$d^*(F(x,z),G(u,v)) + d^*(f(x,z),g(u,v)) \leq \alpha d^*(x,u) + \beta d^*(z,v),$$

for all $(x,z) \in A_x \times A_z$ and $(u,v) \in B_x \times B_z$.

Example 2.1. In the previous definition, suppose that $A_x = A_z = A$, $B_x = B_z = B$, $\alpha' = \frac{\alpha}{2}$, $\beta' = \frac{\beta}{2}$ and

$$F(x,z) = f(x,z), \quad \text{and} \quad G(u,v) = g(u,v).$$

With these assumptions, from relation (2.3) we have

$$2d(F(x,z),G(u,v)) \leq 2\alpha' d(x,u) + (1-2\alpha')d(A,B) + 2\beta' d(z,v) + (1-2\beta')d(A,B),$$

or equivalently

$$d(F(x,z),G(u,v)) \leq \alpha' d(x,u) + \beta' d(z,v) + (1-(\alpha'+\beta'))d(A,B), \tag{2.2}$$

for all $(x,z) \in A \times A$, $(u,v) \in B \times B$. Since $\alpha'+\beta' < 1$, Then (F,G) is a cyclic contraction defined in [2].

Remark 2.2. Note that if (F,G) is a cyclic contraction, from (2.2) we have

$$d(F(x,x),G(u,u)) \leq (\alpha'+\beta')d(x,u) + (1-(\alpha'+\beta'))d(A,B),$$

for all $x \in A$ and $u \in B$. So the cyclic map $T : A \cup B \to A \cup B$ defined by

$$Tz = \begin{cases} F(z,z) & \text{if } z \in A, \\ G(z,z) & \text{if } z \in B. \end{cases}$$

is a cyclic contractin. Therefore, under the conditions of Theorem 4 of [3], according to results of [1], T has a unique best proximity point $x^* \in A$ and $u^* \in B$ that $d(x^*,u^*) = d(A,B)$. It is obvious that (x^*,x^*) is unique coupled best proximity point of F in $A \times A$ and (u^*,u^*) is unique coupled best proximity point of F in $B \times B$. This point confirms Theorem 3.4 of [4].

The next theorem is extend of Theorem 4.1 of [4].

Theorem 2.3. *Let A_x, A_z, B_x and B_z be nonempty convex subsets a uniformly convex Banach space $(X, \|.\|)$ such that $d_x := d(A_x, B_x) > 0$. Let $F : A_x \times A_z \to B_x$, $f : A_x \times A_z \to B_z$, $G : B_x \times B_z \to A_x$ and $g : B_x \times B_z \to A_z$. Suppose that $((F, f), (G, g))$ is a a cyclic contraction ordered pair. Then*

(i) (F, f) has a unique coupled best proximity point $(x^*, z^*) \in A_x \times A_z$ and (G, g) has a unique coupled best proximity point $(x^{**}, z^{**}) \in B_x \times B_z$ such that $F(x^*, z^*) = x^{**}$, $f(x^*, z^*) = z^{**}$, $G(x^{**}, z^{**}) = x^*$ and $g(x^{**}, z^{**}) = z^*$. Moreover for any arbitrary point $(x, z) \in A_x \times A_z$ there hold

$$x_{2n} \to x^*, \quad z_{2n} \to z^*, \quad x_{2n+1} \to x^{**} \quad \text{and} \quad z_{2n+1} \to z^{**}.$$

If in addition $(X, \|.\|)$ has a modulus of convexity of power type with constants $C > 0$ and $q > 1$, then

(ii) a priori error estimates hold

$$\|x^* - x_{2m}\| \leq \frac{\mathcal{P}_{0,1}(x,z)}{1 - \sqrt[q]{\lambda^2}} \sqrt[q]{\frac{\mathcal{P}_{0,1}(x,z) - d}{Cd(A_x, B_x)}} (\sqrt[q]{\lambda})^{2m},$$
$$\|z^* - z_{2m}\| \leq \frac{\mathcal{P}_{0,1}(x,z)}{1 - \sqrt[q]{\lambda^2}} \sqrt[q]{\frac{\mathcal{P}_{0,1}(x,z) - d}{Cd(A_z, B_z)}} (\sqrt[q]{\lambda})^{2m};$$

(iii) a posteriori error estimates hold

$$\|x_{2n} - x^*\| \leq \frac{\mathcal{P}_{2n,2n-1}(x,z)}{1 - \sqrt[q]{\lambda^2}} \sqrt[q]{\frac{\mathcal{P}_{2n,2n-1}(x,z) - d}{Cd(A_x, B_x)}} \sqrt[q]{\lambda},$$
$$\|z_{2n} - z^*\| \leq \frac{\mathcal{P}_{2n,2n-1}(x,z)}{1 - \sqrt[q]{\lambda^2}} \sqrt[q]{\frac{\mathcal{P}_{2n,2n-1}(x,z) - d}{Cd(A_z, B_z)}} \sqrt[q]{\lambda}.$$

where

$$\mathcal{P}_{0,1}(x, z) := \|x_0 - x_1\| + \|z_0 - z_1\|,$$
$$\mathcal{P}_{2n,2n-1}(x, z) := \|x_{2n} - x_{2n-1}\| + \|z_{2n} - z_{2n-1}\|.$$

The next theorem is extend of Theorem 4.2 of [4].

Theorem 2.4. *Let A_x, A_z, B_x and B_z be nonempty subsets of a complete metric space (X, d), $F : A_x \times A_z \to B_x$, $f : A_x \times A_z \to B_z$, $G : B_x \times B_z \to A_x$ and $g : B_x \times B_z \to A_z$. Let there exist non-negative numbers α, β such that $\lambda := \max\{\alpha, \beta\} < 1$ and there holds the inequality*

$$d(F(x, z), G(u, v)) + d(f(x, z), g(u, v)) \leq \alpha d(x, u) + \beta d(z, v), \tag{2.3}$$

for all $(x, z) \in A_x \times A_z$ and $(u, v) \in B_x \times B_z$. Then

(i) There exists a unique pair $(x^*, z^*) \in (A_x \times A_z) \cap (B_x \times B_z)$, which is a common coupled fixed point for the mappings (F, f) and (G, g). Moreover the iteration sequences $\{x_n\}$ and $\{z_n\}$, defined in Definition 2 converge to x^* and z^* respectively.

(ii) a priori error estimates hold

$$\max\{d(x_n, x^*), d(z_n, z^*)\} \leq \frac{\lambda^n}{1-\lambda}(d(x_0, x_1) + d(z_0, z_1));$$

(iii) a posteriori error estimates hold

$$\max\{d(x_n, x^*), d(z_n, z^*)\} \leq \frac{\lambda}{1-\lambda}(d(x_{n-1}, x_n) + d(y_{n-1}, y_n));$$

(iv) The rate of convergence for the sequences of successive iterations is given by

$$d(x_n, x^*) + d(z_n, z^*) \leq \lambda(d(x_{n-1}, x^*) + d(z_{n-1}, z^*)).$$

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Cyclic L-contractions: a generalization of cyclic Meir-Keeler contractions and existence best proximity points

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Abstract

In this paper, we introduce the notion of cyclic L-contraction as a generalization notion of cyclic Meir-Keeler contraction and prove the existence of a best proximity point for these class of functions in the framework of metric spaces with property UC. Also we get an extention of Suzuki, Kikkawa and Vetro's Theorem.

Keywords: Best proximity point, Cyclic L-contractions, Meir-Keeler contractions, property UC.

1 Introduction and preliminaries

Let (X,d) be a metric space and let A and B be nonempty subsets of X. A mapping $T : A\cup B \to A\cup B$ is called a cyclic mapping if $T(A)\subseteq B$ and $T(B)\subseteq A$. $z\in A\cup B$ is called a best proximity point if $d(z,Tz) = d(A,B)$ is satisfied, where $d(A,B) := \inf\{d(x,y) : \ x\in A,\ y\in B\}$. Suzuki, Kikkawab and Vetro in [5] proved the following existence theorem.

Theorem 1.1. *[5] Let (X,d) be a metric space and let A and B be nonempty subsets of X such that (A,B) satisfies the property UC. Assume that A is complete. Let T be a cyclic mapping on $A\cup B$. Assume that there exists $c\in[0,1)$ such that*

$$d(Tx,Ty)\leq c\max\{d(x,y),d(x,Tx),d(y,Ty)\}+(1-c)d(A,B),$$

for every $x\in A$ and $y\in B$. Then there exists a unique best proximity point $z\in A$ such that is fixed point of T^2. Further, for each $x\in A$, $\{T^{2n}x\}$ converges to the best proximity point.

On the other hand, in [1], Di Bari, Suzuki and Vetro introduced the notion of cyclic Meir-Keeler contractions and proved the following existence theorem.

Theorem 1.2. *Let X be a uniformly convex Banach space and let A and B be nonempty subsets of X. Suppose that A is closed and convex. Let $T : A\cup B\to A\cup B$ be a cyclic Meir-Keeler contraction that is, for every $\epsilon>0$, there exists $\delta>0$ such that*

$$d(x,y)<d(A,B)+\epsilon+\delta, \quad \text{implies} \quad d(Tx,Ty)<d(A,B)+\epsilon,$$

for all $x\in A$ and $y\in B$. Then there exists a unique best proximity point in A. Further, for each $x\in A$, $\{T^{2n}x\}$ converges to the best proximity point.

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In this paper, we introduce the notion of cyclic L-contraction as a generalization notion of cyclic Meir-Keeler contraction [3] and prove the existence of a best proximity point for these class of functions in the framework of metric spaces with property UC. Also we get an extention of Suzuki, Kikkawab and Vetro's Theorem.
In the following, we give some preliminaries. We begin with Lim's characterization for Meir-Keeler contractions [2].

Definition 1. *[2] A function $\varphi : [0,\infty) \to [0,\infty)$ is called a L-function if $\varphi(0) = 0$, $\varphi(s) > 0$ for $s \in (0,\infty)$, and for every $s \in (0,\infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ for all $t \in [s, s+\delta]$.*

Lemma 1.3. *[4] Let φ be a L-function. Let $\{s_n\}$ be a nonincreasing sequence of nonnegative real numbers. Suppose $s_{n+1} < \varphi(s_n)$ for all $n \in \mathbb{N}$ with $s_n > 0$. Then $\lim_{n\to\infty} s_n = 0$.*

We finish this section with the following definition and a functional lemma.

Definition 2. *[5] Let A and B be nonempty subsets of the metric space (X,d). Then (A,B) is said to satisfies the property UC, if $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x'_n,y_n) = d(A,B)$, then $\lim_{n\to\infty} d(x_n,x'_n) = 0$.*

Suzuki et al. [5] proved that if A and B be nonempty subsets of uniformly convex Banach space X such that A is convex, then (A,B) has the property UC. Moreover, if A and B be nonempty subsets of a metric space (X,d) such that $d(A,B) = 0$, then (A,B) satisfies the property UC.

Lemma 1.4. *[5] Let A and B be subsets of a metric space (X,d). Assume that (A,B) has the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m,y_n) = d(A,B) \quad \text{or} \quad \lim_{n\to\infty} \sup_{m\geq n} d(x_m,y_n) = d(A,B).$$

Then $\{x_n\}$ is Cauchy.

2 Main results

Let A and B be nonempty subsets of a metric space (X,d) and let T be a cyclic mapping on $A \cup B$. For $x \in A$ and $y \in B$ we define

$$\mathrm{E}(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty)\} - d(A,B),$$

$$\mathrm{E}^{\psi}_{\varphi}(x,y) := \max\big\{\varphi(d(x,y) - d(A,B)), \psi(d(x,Tx) - d(A,B)) \\ , \psi(d(y,Ty) - d(A,B))\big\}.$$

Let us introduce the notion of cyclic L-contraction, which is a generalized notion of cyclic Meir–Keeler contractions.

Definition 3. *Let A and B be nonempty subsets of a metric space (X,d) and let T be a cyclic mapping on $A \cup B$. Then T is called a cyclic L-contraction if there exist nondecreasing and continuous L-functions φ and ψ that $\psi(t) < t$ for all $t \in (0,\infty)$, such that*

$$\mathrm{E}(x,y) > 0 \qquad \textit{implies} \qquad d(Tx,Ty) - d(A,B) < \mathrm{E}^{\psi}_{\varphi}(x,y)$$

and

$$\mathrm{E}(x,y) = 0 \qquad \textit{implies} \qquad d(Tx,Ty) - d(A,B) = 0$$

for all $x \in A$ and $y \in B$.

The following two lemmas obviously hold.

Lemma 2.1. *Suppose $T : A\cup B \to A\cup B$ is a cyclic Meir–Keeler contraction, then T is a cyclic L-contraction.*

Lemma 2.2. *Let A and B be nonempty subsets of a metric space (X, d), let $T : A \cup B \to A \cup B$ be a cyclic L-contraction and φ and ψ be L-functions as in Definition 3. Then for every $(x, y) \in A \times B$, the following hold*

(i) $d(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\}$

(ii) $d(Tx, Ty) - d(A, B) \leq \mathrm{E}_{\varphi}^{\psi}(x, y)$

Lemma 2.3. *Let A and B be nonempty subsets of a metric space (X, d), let $T : A \cup B \to A \cup B$ be a cyclic L-contraction. Define a sequence $\{x_n\}$ in $A \cup B$ by $x_0 \in A$ and $x_n := Tx_{n-1}$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$.*

Lemma 2.4. *Let (X, d), A, B, T and $\{x_n\}$ be as in Lemma 2.3. Suppose sequence $\{x_{2n}\}$ converges to $z \in A$. Then z is a best proximity point, that is, $d(z, Tz) = d(A, B)$ holds.*

Lemma 2.5. *Let A and B be nonempty subsets of the metric space (X, d) such that (A, B) has the property UC. Assume that T is a a cyclic L-contraction. Define a sequence $\{x_n\}$ in $A\cup B$ by $x_0 \in A$ and $x_n := Tx_{n-1}$ for $n \in \mathbb{N}$. Then*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_{2m}, x_{2n+1}) = d(A, B).$$

Lemma 2.6. *Let (X, d), A, B and T be as in Lemma 2.5. Let $z \in A$. Then the following are equivalent*

(i) z is a best proximity point of T.

(ii) z is a fixed point of T^2.

Lemma 2.7. *Let (X, d), A, B and T be as in Lemma 2.5 .Then T has at most one best proximity point in A.*

Theorem 2.8. *Let A and B be nonempty subsets of the metric space (X, d) such that A is complete and (A, B) has the property UC. Assume that T is a a cyclic L-contraction. Then there exists a unique best proximity point in A. Further, for each $x_0 \in A$, $\{T^{2n}x_0\}$ converges to the best proximity point.*

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On existence best proximity points for ϑ-contractions of Ćirić' type

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Abstract

In this paper, we introduce a new class of maps, called ϑ-contractions of Ćirić' type as a generalization of Ćirić's quasi-contractions. Using the concept of P-property on a metric space, we prove convergence and existence results of the best proximity points for this class of contractions. These results extend and improve best proximity point the known results.

Keywords: Best proximity point, nonself mapping, ϑ-contractions of Ćirić' type, P-property

1 Introduction and preliminaries

Let (X,d) be a metric space. Define a distance between two subset $A,B \subseteq X$ by $dist(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$. Let nonself mapping $T : A \to B$, a point $x^* \in A$ is called a best proximity point for T if $d(x^*,Tx^*) = dist(A,B)$. If $dist(A,B) = 0$, x^* is a fixed point of T. Let

$$A_0 := \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\};$$
$$B_0 := \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

In this section we give some basic definitions and concepts which are useful.

Definition 1. *[3] The pair is said to be a semi-sharp proximinal pair if the following condition is satisfied: We say the pair (A,B) have the P-property if*

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Longrightarrow d(x_1,x_2) = d(y_1,y_2),$$

for all $x_1,x_2 \in A$ and $y_1,y_2 \in B$.

Lemma 1.1. *[4] Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing function and I be an identity function defined on $\mathbb{R}_+$. If $I - \vartheta$ is a increasing function, then*

(a) *$\vartheta(t) > 0$, for all $t > 0$;*

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(b) $(I-\vartheta)(t) < t$ *for all* $t > 0$;

(c) ϑ *is continuous.*

From now on for the strictly increasing function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$, we assume that $I - \vartheta$ is also increasing. Let $C(\mathbb{R}_+, \mathbb{R}_+)$ be a space of continuous functions acting from $\mathbb{R}_+$ to $\mathbb{R}_+$. According to Lemma 1.1, these mappings can be displayed with the following subclass of $C(\mathbb{R}_+, \mathbb{R}_+)$:

$$\mathcal{K} := \{\varphi \in C(\mathbb{R}_+, \mathbb{R}_+) : \varphi \text{ is strictly increasing function and } I - \varphi \text{ is increasing}\}.$$

Given nonempty subsets A and B of a metric space (X, d), we set $d^*(x, y) := d(x, y) - dist(A, B)$ for every $x, y \in X$. It is immediately that

$$d^*(x, y) \leq d(x, z) + d^*(z, y)$$

and

$$d^*(x, y) - dist(A, B) \leq d^*(x, z) + d^*(z, y),$$

for all $x, y, z \in X$.

2 Main results

Let A and B be nonempty subsets of the metric space (X, d). In this section, with the aim of extending the concept Ćirić's quasi-contraction related to a self map to a nonself map. Let $T : A \to B$, we introduce the concept of a ϑ-contraction of Ćirić' type. Then we provide conditions for the existence of the best proximity point for it.

To establish our main results of this section, we introduce the following class of cyclic contraction type mappings.

Definition 2. *Let A and B be nonempty subsets of the metric space (X, d) and $T : A \to B$ be a mapping on $A \cup B$. Then $T : A \to B$ is said to be a ϑ-contraction of Ćirić type if there exists $\vartheta \in \mathcal{K}$ such that*

$$d(Tx, Ty) \leq (I - \vartheta)\big(\max\big\{d(x, y), d^*(x, Tx), d^*(y, Ty), d^*(x, Ty), d^*(y, Tx)\big\}\big). \tag{2.1}$$

for all $x, y \in A$.

Example 2.1. A Ćirić's quasi-contraction map in the sense of Theorem 2 of [**?**], is a ϑ-contraction of Ćirić type with $A = B = X$ and $\vartheta(t) = (1 - \lambda)t$ for $t \geq 0$ and $\lambda \in [0, 1)$.

The next lemma is very efficient in proving our main conclusions.

Lemma 2.2. *Let A and B be nonempty subsets of the metric space (X, d) and $T : A \to B$ be a ϑ-contraction of Ćirić type. Let $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$ and (A, B) has the P-property. Then*

(i) *for every $m, n \in \mathbb{N}$ with $n \geq m \geq 0$, we have*

$$0 \leq d(x_m, x_n) \leq (I - \vartheta)^m (M_{x_0}),$$

where

$$M_{x_0} = \vartheta^{-1}\big(d(x_0, x_1)\big);$$

(ii) *the sequence $\{(I - \vartheta)^k (M_{x_0})\}$ converges to 0;*

(iii) $\forall \epsilon > 0, \quad \exists m \in \mathbb{N}: \quad d(x_m, x_n) \leq \epsilon, \quad for \quad n \geq m.$

Theorem 2.3. *Let A and B be nonempty subsets of the metric space (X, d) and $T : A \to B$ be ϑ-contraction of Ćirić type. Let $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$ and (A, B) has the P-property. Then T has a unique best proximity point $x^* \in A$.*

Corollary 2.4. *Let A and B be nonempty subsets of the complete metric space (X, d) and $T : A \to B$ be such that*

$$d(Tx, Ty) \leq \lambda \max \big\{ d(x, y), d^*(x, Tx), d^*(y, Ty), d^*(x, Ty), d^*(y, Tx) \big\}. \tag{2.2}$$

for all $x, y \in A$ and some $\lambda \in [0, 1)$. Let $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$ and (A, B) has the P-property. Then T has a unique best proximity point $x^* \in A$.

Corollary 2.5. *[1, result of Theorem 2.10] [2, Theorem 1] Let T be a self mapping on a complete metric space (X, d) and there exists $\lambda \in [0, 1)$ such that satisfy the Ćirić's quasi-contraction condition*

$$d(Tx, Ty) \leq \lambda \max \big\{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \big\},$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$ such that $\{T^n x_0\}$ converges to x^* for every starting point $x_0 \in X$. Also, the following estimates

$$d(T^n x_0, x^*) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, Tx_0)$$

and

$$d(T^n x_0, x^*) \leq \frac{\lambda}{1 - \lambda} d(T^{n-1} x_0, T^n x_0),$$

hold, for all $x_0 \in A$.

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classifying finite groups with the same order and same prime graph as $PSL(3, p^2)$

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Abstract

In this paper we prove that an infinite series of finite simple groups $PSL_3(p^2)$ are uniquely determined by their order and prime graph in the class of all finite groups.
Keywords: Prime graph, recognition, linear group.

1 Introduction

Let G be a finite group. We denote the set of all prime divisors of $|G|$ by $\pi(G)$, and the set of orders of all elements in G by $\pi_e(G)$ ($\pi_e(G)$ is closed under divisibility and is uniquely determined by the set $\mu(G)$, which it's elements are maximal under divisibility relation). The prime graph of G is a simple graph denoted by $\Gamma(G)$ whose vertex set is $\pi(G)$, and $p \sim q$ in $\Gamma(G)$ or in other words p and q are adjacent, if and only if $pq \in \pi_e(G)$. The connected components of $\Gamma(G)$ are denoted by $\pi_i(G)$, $1 \leq i \leq s(G)$, where $s(G)$ is the number of the connected components of $\Gamma(G)$, we use the notation π_i instead of $\pi_i(G)$. If $|G|$ is even, we assume that $2 \in \pi_1$. The set of order components of G will be denoted by $OC(G) = \{m_1, m_2, \cdots, m_{s(G)}\}$, which $\pi(m_i) = \pi_i$ and each m_i, $i \neq 2$, is called an odd order component. Clearly $|G| = \prod m_i$, $1 \leqslant i \leqslant s(G)$. In Tables 1-3 of [4], the order components of non-abelian simple groups G with $s(G) > 1$ are shown.

The group G is said to be recognizable by prime graph and order if, for every finite group H, $\Gamma(G) = \Gamma(H)$ and $|G| = |H|$ imply $H \cong G$, in other words, G is uniquely determined by its order and prime graph in the class of all finite groups. About the recognition of infinite series of projective special linear groups by prime graph, Khosravi and et.al. in [3], proved that the simple group $PSL_n(3)$, where $n \geq 9$, is quasirecognizable by prime graph; i.e., if G is a finite group such that $\Gamma(G) = \Gamma(PSL_n(3))$, then G has a unique nonabelian composition factor isomorphic to $PSL_n(3)$. Also, in [2], by Khosravi it is proved that if G is a finite group such that $\Gamma(G) = \Gamma(PSL_2(q))$, and k is odd, then $G \cong PSL_2(q)$ and if k is even, then $G \cong PSL_2(q)$ or $G \cong PSL_2(q).2$, the non-split extension of $PSL_2(q)$ by $\mathbb{Z}_2$.

In this paper we prove that an infinite series of finite simple groups $PSL_3(p^2)$ are uniquely determined by their order and prime graph in the class of all finite groups. In other words, we show that $PSL_3(p^2)$ is recognizable by its order and prime graph.

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1.1 Preliminaries

Lemma 1.1. *Let G be a Frobenius group with kernel N and complement H. Then the following statements are valid:*
(1) N is nilpotent, moreover $GK(N)$ is complete;
(2) $|H|$ divides $|N|-1$;
(3) $s(G)=2$ and $S(G)=\{\pi(N),\pi(H)\}$.

Lemma 1.2. *Let G be a 2-Frobenius group. Then G is solvable.*

Lemma 1.3. *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)|\leq 2$.*

Lemma 1.4. *Let G be a group with $s(G)\geq 2$, then G is one of the following groups:*
(a) G is a Frobenius or 2- Frobenius group.
(b) G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, whrer H is a nilpotent group, such that H and $\frac{G}{K}$ are π_1-groups, $\frac{K}{H}$ is a non-abelian simple group and $|\frac{G}{H}|||\mathrm{Aut}(\frac{K}{H})|$. Moreover, any odd order component of G is also an odd order component of $\frac{K}{H}$.

1.2 The structure of $\Gamma(L)$

By Table 1 of [4],

$\pi_1(L_3(p^2))=\pi(p^6(p^2-1)(p^4-1))=\{p\}\cup\pi(p^2-1)\cup\pi(p^2+1)$, and $\pi_2(L_3(p^2))=\pi((p^4+p^2+1)/3)$.

Also, the following proposition give us the set of elements order of $L_3(p^2)$.

Proposition 1.5 (1). *$\mu(L_3(q))=\{q-1,p(q-1)/(3,q-1),(q^2-1)/(3,q-1),(q^2+q+1)/(3,q-1)\}$, where $q=p^f$ is odd.*

Put $q=p^2$, $p\geq 5$. Clearly 3 divides p^2-1. So, by the above proposition,

$$\mu(L_3(p^2))=\{p^2-1,p(p^2-1)/3,(p^4-1)/3,(p^4+p^2+1)/3\}.$$

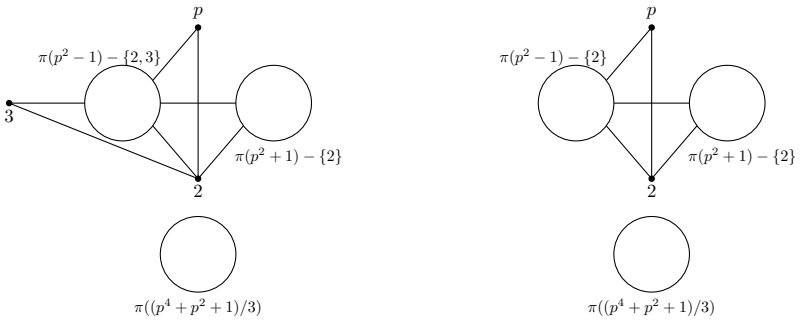
So, if $(p^2-1)_3=3$, then $p\nsim 3$ in $\Gamma(L_3(p^2))$ and also p is not adjacent to any vertices of p^2+1. Therefore, we have drawn the prime graph of $L_3(p^2)$ in two cases, $(p^2-1)_3=3$ and $(p^2-1)_3\neq 3$, as shown in Figure 1. In the figure below each circle represents a complete subgraph of the graph. The existence of an edge between a vertex and a circle means that the given vertex is adjacent to all vertices of the graph contained with in that circle.

2 Main results

Theorem 2.1. *Let G be a finite group, $p\geq 5$ is an odd prime and $q=p^2$. If $|G|=|L_3(q)|$ and $\Gamma(G)=\Gamma(L_3(q))$, then $G\cong L_3(q)$.*

Proof. $|L_3(q)|=q^3(q^2-1)(q^3-1)$, then $|G|=|L_3(p^2)|=p^6(p^4-1)(p^6-1)$, and $\Gamma(G)=\Gamma(L_3(p^2))$, same as Figure 1.

Now we show that G is neither Frobenius nor 2-Frobenius. If G is a solvable group, then it has a solvable Hall $\{p,r,t\}$-subgroup H, where $r\in\pi(p^2+1)$ and $t\in\pi((p^4+p^2+1)/3)$. Since there exists no edge between p, r and t in $\Gamma(G)$, then each element of H has prime power order. So, Lemma 2.3 implies that $|\pi(H)|\leq 2$,



under condition $(p^2-1)_3=3$ under condition $(p^2-1)_3\neq 3$

Figure 1: $\Gamma(L)$

a contradiction. Thus, G is not solvable and Lemma 2.2 implies that G is not 2-Frobenius. Let G be a non-solvable Frobenius group with Frobenius kernel N and complement H. Then $|N|=(p^4+p^2+1)/3$ and $|H|=3p^6(p^2-1)(p^4-1)$. Then by Lemma 2.1, $3p^6(p^2-1)(p^4-1)\mid(p^4+p^2-2)/3$, which is impossible. Hence, G is not a Frobenius group. Now, by Lemma 2.4, G has a normal series $1\trianglelefteq H_1\trianglelefteq K_1\trianglelefteq G$, where H_1 is a nilpotent group such that H_1 and $\frac{G}{K_1}$ are π_1-groups, $\frac{K_1}{H_1}$ is a non-abelian simple group and $\frac{G}{H_1}\leq \mathrm{Aut}(\frac{K_1}{H_1})$. Moreover any odd order components of G is an odd order component of $\frac{K_1}{H_1}$. Therefore $s(\frac{K_1}{H_1})\geq s(G)=2$. Now according to the classification of non-abelian simple groups in Tables 1-3 of [4], we obtain $\frac{K_1}{H_1}\cong L_3(p^2)$. For example, we have checked below that $\frac{K_1}{H_1}$ cannot be isomorphic to the Alternatig group. Suppose that $\frac{K_1}{H_1}\cong A_n$, according to Tables 1 and 3 of [4], the following two cases may occur.

1. Suppose $\frac{K_1}{H_1}\cong A_{p'}$, where $(p^4+p^2+1)/3=p'$ or $(p^4+p^2+1)/3=p'-2$. If $(p^4+p^2+1)/3=p'$, then by $|\frac{K_1}{H_1}|=|A_{p'}|$ and $|\frac{K_1}{H_1}|\mid|G|$ we conclude $\frac{((p^4+p^2+1)/3)!}{2}\leq 3p^6(p^2-1)(p^4-1)$, a contradiction. If $(p^4+p^2+1)/3=p'-2$, then by $p'=(p^4+p^2+7)/3$, we have $\frac{((p^4+p^2+7)/3)!}{2}\leq 3p^6(p^2-1)(p^4-1)$, that is impossible.

2. Suppose $\frac{K_1}{H_1}\cong A_{p'+1}$ or $A_{p'+2}$, then $(p^4+p^2+1)/3=p'$. Since $|A_{p'+1}|\mid|A_{p'+2}|$ and $|A_{p'}|\mid|A_{p'+1}|$, by 1, we get a contradiction. The rest of the cases can be ruled out similarly. Since $\frac{K_1}{H_1}\trianglelefteq\frac{G}{H_1}\leq\mathrm{Aut}(\frac{K_1}{H_1})$, we have $L_3(p^2)\trianglelefteq\frac{G}{H_1}\leq\mathrm{Aut}(L_3(p^2))$. As $|G|=|L_3(p^2)|$, we deduce $|H_1|=1$. Then $G\cong L_3(p^2)$ □

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Recognition of the split extension of $PSL_3(p^2)$ by duality automorphism

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Abstract

In the present paper, we show that almost simple groups $PSL_3(p^2) : 2_2$, where $p \geq 5$, are uniquely determined by their order and prime graph.

Keywords: Prime graph, recognition, linear group, almost simple group.

1 Introduction

Let G be a finite group. We denote by $\omega(G)$ the set of orders of elements of G. This set is closed under divisibility; hence is uniquely determined by a set $\mu(G)$ of elements in $\omega(G)$ which are maximal under divisibility relation. All the prime divisors of $|G|$ is denoted by $\pi(G)$. The prime graph of G is the graph with vertex-set $\pi(G)$, two vertices p and q being joined by an edge whenever G has an element of order pq, and in this case we will write $p \sim q$. The maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$ are denoted by $t(G)$. The number of connected components of $\Gamma(G)$ is denoted by $s(G)$ and the set of $\pi_1(G)$, $\pi_2(G)$, ...,$\pi_{s(G)}(G)$, the connected components of $\Gamma(G)$, is denoted by $S(G)$. We use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, we assume $2 \in \pi_i$. Let $m_i's$ are positive integers with $\pi(m_i) = \pi_i$. These $m_i's$ are called the order components of G. The set of order components of G will be denoted by $OC(G) = \{m_1, m_2, \cdots, m_{s(G)}\}$, then $|G|$ can be expressed as a product of its order components. Also we call m_i, $i \geq 2$, an odd order component. If G is a non-abelian simple group with $s(G) > 1$, its prime graph components are shown in Tables 1-3 of [3]

A group G is an almost simple group, if $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$, for some non-abelian group S. Let A and B be two groups then a split extension is denoted by $A : B$. If L is a finite simple group and $\mathrm{Aut}(L) \cong L : A$, then if B is a cyclic subgroup of A of order n, we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of order n in A, then we will denote them by $L : n_1$, $L : n_2$, etc. We know $\mathrm{Aut}(L_3(p^2)) \cong L_3(p^2) : D_{12}$ where D_{12} denotes the dihedral group of order 12. We remark that D_{12} has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong \mathbb{S}_3$ and one subgroup of order 4 isomorphic to the Klein's four group denoted by 2^2. The field and the duality automorphisms of $L_3(p^2)$ are denoted by 2_1 and 2_2 respectively, and we set $2_3 = 2_1.2_2$(duality∗field which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $L_3(p^2)$.

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Lemma 1.1. *If G is an almost simple group related to $L = L_3(p^2)$, then G is isomorphic to one of the following groups: $L, L:2_1, L:2_2, L:2_3, L:3, L:2^2, L:(D_6)_1, L:(D_6)_2, L:6, L:D_{12} \cong \mathrm{Aut}(L)$.*

Definition 1. *A finite group G is called k-recognizable by order and prime graph if there exist exactly k nonisomorphic groups H satisfying conditions $\Gamma(G) = \Gamma(H)$ and $|G| = |H|$. A 1-recognizable group is usually called a recognizable group.*

Put $L = PSL_3(p^2)$, where $p \geq 5$ is an odd prime. In the present paper, we show that almost simple groups $L:2_2$, are uniquely determined by their order and prime graph.

2 Preliminaries

Lemma 2.1. *Let G be a Frobenius group with kernel N and complement H. Then the following statements are valid:*
(1) N is nilpotent, moreover $GK(N)$ is complete;
(2) $|H|$ divides $|N| - 1$;
(3) $s(G) = 2$ and $S(G) = \{\pi(N), \pi(H)\}$.

Lemma 2.2. *Let G be a 2-Frobenius group. Then G is solvable.*

Lemma 2.3. *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 2.4. *Let G be a group with $s(G) \geq 2$, then G is one of the following groups:*
(a) G is a Frobenius or 2- Frobenius group.
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent group, such that H and $\frac{G}{K}$ are π_1-groups, $\frac{K}{H}$ is a non-abelian simple group and $|\frac{G}{H}|||\mathrm{Aut}(\frac{K}{H})|$. Moreover, any odd order component of G is also an odd order component of $\frac{K}{H}$.

Proposition 2.5. *[1] $\mu(L_3(q)) = \{q-1, p(q-1)/(3,q-1), (q^2-1)/(3,q-1), (q^2+q+1)/(3,q-1)\}$, where $q = p^f$ is odd.*

Now, we find the structure of prime graph of $PSL_3(p^2)$. By Table 1 of [3]:

$\pi_1(L_3(q)) = \pi(q^3(q-1)(q^2-1))$ and $\pi_2(L_3(q)) = \pi((q^2+q+1)/3)$.

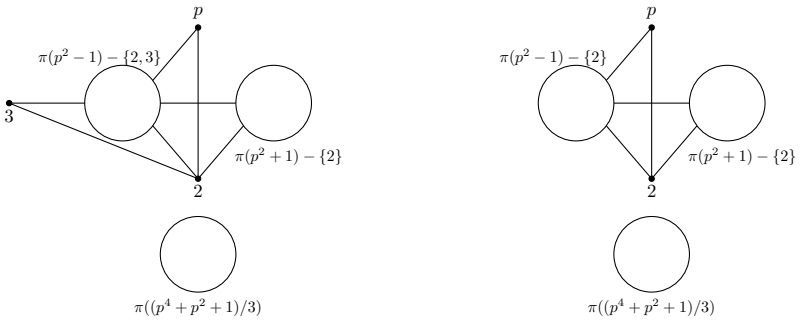
Then $\pi_1(L_3(p^2)) = \{p\} \cup \pi(p^2-1) \cup \pi(p^2+1)$, and $\pi_2(L_3(p^2)) = \pi((p^4+p^2+1)/3)$. Clearly 3 divides p^2-1. So, by proposition 2.5,

$$\mu(L_3(p^2)) = \{p^2-1, p(p^2-1)/3, (p^4-1)/3, (p^4+p^2+1)/3\}.$$

If $(p^2-1)_3 = 3$ (the p-part of n is denoted by n_p, i.e., $n_p = p^t$ if $p^t|n$ and $p^{t+1} \nmid n$), then by $\mu(L_3(p^2))$, $p \nsim 3$ in $\Gamma(L_3(p^2))$ and also p is not adjacent to any vertices of p^2+1. Therefore, we have drawn the prime graph of $L_3(p^2):2_2$ in two cases, $(p^2-1)_3 = 3$ and $(p^2-1)_3 \neq 3$, as shown in Figure 1. In the figure below each circle represents a complete subgraph of the graph. The existence of an edge between a vertex and a circle means that the given vertex is adjacent to all vertices of the graph contained with in that circle.

3 Main results

Theorem 3.1. *Put $L = PSL_3(p^2)$, where $p \geq 5$ is an odd prime. Consider $L:2_2$, the split extension of L by duality automorphism. Let G be a finite group, if $|G| = |L:2_2|$ and $\Gamma(G) = \Gamma(L:2_2)$, then $G \cong L:2_2$.*



under condition $(p^2-1)_3=3$ under condition $(p^2-1)_3\neq 3$

Figure 1: $\Gamma(L:2_2)$

Proof. $|G|=2|L_3(q)|=2q^3(q^2-1)(q^3-1)$. By [2], $\Gamma(L:2_2)$ is not connected, so 2 isn't adjacent to any vertices of π_2. Therefore $\Gamma(L:2_2)=\Gamma(L)$, similar to Figure 1.

Now we show that G is neither Frobenius nor 2-Frobenius. If G is a solvable group, then it has a solvable Hall $\{p,r,t\}$-subgroup H, where $r\in\pi(p^2+1)$ and $t\in\pi((p^4+p^2+1)/3)$. Since there exists no edge between p, r and t in $\Gamma(G)$, then each element of H has prime power order. So, Lemma 2.3 implies that $|\pi(H)|\leq 2$, a contradiction. Thus, G is not solvable and Lemma 2.2 implies that G is not 2-Frobenius. Let G be a non-solvable Frobenius group with Frobenius kernel N and complement H. Since the prime graph of Frobenius kernel is complete, so $|N|=(p^4+p^2+1)/3$ and $|H|=6p^6(p^2-1)(p^4-1)$. Then by Lemma 2.1, $6p^6(p^2-1)(p^4-1)\mid(p^4+p^2-2)/3$, which is impossible. Hence, G is not a Frobenius group. Now, by Lemma 2.4, G has a normal series $1\trianglelefteq H_1\trianglelefteq K_1\trianglelefteq G$, where H_1 is a nilpotent group such that H_1 and $\frac{G}{K_1}$ are π_1-groups, $\frac{K_1}{H_1}$ is a non-abelian simple group and $\frac{G}{H_1}\leq \mathrm{Aut}(\frac{K_1}{H_1})$. Moreover any odd order components of G is an odd order component of $\frac{K_1}{H_1}$. Therefore $s(\frac{K_1}{H_1})\geq s(G)=2$. Now according to the classification of non-abelian simple groups in Tables 1-3 of [3], we obtain $\frac{K_1}{H_1}\cong L$.

Since $\frac{K_1}{H_1}\trianglelefteq\frac{G}{H_1}\leq\mathrm{Aut}(\frac{K_1}{H_1})$, we have $L\trianglelefteq\frac{G}{H_1}\leq\mathrm{Aut}(L)$. As $|G|=2|L|$, we deduce $|H_1|=1$ or 2.

(1) If $|H_1|=1$, then $G\cong L:2_2$.

(2) If $|H_1|=2$, then $H_1\leqslant Z(G)$. Therefore, $2\sim t$ in $\Gamma(G)$, that is a contradiction.

□

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Generalized V-Harmonic Maps: Extending Stability and Applications in Geometry and Physics

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Abstract

Extending the classical theory of harmonic maps into Riemannian manifolds, we introduce the theory of generalized V-harmonic maps. Harmonic maps play a key role in differential geometry and have important links to topology, partial differential equations and mathematical physics owing to their rich geometric and analytic properties. We next investigate generalized stable V-harmonic maps as critical points of generalized V-energy functional. The stability guarantees good-behaved maps under perturbations. Our extended framework gives rise to new stability conditions and determines fundamental properties such as stability of regularity, or energy distribution with respect to oscillation (perturbation) and geometric considerations. Finally, we outline the implications of our results and mention applications for mathematics and physics. The extended notion of V-harmonic maps allows to study different geometric flows, minimal surfaces and variational problems that involve energies minimization. By introducing stability conditions we also enrich the manifestry analytical tools, thus endowing research in the future. Connecting to the existing theories and asking open problems, we intend to show that there is a lot of for other work using this framework and it can be applied to modern geometric problems.
Keywords: V-harmonic maps, V-tension field, Conformal maps, Riemannian manifolds

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Gradient Estimates and Liouville-Type Theorems for V-Harmonic Maps with Potential

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Abstract

In this paper, we study V-Harmonic maps with potential from complete manifolds, which is a newer and more general class of maps than harmonic maps. Many of these maps relate to deep questions in mathematics and physics, like the equilibrium system of ferromagnetic spin chains and Neumann motion. Potential terms also make rich contributions to the mathematical framework, allowing the study of more general geometric and physical phenomena. We derive basic gradient estimates for these maps, focusing on their general properties. We then prove Liouville-type results, which are surprising in comparison to those of classical harmonic maps owing to the presence of potentials. We also establish V-Harmonic maps and their applications to modern geometry and theoretical physics.

Keywords: V-harmonic maps, V-tension field, Conformal maps, Riemannian manifolds

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Stability Analysis of V-Harmonic Maps on Compact Riemannian Manifolds with Applications

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Abstract

This paper studies v-harmonic maps on compact Riemannian manifolds with homogeneous strongly harmonic manifolds as their target spaces. These generalized harmonic maps are of great interest in both geometric analysis and mathematical physics. We concentrate on the stability properties of such maps by considering its Morse index. This is achieved by perturbing the v-harmonic maps along a certain class of conformal vector fields deciding a cosmic time, leading to small deformation results of fractal dimension. The number of negative eigenvalues of the second variation operator, known as the Morse index, is vital in deciding the stability of critical points of energy functionals. These results are also part of the more general theory of harmonic maps by allowing to extend tools and techniques to the v-harmonic setting. Besides investigating variational problems with great assistance from the values of functions and the possible approach thereof, the essence of research on functional inequalities is to provide a versatile analytic tool as well as an instrument for application in this way, including areas such as minimal surfaces, geometric flows, and energy minimization.

Keywords: V-harmonic maps, V-tension field, Conformal maps, Riemannian manifolds

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